Assignment 9, due May 1

1. Prove the **Rank Theorem**: Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be a smooth function, $m > n$. Suppose that at a point $x_0 \in \mathbf{R}^m$ the rank of the tangent map df_{x_0} equals n. Prove that there exist maps ϕ and ψ, such that ϕ maps a neighborhood of x_0 diffeomorphically onto a neighborhood of 0 in $\mathbf{R}^m$, ψ maps a neighborhood of $f(x_0)$ diffeomorphically onto a neighborhood of 0 in $\mathbf{R}^n$ and

$$\psi \circ f \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^n)$$

for $x = (x^1, \ldots, x^m)$ in the range of ϕ.

2. Let $f : M^m \to N^n$ be a smooth map between manifolds of dimensions $m \geq n$ and y—a regular value of f. Use the rank theorem to prove that $f^{-1}(y)$ is a smooth submanifold of dimension $m - n$.

3. Let M be a manifold (without boundary) and $f : M \to \mathbf{R}$—a smooth function, having 0 as its regular value. Prove that the set of points $x \in M$, where $f(x) \geq 0$ is a manifold with boundary and that the boundary is equal $f^{-1}(0)$.

4. Let X be an m-dimensional manifold with boundary ∂X and $f : X \to N$—a smooth function into an n-dimensional manifold (without boundary), $m > n$. Let y be a regular value both for f and for its restriction to ∂X. Prove that $f^{-1}(y)$ is an $m - n$-dimensional manifold with boundary and that its boundary is equal to the intersection of $f^{-1}(y)$ with ∂X.

5*. Prove that a compact, one-dimensional manifold with boundary is diffeomorphic to a finite union of closed intervals and/or circles.

6*. Recall that a manifold is orientable if the coordinate charts can be chosen so that all transition maps have positive Jacobi determinants. This definition extends to manifolds with boundary. Prove that if X is an orientable manifold with boundary, then ∂X is also orientable.