Assignment 4, due Feb 20

1. Prove that a topological space X is contractible to a point if and only if for any topological space Y any two maps of Y into X are homotopic.

2. Prove that a contractible space is path-connected.

3. A subspace X of $\mathbf{R}^n$ is called convex if for every x, $y \in X$ and $t \in [0,1]$, $(1-t)x + ty \in X$. Prove that every such X is contractible to a point.

4. Prove that the subset $S^1 \times \{x_0\}$ is a retract of $S^1 \times S^1$, but that it is not a deformation retract.

5. Let $i : X \to X \times Y$ and $j : Y \to X \times Y$ be maps defined by $i(x) = (x, y_0)$ and $j(y) = (x_0, y)$, where $x_0 \in X$ and $y_0 \in Y$ are fixed points. Prove that the mapping of $\pi(X, x_0) \times \pi(Y, y_0)$ into $\pi(X \times Y, (x_0, y_0)$ defined by $(\beta, \gamma) \mapsto (i_*\beta) \cdot (j_*\gamma)$ is an isomorphism of the product $\pi(X, x_0) \times \pi(Y, y_0)$ onto $\pi(X \times Y, (x_0, y_0)$.

6. Considered the torus T defined as $S^1 \times S^1$ with the base point $a_0 = (1, 1)$. Let α denote the element of $\pi(T, a_0$ defined by the loop $t \mapsto (e^{2\pi it}, 1)$ and β—the element defined by the loop $t \mapsto (1, e^{2\pi it})$. Construct a homotopy between the products $\alpha\beta$ and $\beta\alpha$.