Homework 3, due February 20

Consider a smooth vector field, which in coordinates $x^1 \dots x^n$ has components $\xi^1 \dots \xi^n$. Let F_t denote the flow associated with ξ, i.e. let $F_t(x_0)$ denote the solution of the differential equation

$$\frac{dx}{dt} = \xi(x(t)$$

with the initial condition $x(0) = x_0$. By the fundamental theorem of the theory of ordinary differential equations this equation has a unique solution, defined at least for t close to 0, which is all we need. More precisely, for every x_0, the maps F_t are defined in a neighborhood of x_0 for small $|t|$ and they are diffeomorphisms.

Given a tensor T, we can transform it using the map F_t:

$$(F_tT)^{i_1\dots i_p}_{j_1\dots j_q} = T^{l_1\dots l_p}_{k_1\dots k_q} \frac{\partial x^{k_1}}{\partial x_0^{j_1}} \cdots \frac{\partial x^{k_q}}{\partial x_0^{j_q}} \frac{\partial x_0^{i_1}}{\partial x^{l_1}} \cdots \frac{\partial x_0^{i_p}}{\partial x^{l_p}},$$

where x stands for $F_t(x_0)$. F_tT describes the change of T along the trajectories of the vector field ξ. We define the Lie derivative of T in the direction of ξ as the derivative of the above expression with respect to t at time 0.

Definition: The Lie derivative of a tensor T along a vector field ξ is the tensor

$$L_\xi T^{i_1\dots i_p}_{j_1\dots j_q} = \left[\frac{d}{dt}(F_tT)^{i_1\dots i_p}_{j_1\dots j_q}\right]_{t=0}.$$

1. Prove that $L_\xi T$ is indeed a tensor, of the same type as T.

2. Prove that as $t \to 0$, the entries of the Jacobi matrices entering the definition of $L_\xi T$ satisfy:

$$\frac{\partial x^i}{\partial x_0^j} = \delta^i_j + t\frac{\partial \xi^i}{\partial x_0^j} + o(t);$$
$$\frac{\partial x_0^i}{\partial x^j} = \delta^i_j - t\frac{\partial \xi^i}{\partial x^j} + o(t).$$

3. Use the result of problem 2 to calculate the explicit expression for the components $(L_\xi T)^{i_1\dots i_p}_{j_1\dots j_q}$ of the Lie derivative of T.

4. Show that for a scalar function f, the Lie derivative is the usual directional derivative in the direction of ξ:

$$L_\xi f = \xi^i \frac{\partial f}{\partial x^i} = \partial_\xi f.$$

5. Given another vector field η, find the components of the vector field $L_\xi \eta$. Show that

$$L_\xi \eta = -L_\eta \xi.$$

6. The commutator of the directional derivatives introduced in problem 3 is defined by:

$$[\partial_\xi, \partial_\eta]f = \partial_\xi(\partial_\eta f) - \partial_\eta(\partial_\xi f).$$

Prove that

$$[\partial_\xi, \partial_\eta]f = \partial_{L_\xi \eta} f.$$

Because of this result, the vector field $L_\xi \eta$ is often denoted $[\xi, \eta]$ and called teh commutator of the vector fields ξ and η.

7. Prove the Leibniz formula:

$$L_\xi(f\eta) = fL_\xi\eta + \eta\partial_\xi f,$$

where f is a smooth function.

8*. Let ω be a differential 1-form, ξ and η—vector fields. Prove the Cartan formula:

$$2d\omega(\xi, \eta) = \partial_\xi(\omega(\eta)) - \partial_\eta(\omega(\xi)) - \omega([\xi, \eta]).$$