Homework 2, due February 4

This time, consider the three-dimensional space with a Riemannian metric, which in a fixed coordinate system (x^1, x^2, x^3) has the form

$$ds^2 = E_1\,(dx^1)^2 + E_2\,(dx_2)^2 + E_3\,(dx_3)^2,$$

where E_j are smooth functions of x. Such coordinate system is called triorthogonal with respect to the metric. It does not always exist, not even locally, but many useful coordinate systems have this property—see below.

1. Let e_1, e_2, e_3 denote the unit vectors in the coordinate directions, i.e. the $e_i^j = \delta_i^j$ in the given coordinates. Find the values of the differential forms dx^j on the vectors e_i.

We also have a vector product of two vectors ξ_1 and ξ_2 defined at every point x, by requiring that $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ and extending to all pairs of vectors by linearity in each factor. The mixed product of three vectors is defined now as in Homework 12: $(\xi_1, \xi_2, \xi_3) =< \xi_1, [\xi_2, \xi_3] >$.

2. Given a vector field A, we define, as in Homework 12, a 1-form ω_A and a 2-form ρ_A, requiring that

$$\omega_A(\xi) =< A, \xi >; \quad \rho_A(\xi_1, \xi_2) = (A, \xi_1, \xi_2)$$

for all ξ_1, ξ_2. Express these forms in the x coordinates in terms of the components of the vector field A (and the coefficients E_i of the metric).

3. Show that the cylindrical and spherical coordinates are triorthogonal with respect to the Euclidean metric in $\mathbf{R}^3$, by explicitly calculating the corresponding E_i. Apply the results of problem 2, to express forms corresponding to a given vector field in these coordinate systems.

4. Let f be a smooth function of x. Define the gradient of f as the unique vector field ∇f such that

$$\omega_{\nabla f} = df,$$

where $df = \frac{\partial f}{\partial x^i}\,dx^i$ is the differential of f. Find the components of the gradient of f in the basis e_1, e_2, e_3 defined in problem 1. Apply this to cylindrical and spherical coordinate systems.

5. Prove that the above definition of the gradient agrees with a more general definition given in problem 3 of Homework 10. This is why results of the previous problem are the same as those obtained there.

6. Define the curl of a vector field A as the unique $curlA$ such that

$$d\omega_A = \rho_{curlA}.$$

Calculate the components of $curlA$ in terms of the components of A and the coefficients E_i. Apply this to Cartesian, cylindrical and spherical coordinates.

7. Define the divergence of a vector field by the condition

$$d\rho_A = divAV,$$

where V denotes the volume form:

$$V = \sqrt{E_1 E_2 E_3}\,dx^1 \wedge dx^2 \wedge dx^3.$$

Calculate the divergence of a vector field A, knowing its components.

8. The Laplace operator is defined by

$$\Delta f = div(\nabla f).$$

Find its expression in the coordinates x^i. Argue that this definition is independent on the triorthogonal coordinate system (with respect to a given metric). Use this observation to calculate the standard laplacian in cylindrical and (again) spherical coordinates.

9. Use exterior differentiation and exterior multiplication of forms to prove the identities (valid in any triorthogonal coordinate system; A, B are vector fields, a—a function):

$$div([A,B]) =< curl A, B > - < curl B, A >;$$

$$curl(aA) = [\nabla a, A] + a\, curl A;$$

$$div(aA) =< \nabla a, A > + a\, div A;$$

$$curl\, \nabla a = 0;$$

$$div\, curl\, A = 0.$$