Homework 1, due January 27

In this series of problems we consider the Euclidean space $\mathbf{R}^3$ with a fixed coordinate system (x^1, x^2, x^3). A vector simply means an element of $\mathbf{R}^3$, a 1-form—a linear functional on $\mathbf{R}^3$ etc., without any requirements concerning transformations under coordinate changes. Recall that in the Euclidean coordinates the scalar product of vectors $\xi = x^i$ and and $\eta = y^j$ equals

$$< \xi, \eta >= \delta_{ij} x^i y^j = \sum_{i=1}^{3} x^i y^i.$$

1. Given a vector field $A(x)$ in a region of space D, we define a 1-form, which acts on vectors by scalar multiplication: at every $x \in D$

$$\omega_A(\xi) =< A(x), \xi > .$$

Prove that every 1-form in D is of this form. Find the expression of ω_A in coordinates.

2. Define a 2-form ρ_A, acting on pairs of vectors by

$$\rho_A(\xi_1, \xi_2) =< A(x), [\xi_1, \xi_2] >,$$

where $[\xi_1, \xi_2]$ denotes the vector product. Note that the right-hand side is the oriented volume of the parallelpiped with sides A, ξ_1 and ξ_2. This quantity is also called the mixed product of the three vectors and denoted (A, ξ_1, ξ_2). Again, prove that every 2-form in D can be written as ρ_A for some A and find the relation between components of A and those of ρ_A.

3. Prove that above correspondences between vector fields and differential forms depend only on the choice of the inner product and orientation in $\mathbf{R}^3$.

4. Prove that with the above notation

$$\omega_A \wedge \omega_B = \rho_{[A,B]},$$

so that vector product can be regarded as a special case of exterior product of 1-forms.

5. Prove that

$$\omega_A \wedge \rho_B =< A, B > dx^1 \wedge dx^2 \wedge dx^3,$$

so that the scalar product can also be represented in terms of exterior multiplication of forms.