NONGAUSSIAN LIMITING BEHAVIOR
OF THE PERCOLATION THRESHOLD IN A LARGE SYSTEM

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Abstract

We study short-range percolation models. In a finite box we define the percolation threshold as a random variable obtained from a stochastic procedure used in actual numerical calculations, and study the asymptotic behavior of these random variables as the size of the box goes to infinity. We formulate very general conditions under which in two dimensions rescaled threshold variables cannot converge to a Gaussian and determine the asymptotic behavior of their second moments in terms of a widely used definition of correlation length. We also prove that in all dimensions the finite-volume percolation thresholds converge in probability to the percolation threshold of the infinite system. The convergence result is obtained by estimating the rate of decay of the limiting distribution function's tail in terms of the correlation length exponent ν. The proofs use exponential estimates of crossing probabilities. Substantial parts of the proofs apply in all dimensions.

1. Introduction and Main Results

Many problems in condensed matter physics lead to study of clusters of random objects. Mathematical analysis of such phenomena employs percolation models [20]. In the most widely used independent bond percolation model we consider the d-dimensional hypercubic lattice Z^d ($d \geq 2$). Each bond (i.e. a segment connecting two nearest-neighbor sites) of the lattice is open with probability p and closed with probability $1-p$ and the bonds are open or closed independently of each other. We thus get a product probability measure on the space of all bond configurations. This space can be thought of as a product of countably many copies of the set $\{0,1\}$, where 0 represents a closed and 1—an open bond. For each configuration, the connected cluster of a point $x \in Z^d$ is defined as the set of all points $y \in Z^d$, which can be connected to x by a path consisting of open bonds. Percolation theory studies probabilistic properties of connected clusters—the distributions of their sizes, shapes etc. In particular, one of the crucial questions is whether there exist infinite connected clusters. Universality of many features of the model in the vicinity of the percolation threshold (see below and [10], [17], [20]) is another topic of fundamental importance. Before we discuss these questions, let us however give an example of another percolation model.

In the independent site percolation model, we assign independent random variables to the sites of the lattice Z^d (as opposed to bonds in the previous model). Each site is occupied with probability p and vacant with probability $1-p$, independently of other sites. A connected cluster of $x \in Z^d$ is now defined as the set of all points $y \in Z^d$, which can be connected to x by a nearest-neighbor path, consisting of occupied sites (if x is vacant, its cluster is empty).

There exist other interesting and useful percolation models, in which connected clusters are defined based on other notions of connectedness [15]. In the remainder of the paper we focus on the independent bond model, but all results of the paper remain valid, with no major changes in proofs, for a general class of short-range independent percolation models, including the independent site percolation model introduced above. In particular, the main standard results from percolation theory, which we now proceed to state, apply to such a general class.

Theorem 1.1: There exists a unique critical density $0 < p_c < 1$ such that when $p < p_c$ with probability one there are no infinite connected clusters, while for $p > p_c$ with probability one there is a (unique) infinite connected cluster.

Proof of this classical result can be found in [10].

The critical density p_c is also called the percolation threshold, the existence of the infinite cluster being referred to as the phenomenon of percolation. It has been defined above, in terms of configurations of the infinite system of bonds. It is convenient to use a characterization of p_c in terms of the limiting behavior of finite-volume quantities. Let Λ_L denote the d-dimensional box (i.e. hypercube) $[0, L]^d$. A path of bonds contained in Λ_L is an open left-to-right (L-R) crossing of Λ_L if it consists entirely of open bonds and connects the left face of Λ_L to its right face (with the obvious meaning of "left" and "right"). Let $\pi_L(p)$ denote the probability that there is an open L-R crossing in Λ_L. Since existence of an open L-R crossing is an increasing event of the bond configuration [10], it follows from the Russo'a formula [10], that for any fixed L, π_L is and increasing function of p. The following fundamental theorem combines several results by various authors (Russo, Seymour, Welsh, Kesten, Aizenman, Newman, Barsky, Menshikov) and is by now well-known (see [10]).

Theorem 1.2:

(1) for $p < p_c$ $\quad \lim_{L\to\infty} \pi_L(p) = 0$;

(2) for $p > p_c$ $\quad \lim_{L\to\infty} \pi_L(p) = 1$;

(3) when $d = 2$, $\liminf_{L\to\infty} \pi_L(p_c) > 0$ and $\limsup_{L\to\infty} \pi_L(p_c) < 1$.

In the case of bond percolation, part (3) follows easily from a duality argument and the well-known result that $p_c = \frac{1}{2}$ (see [10], [15]). We emphasize that it is also true for other two-dimensional independent short-range models, e.g. for the independent site percolation, for which the value of p_c is not known exactly. The rest of the paper deals with an algorithm which is used to determine p_c numerically. We shall study the properties of the algorithm using the above theorem.

Let $n(L)$ denote the total number of bonds in Λ_L. To leading order in L as $L \to \infty$, $n(L)$ behaves like cL^d, where $c > 0$ is independent of L. An explicit expression for $n(L)$ can be easily calculated, but for us it plays no role; in the sequel, we will denote $n(L)$ simply by n, suppressing the dependence on L. For a fixed L, let us consider the set Ω_n, of all ordered sequences of all the n bonds in Λ_L. Ω_n has $n!$ elements. Let P denote the probability measure on Ω_n, assigning equal probabilities $\frac{1}{n!}$ to all elements of Ω_n. The (finite) probability space

(Ω_n, P) can be thought of as a mathematical model of the following procedure. Initially all the bonds in Λ_L are closed. We choose a bond at random and open it, then choose a bond from among the remaining (closed) bonds and open it, etc. At every stage each currently closed bond has an equal chance to be opened. The final product of the procedure is a sequence $\omega = (\omega_1, \ldots, \omega_n)$ of bonds in Λ_L. With this interpretation in mind, the following definition of a finite-volume percolation threshold becomes natural.

Definition 1.1: For $\omega \in \Omega_n$, let $i_c(n)$ be the smallest number i such that some of the bonds in the set $\{\omega_1, \ldots, \omega_i\}$ form an L-R crossing of Λ_L. Let

$$p_c^{(L)}(\omega) = \frac{i_c}{n}. \tag{1.1}$$

The random variable $p_c^{(L)}$ is called the percolation threshold in Λ_L. It is the fraction of the open bonds present when the first L-R connection is established.

The results of this paper are concerned with convergence of the variables $p_c^{(L)}$ as $L \to \infty$ and with the asymptotic properties of their distributions. Our main results are Theorems 1.3., 1.7. and 1.8. We now proceed to state them, introducing on the way some auxiliary facts and concepts. The first result, proven in the next section can be called a "Law of Large Numbers" and justifies the name "finite-volume percolation threshold":

Theorem 1.3: for every $d \geq 2$ the sequence $p_c^{(L)}$ converges in probability to the infinite-volume percolation threshold p_c, i.e.

$$\forall \delta > 0 \quad \lim_{L \to \infty} P[|p_c^{(L)} - p_c| > \delta] = 0. \tag{1.2}$$

Proof of Theorem 1.3. is the content of section 2.

Theorems 1.7. and 1.8. below deal with the magnitude of the fluctuations of the random variables $p_c^{(L)}$ and with possible limiting distributions of variables obtained from $p_c^{(L)}$ by suitable rescaling. While they are stated and proved for two-dimensional systems, parts of their proofs apply in all dimensions, which is why the discussion of the critical behavior will be presented for an arbitrary $d \geq 2$.

A central concept in the theory of critical behavior is that of the correlation length [18]. Intuitively speaking, the correlation length is a minimum size at which a finite volume system can be distinguished from the critical one. At a rigorous level, correlation lengths in percolation models have been defined in several different ways, of which we will now discuss two. See also [5], [16] and [19]. For a pair of lattice sites x and y let $\tau_{x,y}(p)$ denote the probability that x and y are connected by an open path. In particular, for $x = (0, \ldots, 0)$ and $y = (L, \ldots, 0)$, we will write τ_L instead of $\tau_{x,y}$.

Definition 1.2: For $p < p_c$ we define the correlation length $\zeta(p)$ by the formula

$$\zeta(p)^{-1} = -\lim_{L \to \infty} \frac{\log \tau_L(p)}{L}. \tag{1.3}$$

Existence of the limit follows from standard subadditivity arguments, see [10].

For our purposes, it is more convenient to use a quantity defined in terms of the crossing probability π_L, rather than the connectivity function τ_L. The following proposition shows how to obtain an asymptotically equivalent definition of a correlation length; we first introduce the concept of asymptotic equivalence:

Definition 1.3: we say that two functions $f(p)$ and $g(p)$ are asymptotically equivalent as $p \to p_c$, and write $f \asymp g$, if there exist two positive constants a and A, such that in a certain neighborhood of p_c we have

$$a \leq \frac{f(p)}{g(p)} \leq A. \tag{1.4}$$

Proposition 1.1: Let $\xi(p)$ be defined by the formula

$$\xi^{-1}(p) = -\limsup_{L \to \infty} \frac{\log \pi_L(p)}{L}. \tag{1.5}$$

Then $\xi \asymp \zeta$ as $p \to p_c$.

Proof: Only standard techniques (which can be found e.g. in [10]) are used in the following proof, which we include for completeness. For an upper bound on $\frac{\xi}{\zeta}$, note that $\pi_L(p)$ is bounded above by the sum of $\tau_{x,y}$ over x and y in the left and right faces of Λ_L respectively. For any such pair (x, y), let x' be the reflection of x with respect to the hyperplane containing right wall of Λ_L. FKG inequality implies that $\tau_{x,y}\tau_{y,x'} \leq \tau_{x,x'}$, i.e. $\tau_{x,y}^2 \leq \tau_{2L}$ and the desired bound follows. To prove the opposite inequality, let B_L denote the box of linear size L centered at 0 (a translate of Λ_L). Clearly, τ_L is bounded above by the probability that 0 is connected by an open path to the boundary of B_L. This is, in turn, bounded by $2d$ times the probability that 0 is connected to a given face of that boundary by an open path lying in B_L. Finally, if we have such open paths connecting 0 both to the left and to the right face of B_L, they form an L-R crossing of B_L. FKG inequality, now implies that $\pi_L \geq \frac{\tau_L^2}{4d^2}$, which ends the proof.

It is clear from the above proof that taking $\liminf$ instead of the $\limsup$ in (1.5), we would obtain an asymptotically equivalent quantity .

The following different definition of a correlation length has been introduced in [4].

Definition 1.4: Fix a number $\epsilon > 0$. For $p < p_c$, let $L_0(p, \epsilon)$ be the smallest L for which $\pi_L(p) \leq \epsilon$; for $p > p_c$ let $L_0(p, \epsilon)$ be the smallest L for which $1 - \pi_L(p) \leq \epsilon$.

While L_0 depends, of course, on the choice of ϵ, the following statement proved in [16], shows that the asymptotic behavior of L_0 as $p \to p_c$ does not depend on p in a significant way.

Theorem 1.4: There exists $\epsilon_0 > 0$, such that for any $\epsilon_1, \epsilon_2 \leq \epsilon_0$,

$$L_0(p, \epsilon_1) \asymp L_0(p, \epsilon_2) \tag{1.6}$$

as $p \to p_c$, for two-dimensional independent short-range percolation models.

From now on, we will suppress in the notation the dependence of L_0 on ϵ and write simply $L_0(p)$. Another result from [16], which will be important for us is:

Theorem 1.5: In two dimensions, and for small ϵ

$$L_0(p_c - y, \epsilon) \asymp L_0(p_c + y, \epsilon) \tag{1.7}$$

as $y \downarrow 0$.

From now on, whenever we discuss two-dimensional models, we will assume that ϵ is small enough for the two above asymptotic relations to hold. We will also use a very recent result:

Theorem 1.6: For $d = 2$,

$$L_0(p) \asymp \xi(p) \tag{1.8}$$

as $p \to p_c$.

This new result has been communicated to the authors by K. Alexander [2]. We emphasize that for $d > 2$ no analogous result is known. In general, different definitions of correlation length are known to be equivalent only up to possible logarithmic corrections (see [4] and also [19]).

We shall now discuss some assumptions on the behavior of the correlation length(s) as $p \to p_c$. Rigorous mathematical study of the function ξ (or L_0) is a very hard problem. At the same time a powerful physical theory—the renormalization group method [18]—offers predictions concerning critical behavior in percolation and other models. While only some of these predictions have been rigorously verified, many others are virtually universally trusted. Among such noncontested statements is existence of critical exponents. In what follows, we restrict the discussion to the critical exponent ν and to the correlation length L_0. Roughly speaking, it is believed that as $p \to p_c$, $L_0(p)$ behaves as a power of $|p - p_c|$, possibly with some corrections of an order of magnitude lower than any power. More precisely, it is believed that both limits

$$\nu_- = - \lim_{p \to p_c -} \frac{\log L_0(p)}{\log(p_c - p)} \tag{1.9}$$

and

$$\nu_+ = - \lim_{p \to p_c +} \frac{\log L_0(p)}{\log(p - p_c)} \tag{1.10}$$

exist. Their values have been calculated numerically for various models ([20], [23]) and found to depend only on the dimensionality of the system (universality); moreover with high accuracy $\nu_+ = \nu_-$ (note that in two dimensions the last statement follows from the relation $L(p_c - y, \epsilon) \asymp L_0(p_c + y, \epsilon)$); this common value is denoted by ν and called the correlation length exponent. While existence of ν has not been rigorously proven, we think it is useful to assume it in some form and prove rigorous results contingent on this assumption. The assumption we make implies existence of critical exponents. We do not want to favor any explicit form of $L_0(p)$ (like $|p - p_c|^{-\nu}$ or $|p - p_c|^{-\nu} \log \frac{1}{|p - p_c|}$ etc.), so we will make a much less restrictive assumption, consistent with all explicit formulas postulated or heuristically derived in the literature. See [24] for a recent study of critical behavior of percolation models, where some results use assumptions related to ours.

<u>Assumption 1.1</u> (regularity): $L_0(p + \delta)$ is a regularly varying function of δ, as $\delta \to 0$ from above and from below. This means, by definition [9] that for every $x \neq 0$, the limit $\lim_{t \to 0} \frac{L_0(p_c + tx)}{L_0(p_c + t)}$ exists.

Regular variation implies [9] that there exist two exponents, ν_- and ν_+, such that

$$L_0(p_c + \delta) = \delta^{-\nu_+} S(\delta); \qquad \delta > 0 \tag{1.11}$$

and

$$L_0(p_c + \delta) = (-\delta)^{-\nu_-} S(\delta); \qquad \delta < 0, \tag{1.12}$$

where S is a slowly varying function of δ at 0, i.e. for every $x \neq 0$,

$$\lim_{t\to 0} \frac{S(tx)}{S(t)} = 1. \tag{1.13}$$

The last statement clearly implies that for any $\epsilon > 0$

$$t^{-\epsilon} < S(t) < t^{\epsilon}, \tag{1.14}$$

if $|t|$ is sufficiently small (depending on ϵ). It also follows from representations (1.11) and (1.12) that the limit of $\frac{L_0(p_c+tx)}{L_0(p_c+t)}$ (whose existence is postulated in the definition of a regularly varying function) equals $(-x)^{-\nu_-}$ for $x < 0$ and $x^{-\nu_+}$ for $x > 0$. In view of (1.11) and (1.12) we say that $L_0(p+\delta)$ is a regularly varying function of δ with the exponent ν_- as $\delta \to 0-$ and with the exponent ν_+ as $\delta \to 0+$.

The existence of critical exponents thus follows from the regularity assumption, which in particular admits all power laws with logarithmic corrections. Such corrections are indeed known to appear in critical phenomena, especially at upper critical dimensions [18]. For example, the correlation length in weakly coupled four-dimensional ϕ^4 lattice field theory satisfies the asymptotic relation $\xi(\delta) \asymp \delta^{-\frac{1}{2}}(\log\delta)^{\frac{1}{6}}$ [13]. We remark here that Assumption 1.1 takes into account not only the size but also the nature of the possible correction to a power law. All such corrections discussed in physical literature (logarithms and their powers) satisfy the assumption. However, as pointed out by the referee of this paper, from the mathematical point of view not all power laws with corrections of at most logarithmic order satisfy the assumption. For example, the monotone function $\delta^{-\nu}(2+\cos\frac{1}{\delta})$ is not a regularly varying function of δ as $\delta \to 0$, even though it differs from a pure power only by a factor bounded from below and from above (by 1 and 3). While there are no physical grounds to expect such form of the correlation length, no rigorous argument excluding it is presently available.

The concept of a correlation length leads naturally to that of a critical interval:

Definition 1.5: For a positive L, define

$$p_c^-(L) = \inf\{p : L_0(p) > L\} \tag{1.15}$$

and

$$p_c^+(L) = \sup\{p : L_0(p) > L\} \tag{1.16}$$

We call $[p_c^-(L), p_c^+(L)]$ the critical interval and its width,

$$\delta_0(L) = p_c^+(L) - p_c^-(L) \tag{1.17}$$

—the critical width corresponding to the length scale L. We will also use the one-sided critical widths

$$\delta_-(L) = p_c - p_c^-(L); \tag{1.18}$$

$$\delta_+(L) = p_c^+(L) - p_c. \tag{1.19}$$

Of course, $\delta_0(L) = \delta_-(L) + \delta_+(L)$.

$p_c^+(L)$ and $p_c^-(L)$ play the role close to that of inverse functions of L_0 on the intervals $p < p_c$ and $p > p_c$ respectively. They are defined for arbitrary positive real, and not only for integer values of L. Note that L_0 is not a strictly increasing function (it assumes only integer values) and therefore it has no inverse function. Moreover, since L_0 does not have to assume all integer values, we cannot even guarantee that $L_0(p_c^\pm(L)) = L$. The following proposition shows, however, that the last equation is at least approximately correct and shows that $\delta_\pm$ inherit the regular variation property from L_0.

Proposition 1.2: for any integer L

$$L_0(p_c^\pm(L)) \leq L. \tag{1.20}$$

As $L \to \infty$,

$$\frac{L_0(p_c^\pm(L))}{L} \to 1. \tag{1.21}$$

Moreover, $\delta_\pm$ is a regularly varying function of L with the exponent $-\frac{1}{\nu_\pm}$, i.e. for any $t > 0$

$$\lim_{L\to\infty} \frac{\delta_\pm(tL)}{\delta_\pm(L)} = t^{-\frac{1}{\nu_\pm}}. \tag{1.22}$$

The last statement is equivalent to saying that (compare (1.11) and (1.12))

$$\delta_\pm(L) = L^{-\frac{1}{\nu_\pm}} S_\pm(L), \tag{1.23}$$

where $S_\pm$ are slowly varying functions of L as $L \to \infty$ [9].

We prove the proposition in Appendix 2.

The next theorem estimates fluctuations of $p_c^{(L)}$ in terms of the critical width $\delta_0(L)$.

Theorem 1.7: Let $d = 2$ and suppose the Assumption 1.1. holds. There exist positive constants c and C, independent of L, such that for every L

$$c\delta_0^2(L) \leq E[(p_c^{(L)} - p_c)^2] \leq C\delta_0^2(L). \tag{1.24}$$

Proof will be given in section 3.

Even assuming their existence, not much is known about the values of the critical exponents ν_- and ν_+ (one known inequality, $\nu > 1$, rigorously established in two dimensions, will be used in the proof of Theorem 1.7). On the other hand, extensive numerical studies give consistent values. As mentioned above, these studies strongly suggest that $\nu_- = \nu_+$. We emphasize that in two dimensions, once the critical exponents $\nu_\pm$ exist, they have to be identical, by virtue of Theorem 1.5. Also, the numerical value of $\nu = \nu_- = \nu_+$ in two dimensions is 1.33, a result further confirmed by conformal field theory calculations [7] (see also [22]), which yield an exact value $\nu = \frac{4}{3}$ (there is, however, no rigorous proof of the last equality). The following statement, used in the proof of the next theorem is thus a generally believed assumption:

Assumption 1.2: In two dimensions $\nu < 2$.

Remark 1.1: It is clear from the definition of L_0 that ν is a monotone decreasing function of the dimension (if it exists) and therefore Assumption 1.2 implies that $\nu < 2$ in all dimensions $d \geq 2$.

Theorem 1.8: Suppose the Assumptions 1.1 and 1.2 hold. Let $\sigma_L = \sqrt{Var[p_c^{(L)}]}$ be the standard deviation of $p_c^{(L)}$ ($Var[X] = E[(X - EX)^2]$ denotes the variance of a random variable X). Then, in $d = 2$ the sequence of random variables $\frac{p_c^{(L)} - E[p_c^{(L)}]}{\sigma_L}$ does not converge in distribution to a nondegenerate Gaussian variable (nor does it have a subsequence convergent in distribution to such a variable).

The proof is given in section 4.

Notation: Throughout the paper we follow the usual custom of denoting arbitrary positive constants in estimates by c. The actual value of c may vary from one equation to another and, sometimes, when this does not cause a confusion, also within one equation. c may depend on the dimensionality of the model, but not on the size of the system L or on the value of bond density p. Also, it may change its value when the arguments are applied to models other than the independent bond model.

2. Law of Large Numbers and Large Deviation Bounds

In this section we prove Theorem 1.3. The usual coupling of percolation models with different values of p ([10], pp. 10-11) could be used to obtain an alternative proof. The advantage of the proof below is that it yields explicit bounds on large deviation probabilities, which are also used later in section 3.

The idea of the proof is to estimate the probability that $p_c^{(L)}$ deviates from p_c by probabilities of crossing events in the independent bond percolation model with appropriately chosen bond density p.

Notation: we will use the following notation for events in the space of bond configurations in the box Λ_L: Γ will denote the event that there exists an L-R open crossing of Λ_L, A_k—the event that there are exactly k open bonds and $\bar{A}_k$—the event that there are at least k open bonds, i.e. $\bar{A}_k$ is the disjoint union of A_l with $l = k, k+1, \ldots, n$. P_p will denote the probability measure on bond configurations in Λ_L in the independent bond model, i.e. a product of n Bernoulli measures with probability p of any given bond being open. P will denote the uniform measure on the set Ω_n of ordered bond sequences introduced in section 1. $P[E|F]$ will denote the probability of E conditioned on F and similarly for P_p. A complement of a set S will be denoted by S^c. Finally, $\#(S)$ will denote the number of elements in a set S.

Proof of Theorem 1.3: fix a $\delta > 0$. We will only prove that

$$\lim_{L\to\infty} P[p_c^{(L)} < p_c - \delta] = 0. \tag{2.1}$$

The proof that

$$\lim_{L\to\infty} P[p_c^{(L)} > p_c + \delta] = 0 \tag{2.2}$$

is analogous. Choose m so that

$$p_c - \delta \le \frac{m}{n} < p_c - \frac{3\delta}{4}. \tag{2.3}$$

This is always possible for large L (here $n = n(L)$). Now, $p_c^{(L)} \le \frac{m}{n}$ for a given ordered sequence of bonds implies that the first m bonds in the sequence contain an L-R crossing of Λ_L. Since all ordered sequences have the same probability, we obtain

$$P[p_c^{(L)} \le p_c - \delta] \le P[p_c^{(L)} \le \frac{m}{n}] = \frac{\#(A_m \cap \Gamma)}{\#(A_m)} \tag{2.4}$$

and multiplying the numerator and the denominator by $p^m(1-p)^{n-m}$, we obtain

$$P[p_c^{(L)} \le \frac{m}{n}] = \frac{P_p[\Gamma \cap A_m]}{P_p[A_m]} = P_p[\Gamma | A_m]. \tag{2.5}$$

We emphasize that (2.5) holds with any choice of $p \in (0,1)$. Let us choose $p = p_c - \frac{\delta}{2}$. Then $P_p[C] = \pi_L(p_c - \frac{\delta}{2})$. In order to obtain from here a good estimate on $P_p[\Gamma|A_m]$, let us note first that

$$P_p[\Gamma|A_m] \le P_p[\Gamma|\bar{A}_m]. \tag{2.6}$$

This is intuitively obvious, since conditioning on an event with more open bonds should make an open crossing more likely. For a formal proof, note that for any i all configurations in A_i have the same probability $p^i(1-p)^{(n-i)}$ and since $\#(A_i) = \binom{n}{i}$, $i\#(A_i) = (n-i+1)\#(A_{i-1})$. Since Γ is an increasing event, i.e. adding open bonds to a configuration in Γ we obtain another element of Γ, we have

$$P_p[\Gamma|A_i] = \frac{\#(\Gamma \cap A_i)}{\#(A_i)} \ge \frac{(n-i+1)\#(\Gamma \cap A_{i-1})}{i\#(A_i)} = P_p[\Gamma|A_{i-1}], \tag{2.7}$$

Taking a convex combination of inequalities (2.7) with $i = m, \dots, n$ we get (using $\bar{A}_m = \bigcup_{i=m}^n A_i$):

$$\begin{aligned} P_p[\Gamma|\bar{A}_m] &= \frac{\sum_{i=m}^n P_p[\Gamma \cap A_i]}{\sum_{j=m}^n P_p[A_j]} \\ &= \sum_{i=m}^n \frac{P_p[\Gamma \cap A_i]}{P_p[A_i]} \frac{P_p[A_i]}{\sum_{j=m}^n P_p[A_j]} \\ &\ge P_p[\Gamma|A_m] \sum_{i=m}^n \frac{P_p[A_i]}{P_p[\bar{A}_m]} = P_p[\Gamma|A_m] \end{aligned} \tag{2.8}$$

which proves (2.6). With our choice of p, a standard bound on the probability of a large deviation for a sum of i.i.d. random variables based on the exponential Chebyshev inequality (see [8], ch.1, section 9) implies

$$P_p[\bar{A}_m] \ge 1 - e^{-c\delta^2 n} \tag{2.9}$$

with a strictly positive constant c. It follows that (using (2.4), (2.5), (2.6) and (2.9))

$$P[p_c^{(L)} \le p_c - \delta] \le P_p[\Gamma|A_m] \le P_p[\Gamma|\bar{A}_m] \le \frac{P_p[\Gamma]}{P_p[\bar{A}_m]} \le cP_p[\Gamma] = c\pi_L(p), \tag{2.10}$$

with $p = p_c - \frac{\delta}{2}$, where c is an absolute constant. Since the last quantity goes to zero when $L \to \infty$ (Theorem 1.2), the theorem is proven. It follows from the presented proof (using Proposition A1.1) that convergence to p_c takes place with exponential bounds on large deviation probabilities. These exponential bounds will be used in the next section.

The following corollary shows that a law of large numbers also holds when we subtract the means from $p_c^{(L)}$.

Corollary: $p_c^{(L)} - E[p_c^{(L)}]$ converges to 0 in probability.

Proof: For any $\eta > 0$

$$|p_c - E[p_c^{(L)}]| \leq E[|p_c - p_c^{(L)}|] \leq \eta P[|p_c^{(L)} - p_c| \geq \eta] + \eta. \tag{2.17}$$

This implies that

$$\lim_{L\to\infty} E[p_c^{(L)}] = p_c \tag{2.18}$$

and the corollary follows.

3. Order of Fluctuations

In this section we prove Theorem 1.7.

Our proof of the lower bound in Theorem 1.7. does not use the assumption $d = 2$ except for the inequality $\nu > 1$. It can be carried out in any dimension, in which the inequality $\nu > \frac{2}{d}$ is satisfied and we will present it in this generality. The condition $\nu > \frac{2}{d}$ was introduced by Harris in [14], in relation to fluctuations of the critical temperature of finite magnetic systems with site dilution— a quantity closely related to the percolation threshold studied here. The weak inequality $\nu \geq \frac{2}{d}$ was subsequently proven for a large class of systems, including percolation models in all dimensions $d \geq 2$ [5]. In two dimensions the inequality $\nu > 1$ has been rigorously established in [16]. However, since ν is known to be $\frac{1}{2}$ in high dimensions [12], the strict inequality holds when d is large. The numerical value of ν in three dimensions is 0.9 [23], which would make the inequality strict also in three dimensions. The following assumption is therefore well-founded:

<u>Assumption 3.1</u> (Harris condition): $\nu > \frac{2}{d}$.

The following proposition contains the lower bound claimed in Theorem 1.4.:

<u>Proposition 3.1</u>: Suppose Assumption 3.1 holds (as mentioned above, this is known to be true in two dimensions). Then there exists a constant $c > 0$, independent of L, such that for all L

$$E[(p_c^{(L)} - p_c)^2] \geq c[\delta_0(L)]^2. \tag{3.1}$$

Proof: We will use the following integral representation of the second moment of a random variable X in terms of its distribution function (see [8], ch.1, Lemma 5.7):

$$E[X^2] = \int_0^\infty 2yP[|X| \geq y]\, dy. \tag{3.2}$$

With $X = p_c^{(L)} - p_c$, we obtain

$$E[(p_c^{(L)} - p_c)^2] = \int_0^\infty 2\delta P[|p_c^{(L)} - p_c| \geq \delta]\, d\delta =$$

$$\int_0^{p_c} 2\delta P[p_c^{(L)} \leq p_c - \delta]\, d\delta + \int_0^{1-p_c} 2\delta P[p_c^{(L)} \geq p_c + \delta]\, d\delta. \tag{3.3}$$

<u>Notation</u>: In what follows the first and the second term on the right-hand side of (3.3) will be denoted by I_1 and I_2 respectively.

Let $0 < \delta < p_c$. Take an m so that

$$p_c - \frac{5}{4}\delta \leq \frac{m}{n} \leq p_c - \delta \tag{3.4.}$$

This is possible for a fixed L, whenever $\delta > \frac{4}{n(L)}$ (note that $n(L)$ behaves like cL^d). Proceeding as in the proof of Theorem 1.3. (using (2.5) and (3.4)), we obtain for an arbitrary p

$$P[p_c^{(L)} \leq p_c - \delta] \geq P_p[\Gamma | A_m] \geq P_p[\Gamma | \tilde{A}_m], \tag{3.5}$$

where $\tilde{A}_m$ denotes the event that a configuration in Λ_L has at most m open bonds. The inequality $P_p[\Gamma|A_m] \geq P_p[\Gamma|\tilde{A}_m]$ is proven similarly to (2.6). Using the obvious inequality $P_p[A \cap B] \geq P_p[A] - P_p[B^c]$, we get

$$P_p[\Gamma|\tilde{A}_m] \geq \frac{P_p[\Gamma] - P_p[\tilde{A}_m^c]}{P_p[\tilde{A}_m]}. \tag{3.6}$$

Let us choose $p = p_c - \frac{3}{2}\delta$. Just like in (2.9), a standard large deviation estimate implies that with this choice

$$P_p[\tilde{A}_m] \geq 1 - e^{-nc\delta^2} \tag{3.7}$$

with a positive constant c independent of n and δ. Combining (3.4) and (3.6) and noting that $P_p[C] = \pi_L(p)$, we arrive at the inequality

$$P[p_c^{(L)} \leq p_c - \delta] \geq c'[\pi_L(p_c - \frac{3}{2}\delta) - e^{-nc\delta^2}], \tag{3.8}$$

where $c' > 0$ is an absolute constant. Thus

$$I_1 \geq c' \int_0^{p_c - cL^{-d}} \delta[\pi_L(p_c - \frac{3}{2}\delta) - e^{-nc\delta^2}]\, d\delta. \tag{3.9}$$

The reason for subtracting cL^{-d} from p_c in the upper limit of integration is that we have used $\delta > \frac{4}{n(L)}$ to prove (3.8). Since for $p > p_c^-(L)$ (see (1.15)), $\pi_L(p) \geq \epsilon$ (where ϵ is chosen as described in Definition 1.4.), the last expression is bounded below by

$$c' \int_{cL^{-d}}^{\frac{2}{3}\delta_-(L)} \delta\epsilon\, d\delta - c' \int_0^{p_c} e^{-cL^d\delta^2} \delta\, d\delta \geq c''[\delta_-(L)]^2 - c' \int_0^{p_c} e^{-cL^d\delta^2} \delta\, d\delta. \tag{3.10}$$

To estimate the first term we used the fact that $L^{-d}\delta_-(L) \to 0$, which follows from (1.23) together with the Harris condition $\nu > \frac{2}{d}$. Changing the variable of integration to $u = c^{\frac{1}{2}} L^{\frac{d}{2}} \delta$, we estimate the integral in the second term as follows

$$\int_0^{p_c} e^{-cL^d\delta} \delta\, d\delta \leq \frac{1}{cL^d} \int_0^{\infty} e^{-u^2} u\, du = O(\frac{1}{L^d}) \tag{3.11}$$

as $L \to \infty$. Harris condition implies now that the first term dominates the second one, so we obtain

$$I_1 \geq c[\delta_-(L)]^2. \tag{3.12}$$

A similar proof shows that

$$I_2 \geq c[\delta_+(L)]^2. \tag{3.13}$$

We just sketch the argument, which is analogous to the bound on I_1: choosing m so that this time $p_c + \delta \leq \frac{m}{n} < p_c + \frac{5}{4}\delta$, we have, with $p = p_c + \frac{3}{2}\delta$,

$$P[p_c^{(L)} \geq p_c + \delta] \geq P_p[\Gamma^c | A_m] \geq P_p[\Gamma^c | \bar{A}_m] \geq \frac{P_p[\Gamma^c] - P_p[\bar{A}_m^c]}{P_p[\bar{A}_m]}. \tag{3.14}$$

From here, using steps analogous to (3.7)-(3.11), we obtain the desired bound (3.13), which, together with (3.12) proves the theorem.

Remark 3.1: We have actually proven two separate bounds—on I_1 and on I_2. In two dimensions, in view of the equivalence relation (1.6), we have $\delta_-(L) \asymp \delta_+(L)$ and, consequently, both bounds have the form $c[\delta_0(L)]^2$. This will be essential in section 4.

The next proposition takes care of the upper bound in Theorem 1.7. Unlike the previous one, it is strictly limited to two dimensions.

Proposition 3.2: Suppose Assumption 1.1. holds. Then there exists a constant $C < +\infty$, independent of L, such that for all L

$$E[(p_c^{(L)} - p_c)^2] \leq C[\delta_0(L)]^2. \tag{3.15}$$

Proof: we use again the integral representation (3.3) of the second moment. Again, we shall just prove the bound for I_1; the bound for I_2 can be handled similarly. In Appendix 1 we show that for $p < p_c$

$$\pi_L(p) \leq e^{-c\frac{L}{L_0(p)}}, \tag{3.16}$$

with a constant $c > 0$ independent of p. It follows (using first the inequality (2.10)) that

$$I_1 = \int_0^{p_c} 2\delta P[p_c^{(L)} \leq p_c - \delta]\, d\delta \leq c\int_0^{p_c} \delta \pi_L(p_c - \frac{\delta}{2})\, d\delta \leq c\int_0^{\delta_-(L)} \delta\, d\delta + c\int_{\delta_-(L)}^{p_c} \delta e^{-c\frac{L}{L_0(p_c-\delta)}}\, d\delta. \tag{3.17}$$

The first term is proportional to $\delta_0(L)^2$ and we just need to estimate the second term. Using the regularity assumption, together with (1.11), (1.12) and (1.20), we can bound it by

$$c\int_{\delta_-(L)}^{p_c} \delta \exp[-c\frac{L_0(p_c - \delta_-(L))}{L_0(p_c - \delta)}]\, d\delta = c\int_{\delta_-(L)}^{p_c} \delta \exp[-c(\frac{\delta}{\delta_-(L)})^{\nu} \frac{S(\delta_-(L))}{S(\delta)}]\, d\delta. \tag{3.18}$$

Changing the variable to $z = \frac{\delta}{\delta_-(L)}$, we obtain

$$c\delta_-(L)^2 \int_1^{\frac{p_c}{\delta_-(L)}} z \exp[-cz^\nu \frac{S(\delta_-(L))}{S(z\delta_-(L))}]\, dz. \tag{3.19}$$

It follows from the proposition proven in Appendix 2 that the integral in the last formula is bounded as $L \to \infty$ and therefore, Proposition 3.2. is proven.

4. The distribution of $p_c^{(L)}$ for large L.

In this section we study possible limit theorems for the sequence of finite-volume percolation thresholds $p_c^{(L)}$ and, in particular, we prove Theorem 1.8. Under assumptions made in section 1, we show that no nontrivial Gaussian limiting behavior is possible in two dimensions (of course, choosing an appropriate normalization, we can obtain in the limit a degenerate distribution, which is, by definition, Gaussian; this is the case e.g. in Theorem 1.3). Let

$$\eta_L = \frac{p_c^{(L)} - p_c}{\delta_0(L)}. \tag{4.1}$$

Our strategy is to prove that this sequence of random variables is relatively compact and therefore contains a subsequence converging in distribution. Next, we will show that the limit of each such convergent subsequence has a nondegenerate distribution, which is not normal. Finally, we will extend the result about nongaussian limiting behavior to other normalizations of $p_c^{(L)}$ using the Convergence of Types Theorem ([8], ch. 2, Theorem 7.16).

Proposition 4.1: The sequence η_L contains a subsequence convergent in distribution.

Proof: It follows from Proposition 3.2. that second moments of η_L are bounded by C. The proposition follows from a well known relative compactness criterion (see [9] vol. II ch. 8 or [8], ch. 2, Theorem 2.7).

Proposition 4.2: No subsequence of η_L converges in distribution to a constant random variable.

Proof: Proceeding exactly as in the proof of Proposition 3.2., we can show that the sequence of fourth moments of η_L is bounded by a constant (using Proposition A2.1 with $k=3$; in fact a similar proof shows that for any m the sequence $E[|\eta_L|^m]$ is bounded). This implies that if $\eta_{L_K} \xrightarrow{\mathrm{d}} \eta$, then for any function $f(\eta)$ which grows at $\pm\infty$ slower than $|\eta|^4$,

$$E[f(\eta_L)] \to E[f(\eta)]. \tag{4.2}$$

(see [8], ch.2, Exercise 2.5). In particular, taking $f(\eta) = \eta^2$, $f(\eta) = \eta^2 I_{R_+}$ and $f(\eta) = \eta^2 I_{R_-}$ (where I_{R_+} and I_{R_-} denote the indicator functions of the positive and of the negative half-line respectively), we see that

$$E[\eta^2] = \lim_{L\to\infty} E[\eta_L^2], \tag{4.3}$$

$$E[\eta^2 I_{R_+}(\eta)] = \lim_{L\to\infty} E[\eta_L^2 I_{R_+}(\eta_L)] \tag{4.4}$$

and

$$E[\eta^2 I_{R_-}(\eta)] = \lim_{L\to\infty} E[\eta_L^2 I_{R_-}(\eta_L)] \tag{4.5}$$

We have seen in the proof of Proposition 3.1. that the right hand sides of the three above equations are strictly positive and, therefore, so are the left-hand sides. (4.3) clearly implies that η cannot be almost surely equal to 0 and (4.4) together with (4.5) show that its distribution cannot be concentrated at any other single point. η is thus a nondegenerate random variable.

Proposition 4.3: No subsequence η_L converges in distribution to a normal random variable.

Proof: Let $\eta_{L_k} \xrightarrow{\mathrm{d}} \eta$. For any y which is a continuity point of the distribution function of η we have, using (3.8),

$$P[\eta < y] = \lim_{L\to\infty} P[\eta_L < y] = P[p_c^{(L)} < p_c + \delta_0(L)y] \geq c[\pi_L(p_c + \frac{3}{2}\delta_0(L)y) - \exp(-cL^d(\delta_0(L)y)^2)]. \tag{4.6}$$

We need to assume that y is a continuity point of the limiting distribution function, since we do not know that the distribution of η is continuous (see Remark 4.1). Of course, all y except for countably many satisfy this condition. Let us choose a negative y with a large absolute value. Using the lower bound in Proposition A1.1 (Appendix 1), the right-hand side of (4.6) can be bounded below by

$$c[\exp(-c\frac{L}{L_0(p_c + \frac{3}{2}\delta_0(L)y)}) - \exp(-cL^d(\delta_0(L)y)^2)]. \tag{4.7}$$

Let us remind that in two dimensions $\delta_0(L) \asymp \delta_-(L) \asymp \delta_+(L)$ (see Remark 3.1). Using (1.21) and the representation (1.12) of the regularly varying function L_0, we obtain:

$$\exp(-c\frac{L}{L_0(p_c + \frac{3}{2}\delta_0(L)y)}) \asymp \exp(-c\frac{L_0(p_c - \delta_0(L))}{L_0(p_c + \frac{3}{2}\delta_0(L)y)}) = \exp(-cy^\nu \frac{S(\delta_0(L)y)}{S(\delta_0(L))}). \tag{4.8}$$

Since S is a slowly varying function, the limit of the last expression as $L \to \infty$ equals Ke^{-cy^ν}. On the other hand, Proposition 1.2 implies (in two dimensions) that $\delta_0(L) = L^{-\frac{1}{\nu}}S_1(L)$, where S_1 is a slowly varying function of L. Consequently, $L^d\delta_0^2(L) = L^{d-\frac{2}{\nu}}S_1(L) \to \infty$ and, therefore, for sufficiently large L

$$P[\eta_L \leq y] \geq \frac{c}{2}\exp(-2cy^\nu). \tag{4.9}$$

The inequality $\nu < 2$ (Assumption 1.2) implies now that the distribution of η cannot be Gaussian. Note, that the right tail of the distribution of η can be estimated in the same way, using an appropriate analog of Proposition A1, satisfied in two dimensions by duality.

Theorem 4.4. Let b_L and a_L be arbitrary sequences of real numbers, with $b_L > 0$. No subsequence of $\frac{p_c^{(L)}-a_L}{b_L}$ converges in distribution to a nondegenerate normal random variable.

Proof: This is a standard application of the Convergence of Types Theorem (see [8], ch.2, Theorem 7.16). We have

$$\frac{p_c^{(L_k)} - a_{L_k}}{b_{L_k}} = \frac{\delta_0(L_k)}{b_{L_k}}\eta_{L_k} + \frac{p_c - a_{L_k}}{b_{L_k}}, \tag{4.10}$$

so that the random variables $\frac{p_c^{(L_k)}-a_{L_k}}{b_{L_k}}$ are related to η_L by affine transformations. If both sequences converge in distribution to nondegenerate random variables, the Convergence of Types Theorem implies that $\frac{\delta_0(L_k)}{b_{L_k}}$ converges to a nonzero constant and $\frac{p_c-a_{L_k}}{b_{L_k}}$ converges to a constant and, consequently, if the limit of $\frac{p_c^{(L_k)}-a_{L_k}}{b_{L_k}}$ is normal, so is that of η_{L_k}. This ends the proof.

Theorem 1.8. is an immediate corollary of Theorem 4.4.

Remark4.1: We briefly discuss here the issue of continuity of the limiting distribution function and its relation to assumptions made in literature. If we assume that in addition to the relation $\xi \asymp L_0$ (proven in two dimensions) the correlation length ξ behaves as a pure power, $\xi(p) \asymp |p-p_c|^{-\nu}$, and that

$$\lim_{L\to\infty} L^{\frac{1}{\nu}}\frac{d\pi_L}{dp} \tag{4.11}$$

exists (these assumption are not used in the rest of this paper; see [17] for discussion of numerical results), then equicontinuity of the functions $\pi_L(\delta_0(L)y)$ implies continuity of the limit $F(y)$ of this sequence (or any limit of its subsequence) by the Arzelà-Ascoli theorem. On the other hand, bounds of Section 2 show that

$$F(\frac{y}{2}) \le \liminf_{L\to\infty} P[\eta_L \le y] \le \limsup_{L\to\infty} P[\eta_L \le y] \le F(\frac{3y}{2}) \tag{4.12}$$

and an obvious modification shows that for an arbitrary $\alpha > 0$

$$F((1-\alpha)y) \le \liminf_{L\to\infty} P[\eta_L \le y] \le \limsup_{L\to\infty} P[\eta_L \le y] \le F((1+\alpha)y). \tag{4.13}$$

which implies that any limit of the distribution functions $P[\eta_L \le y]$ is continuous.

While Theorems 1.7. and 1.8 are limited to two dimensions, because of the use of the Rescaling Lemma and duality, large parts of their proofs remain valid in higher dimensions. Plausible consequences for the behavior of the finite-volume percolation threshold, based on available rigorous and numerical information about the value of the correlation length exponent ν were discussed by the present authors in [3]. As pointed out there, since the numerical value of ν in three dimensions is close to 0.9 and smaller than that in higher dimensions (in particular, $\nu = \frac{1}{2}$ above 5 dimensions, according to a well known prediction, see [12]), one may expect the limiting distribution to be even more radically non-Gaussian in higher dimensions (see Remark 1.1). In [21] the moments of the limiting distributions have been studied numerically and found to behave in a nongaussian way. In three dimensions the limiting distribution was even found (numerically) to be asymmetric [11]. Both results support the predictions of [3]. Such a nongaussian behavior in high dimensions would stand in a sharp contrast to the behavior of such extensive (additive) quantities like magnetization or energy of a spin system, which are known to be Gaussian high dimensions. However, the finite-volume percolation threshold is a highly nonlocal function of the bond variables, to which an analogy with the central limit theorem for sums of weakly dependent random variables needs not to apply.

Appendix 1: Exponential bounds on crossing probabilities.

In this appendix we use the technique of rescaling [1] to prove bounds on crossing probabilities, used in sections 3 and 4.

Proposition A1.1: Let $d = 2$. There exist strictly positive constants k, c_1 and c_2, independent of p such that for $p < p_c$

$$ke^{-c_1 \frac{L}{L_0(p)}} \leq \pi_L(p) \leq e^{-c_2 \frac{L}{L_0(p)}}. \tag{A1.1}$$

Remark: The lower bound was communicated to us together with its proof by L. Chayes.

Proof: Let $\pi_{L,M}$ denote the probability that there is an open crossing in a given rectangle $L \times M$ (L units in the horizontal and M units in the vertical direction). Consider the dual percolation model (see [15], [10]). This is also a two-dimensional independent percolation model (bond percolation is self-dual, in the sense that its dual is also a bond percolation model; the dual of site percolation is known as star percolation etc.). The dual of a model with density p (of open bonds, occupied sites etc.) has density $1-p$. If we denote the crossing probabilities of the dual model by $\pi^*_{L,M}$, then

$$\pi_{L,M}(p) + \pi^*_{M,L}(1-p) = 1. \tag{A1.2}$$

It is convenient to use crossing probabilities of elongated boxes, in view of the following

Rescaling Lemma: there exists an $\epsilon_0 > 0$ such that for $p < p_c$. Then, if $\pi_{L,2L}(p) \leq \epsilon_0$, we have

$$\pi_{2L,4L} \leq \pi^2_{L,2L}. \tag{A1.3}$$

A proof of the Rescaling Lemma can be found e.g. in [1]. It is written there for bond percolation, but no changes are necessary to cover other independent two-dimensional models considered here.

Clearly, $\pi_L(p) \leq \pi_{L,2L}(p)$. On the other hand, using FKG inequality and the Russo-Seymour-Welsh lemma [10] for the dual model, we have

$$\pi^*_{2L,L}(1-p) \geq \pi^*_L(1-p)(\pi^*_{\frac{3}{2}L,L}(1-p))^2 \geq \pi^*_L(1-p)(1-\sqrt{1-\pi^*_L(1-p)})^6 \geq (1-\sqrt{1-\pi^*_L(1-p)})^7. \tag{A1.4}$$

We have used here a well-known technique of creating and L-R crossing of a $2L \times L$ box from L-R crossings of two overlapping $\frac{3}{2}L \times L$ boxes and a vertical crossing of a (middle) $L \times L$ box [10]. Since $(1-\sqrt{\pi_L(p)})^7 \geq 1 - 7\sqrt{\pi_L(p)}$ (Bernoulli inequality), using the duality relation (A1.3), we obtain

$$\pi_{L,2L}(p) \leq 7\sqrt{\pi_L(p)}. \tag{A1.5}$$

We now introduce an analog of L_0 using crossing probabilities of elongated boxes:

$$L_1(p,\epsilon) \overset{\text{def}}{=} \min\{L : \pi_{L,2L}(p) \leq \epsilon\}. \tag{A1.6}$$

Comparing (A1.6) to the Definition 1.4, we see, using (A1.5) that

$$L_0(p,\epsilon) \leq L_1(p,\epsilon) \leq L_0(p, \frac{\epsilon^2}{49}), \tag{A1.7}$$

which, in view of Theorem 1.4, proves that

$$L_0(p) \asymp L_1(p). \tag{A1.8}$$

Provided that ϵ in the definition of L_1 has been chosen sufficiently small, the Rescaling lemma implies

$$\pi_{2^k L_1(p), 2^{k+1}L_1(p)}(p) \leq \pi_{L_1(p),2L_1(p)}^{2^k}, \tag{A1.9}$$

which yields an upper bound

$$\pi_{L,2L}(p) \leq e^{-c\frac{L}{L_1(p)}} \tag{A1.10}$$

for L of the form $2^k L_1(p)$ (with $c = -\log\epsilon > 0$) and hence also (perhaps with a different constant c) for arbitrary L. Since $\pi_L(p) \leq \pi_{L,2L}(p)$ this proves the desired upper bound, in view of the asymptotic equivalence of L_0 and L_1. To prove the lower bound, note first that the above inequalities between $\pi_L(p)$ and $\pi_{L,2L}(p)$ imply that the correlation length $\xi_1(p)$ defined by

$$\xi_1^{-1}(p) = -\limsup_{L\to\infty} \frac{\log \pi_{L,2L}(p)}{L} \tag{A1.11}$$

is asymptotically equivalent to ξ defined in (1.5) (as for ξ, the choice of $\liminf$ would also yield an asymptotically equivalent quantity). It follows from (A1.11) that for large L

$$\frac{\log \pi_{L,2L}(p)}{L} \geq -2\xi_1^{-1}(p), \tag{A1.12}$$

which implies

$$\pi_{L,2L}(p) \geq e^{-2\frac{L}{\xi_1(p)}} \tag{A1.13}$$

for large L. In view of the asymptotic equivalence of ξ_1 to ξ (and, therefore, also to L_0), this implies the desired lower bound in (A1.1) (the constant k may be necessary to accommodate the finite number of values of L for which (A1.13) does not hold).

Appendix 2. Regularly varying functions

In this Appendix we state a fundamental representation theorem for slowly varying functions and use it to prove a bound necessary in the proof of Proposition 3.2. We then give a proof of Proposition 1.2.

Theorem: Let $S(x)$ be a slowly varying function, $x \geq 0$. Then there exist functions a and ϵ, such that $\lim_{x\to 0} \epsilon(x) = 0$; $\lim_{x\to 0} a(x) = c < \infty$ and for $x \leq 1$

$$S(x) = a(x) \exp\left(\int_x^1 \frac{\epsilon(y)}{y}\, dy\right). \tag{A2.1}$$

This theorem is proven in [9]; one has to change variables from x to $\frac{1}{x}$ to obtain the present formulation. It also follows from the proof in [9] that a and ϵ can be chosen so that $c > 0$.

Proposition A2.1: Let $S(z)$ be a slowly varying function, $(z \geq 0)$, bounded above on intervals $[z_0, \infty]$ for all $z_0 > 0$. c and ν—positive constants and k—a positive integer. Let

$$J(t) = \int_0^\infty z^k \exp[-cz^\nu \frac{S(t)}{S(zt)}]\, dz. \tag{A2.2}$$

Then

$$\limsup_{t\to 0} J(t) < \infty. \tag{A2.3}$$

Proof: We use the representation (A2.1), where with no loss of generality we can assume that $\epsilon(x) \geq C$ where C is a (negative) constant. Fix a $\mu \in (0, \nu)$. There exists x_0 such that for $x \leq x_0$ we have $\epsilon(x) \geq -\mu$. Now let $z \leq \frac{x_0}{t}$. We have

$$\frac{S(t)}{S(tz)} = \frac{a(t)}{a(tz)} \exp\left(\int_t^1 \frac{\epsilon(y)}{y}\, dy - \int_{tz}^1 \frac{\epsilon(y)}{y}\, dy\right) = \frac{a(t)}{a(tz)} \exp\left(\int_t^{tz} \frac{\epsilon(y)}{y})\, dy\right) \geq K \exp(-\mu \log z), \tag{A2.4}$$

where K is a positive constant. Hence, for $z \leq \frac{x_0}{t}$,

$$z^\nu \frac{S(t)}{S(tz)} \geq K z^{\nu-\mu}. \tag{A2.5}$$

As a result we obtain a finite, t-independent bound on a part of our integral:

$$\int_1^{\frac{x_0}{t}} z^k \exp(-z^\nu \frac{S(t)}{S(tz)})\, dz \leq \int_1^{\frac{x_0}{t}} z^k \exp(-K z^{\nu-\mu})\, dz \leq \int_1^\infty z^k \exp(-K z^{\nu-\mu})\, dz. \tag{A2.6}$$

The remaining part of the integral can be estimated by a similar application of the Karamata theorem or directly as follows: when $zt > x_0$, we have $S(zt) < c^{-1}$, where c is a positive constant (by assumption). Also, for small t we have $S(t) \geq t^\mu \geq x_0^\mu z^{-\mu}$. Hence for small t the remaining part of the integral (from $\frac{x_0}{t}$ to ∞ is bounded by

$$\int_1^\infty z^k \exp(-c x_0^\mu z^{\nu-\mu})\, dz < \infty \tag{A2.7}$$

and the proof is finished.

Proof of Proposition 1.2: To prove (1.20), let $p_n \uparrow p_c^-(L)$ with $p_n < p$. Then, by definition of $p_c^-(L)$, $L_0(p_n) \leq L$, i.e. (see Definition 1.4) $\exists L_n \leq L : \pi_{L_n}(p_n) \leq \epsilon$. Since there are only finitely many integers between 0 and L, L_n is equal to some $M \leq L$ for infinitely many n. Taking a subsequence and letting n to infinity, we obtain, using continuity of π_L as a function of p, that $\pi_M(p_c^-(L)) \leq \epsilon$, which, in view of Definition 1.4 implies (1.20) for $p_c^-(L)$. The proof for $p_c^+(L)$ is analogous. (1.21) will be proven for δ_-; the proof for δ_+ is analogous. For any $\theta < 1$, the definition of δ_- (see (1.15) and (1.18)) implies that

$$L_0(p_c - \theta\delta_-(L)) > L. \tag{A2.8}$$

Hence, using (1.13),

$$\liminf_{L\to\infty} \frac{L_0(p_c^-(L))}{L} \geq \liminf_{L\to\infty} \frac{L_0(p_c - \delta_-(L))}{L_0(p_c - \theta\delta_-(L))} = \theta^\nu \qquad (A2.9).$$

Taking the limit $\theta \to 1$ implies $\liminf_{L\to\infty} \frac{L_0(p_c^-(L))}{L} \geq 1$ which in combination with (1.20) proves (1.21). To prove the last statement of the proposition, we need to show that for any $t > 0$

$$\frac{\delta_-(tL)}{\delta_-(L)} \to t^{-\frac{1}{\nu}}. \tag{A2.10}$$

We argue by contradiction. If (A2.10) does not hold, then either

$$\limsup_{L\to\infty} \frac{\delta_-(tL)}{\delta_-(L)} \geq (1+\alpha) t^{-\frac{1}{\nu}} \tag{A2.11}$$

or

$$\liminf_{L\to\infty} \frac{\delta_-(tL)}{\delta_-(L)} \leq (1-\alpha) t^{-\frac{1}{\nu}} \tag{A2.12}$$

for some $\alpha > 0$. In the first case, regular variation of L_0 with the exponent $-\nu$ implies that

$$\liminf_{L\to\infty} \frac{L_0(p_c - \delta_-(tL))}{L_0(p_c - \delta_-(L))} \leq \liminf_{L\to\infty} \frac{L_0(p_c - (1+\alpha)t^{-\frac{1}{\nu}}\delta_-(L))}{L_0(p_c - \delta_-(L))} = (1+\alpha)^{-\nu} t. \tag{A2.13}$$

We have used here the fact that for $p < p_c$, L_0 is a (not strictly) increasing function of p, which follows from Definition 1.4 and monotonicity of π_L in p. But (1.21) implies that

$$\liminf_{L\to\infty} \frac{L_0(p_c - \delta_-(tL))}{L_0(p_c - \delta_-(L))} = t, \tag{A2.14}$$

so (A2.11) leads to a contradiction. Similarly, (A2.12) implies that

$$\limsup_{L\to\infty} \frac{L_0(p_c-\delta_-(tL))}{L_0(p_c-\delta_-(L))} \geq (1-\alpha) - \nu t, \tag{A2.15}$$

which again contradicts (A2.14). This proves (A2.10) and the last part of the theorem for δ_-. The proof for δ_+ is similar.

Acknowledgements

We are pleased to thank K. Alexander, L. Chayes, H. Kesten, A. Kupiainen and D. Stauffer for very helpful discussions and for providing us with necessary references. We are particularly grateful to H. Kesten who read the manuscript and made several valuable suggestions. We are also very grateful to the referee whose comments and constructive criticism of the first version of the paper led to significant improvements in organization of proofs and helped us clarify the status of Assumption 1.1. Part of the work was done while L.B. was visiting the Department of Mathematics of the University of Arizona and he is grateful for the hospitality received there.

References

1. Chayes, J.T. and Chayes, L.: Percolation and random media. In: Osterwalder, K. and Stora R. (eds.): Critical Phenomena, Random Systems, Gauge Theories. North Holland 1986.

2. Alexander, K. Private communication.

3. Berlyand, L., Wehr, J.: The probability distribution of the percolation threshold in a large system. J. Phys. A: Math. Gen. 28, 7127 (1995).

4. Chayes, J.T., Chayes, L. and Fröhlich, J.: The low-temperature behavior of disordered magnets. Commun. Math. Phys. 100, 399 (1985).

5. Chayes, J.T., Chayes, L., Fisher, D. and Spencer, T.: Finite-size scaling and correlation lengths for disordered systems. Phys. Rev. Lett. 57, 3002 (1986).

6. Chayes, L. Private communication.

7. Saleur, H.: Conformal invariance for polymers and percolation. J. Phys. A: Math. Gen. 20, 455 (1987).

8. Durrett, R.: Probability. Theory and Examples. Second Editiion, Wadsworth 1996.

9. Feller, W.: An Introduction to Probability Theory and Its Applications, v.2. Wiley 1971.

10. Grimmett, G.: Percolation. Springer 1989.

11. Haas, U.: The distribution of percolating concentrations in finite systems. Physica A 215, 247 (1995).

12. Hara, T. and Slade, G.: Mean-field critical behaviour for percolation in high dimensions. Commun. Math. Phys. 128, 333 (1991).

13. Hara, T. and Tasaki, H.: A rigorous control of logarithmic corrections in four-dimensional ϕ^4 spin systems II. J. Stat. Phys. 47, 98 (1987).

14. Harris, A.B.: Upper bounds for the transition temperatures of generalized Ising models. J. Phys. C 7, 3082 (1974).

15. Kesten, H.: Percolation Theory for Mathematicians. Birkhäuser (1982).

16. Kesten, H.: Scaling relations for 2D-percolation. Commun. Math. Phys. 109, 109 (1987).

17. Langlands, R., Pouliot, P., Saint-Aubin, Y.: Conformal invariance in 2D percolation. Bull. Am. Math. Soc. 30, 1(1994).

18. Ma, S.-k.: Modern Theory of Critical Phenomena, Benjamin 1976.

19. Nguyen, B.G.: Correlation length and its critical exponent for percolation processes. J. Stat. Phys. 46, 517 (1987).

20. Stauffer, D. and Aharony, A.: Introduction to Percolation Theory, Taylor and Francis 1992.

21. Ziff, R.: Spanning probabilities in 2D percolation. Phys. Rev. Lett. 69, 2670 (1992); see also Phys. Rev. Lett. 72, 1942 (1994).

22. Nienhuis, B.: Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas. J. Stat. Phys. 34, 731 (1984).

23. Isichenko, M.B.: Percolation, statistical topography and transport in random media. Rev. Mod. Phys. 64, 961 (1992).

24. Borgs, C., Chayes, J.T., Kesten, H., Spencer, J.: The birth of the infinite cluster: finite-size scaling in percolation. In preparation.