HOMEWORK 3, due December 15

1. **Adjoint operators.** Let $\mathcal{H}$ be a complex Hilbert space. Denote l_x the linear functional consisting in multiplication by x:

$$l_x(y) = (y, x).$$

By Riesz-Fréchet theorem, this sets a one-to-one correspondence between $\mathcal{H}$ and its dual $\mathcal{H}'$. We can use it to represent dual operators as operators on $\mathcal{H}$: let A be a bounded operator. Given $l = l_x \in \mathcal{H}'$, let z be the unique vector in $\mathcal{H}$ such that $A'(l_x) = l_z$. Letting $A^*x = z$, prove that A^* is a bounded linear operator. Prove:

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*;$$

$$(AB)^* = B^* A^*$$

Show that spectrum of A^* is the complex conjugation of the spectrum of A. An operator is called self-adjoint if $A^* = A$. When is a product of two self-adjoint operators self-adjoint? What do we know about the spectrum of a self-adjoint operator?

2. **Hellinger-Toeplitz theorem**. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator on a Hilbert space $\mathcal{H}$, satisfying: $(Ax, y) = (x, Ay)$ for all x, y. Prove that A is a bounded operator. That is, a self-adjoint operator defined on the whole space has to be continuous. This is a motivation for theory of unbounded operators we will develop in the second semester.

3. a) Prove that every bounded operator in a Hilbert space $\mathcal{H}$ can be written as a linear combination of two self-adjoint operators. A bounded operator V in $\mathcal{H}$ is called unitary if it is invertible and $V^{-1} = V^*$. Prove that V preserves the inner products. b)* Prove that every bounded operator can be written as a linear combination of four unitary operators.

4. Let M and V be bounded linear operators on a Banach space X. Assume V is an isometry: $|Vx| = |x|$ for every $x \in X$. Prove that the operators M and VMV^{-1} have the same spectrum.

5. Find the dual and the adjoint of an integral operator with a square-integrable kernel $K(s,t)$ from L^2 to L^2.

6. Let f be a bounded, measurable, complex-valued function on $\mathbf{R}$. Consider the multiplication operator defined on $L^p(\mathbf{R}$ $(1 \le p < \infty)$ by

$$T_f x(t) = f(t)x(t).$$

Find the spectrum of T_f. To avoid measure theory, you can assume f is continuous. For $p = 2$, when is T_f self-adjoint?

7. **Difference operators.** Consider the space $l^2(\mathbf{Z}^d)$ consisting of functions $\phi : \mathbf{Z}^d \to \mathbf{C}$, such that $\sum |\phi(x)|^2 < \infty$, where $\mathbf{Z}^d$ denotes the integer lattice in d dimensions (the set of $x \in \mathbf{R}^d$ with integer coordinates). Prove that l^2 is a Hilbert space with the inner product

$$(\phi, \psi) = \sum_{x \in \mathbf{Z}^d} \phi(x)\overline{\psi(x)}.$$

Let ∂_j be the operator on l^2 defined by

$$\partial_j(\phi)(x) = \phi(x + e_j) - \phi(x),$$

where e_j is the unit vector in the j-th direction. Find the adjoint of ∂_j. Find the explicit expression for the lattice Laplacian:

$$\Delta = -\sum_{j=1}^{d} \partial_j^* \partial_j.$$

8*. Find the spectrum of Δ of problem 7.

9. Let f be a continuous function with $\int_0^\infty |f(t)|\, dt < \infty$. a) Prove that the operator

$$(Kx)(t) = \int_0^\infty f(t+s)x(s)\, ds$$

is a bounded operator from $L^2(0,\infty)$ to $L^2(0,\infty)$. b)* Prove that K is compact.

10. Prove that the spectrum of a unitary operator lies in the unit circle.

11. Let X, Y and Z be Banach spaces and $A : X \to Y$, $B : Y \to Z$—linear operators with B bounded and injective. Prove that if BA is bounded, then so is A.

12. Let X be a Banach space and A_n a sequence of bounded operators, which are invertible (as elements of $\mathcal{L}(X)$). Suppose that $|A_n - A| \to 0$. Prove that A is invertible if and only if the operator norms $|A_n^{-1}|$ are bounded. Prove that in this case A_n^{-1} converges to A^{-1} in the operator norm.

13. Let $p(z) = (z - M)^{-1}$ be the resolvent of an element M of a Banach algebra (p is defined on the resolvent set $\rho(M)$). Prove that $\frac{dp}{dz} = -p(z)^2$.

14. For an element M of a Banach algebra, prove that

$$\lim_{z\to\infty} (z - M)^{-1} = 0$$

and

$$\lim_{z\to\infty} z(z - M)^{-1} = I,$$

where I is the unit of the algebra.

15. **Nuclear operators.** Let X and Y be Banach spaces. Suppose $l_n \in X'$ and $y_n \in Y$ are such that $\sum_n |l_n||y_n| < \infty$. Prove that the operator $K : X \to Y$ defined by

$$Kx = \sum_n l_n(x)y_n$$

is compact.

16*. Prove that a compact operator maps weakly convergent sequences to strongly convergent sequences. That is, assuming that $K : X \to Y$ is compact and that, for some $x, x_n \in X$, $l(x_n) \to l(x)$ for every $l \in X'$, prove that $|Kx_n - Kx| \to 0$.

17. **HAVE FUN!!**