HOMEWORK 2, due November 7

1. Prove that the unit ball $B = \{x : |x| \leq 1\}$ in a normed vector space is convex: $x, y \in B$ implies that $(1-\lambda)x + \lambda y \in B$ for any $\lambda \in [0, 1]$.

2. Prove that none of the following spaces is a Hilbert space, i.e. it is impossible to define an inner product on the space so that $(x, x)^{\frac{1}{2}}$ is the original norm:

a) l^p for $p \neq 2$; b) $L^p[0, 1]$ for $p \neq 2$; c) $C[0, 1]$ with the supremum norm.

3. Show that there are two linearly independent vectors x and y in l^∞ such that $|x| = |y| = 1$ and $|x+y| = 2$.

4. Prove that if a normed vector space X contains linearly independent vectors x and y such that $|x + y| = |x| + |y|$, then there is a line segment (bigger than a single point) contained in the unit sphere of X, $S = \{x : |x| = 1\}$.

5. Prove that there are no nontrivial line segments contained in the unit sphere of normed vector space if and only if every subspace contains a unique closest element to a given vector.

5. Prove that in l^p, $1 < p < \infty$, there are no nontrivial line segments contained in the unit sphere.

6. Let c be the vector space consisting of sequences $x = (x_1, x_2, \ldots)$ of complex numbers converging to zero. Find the dual of c.

7. Two norms $|\cdot|_1$ and $|\cdot|_2$ on a vector space X are called equivalent if there exist positive constants a and A such that for every $x \in X$ $a|x|_1 \leq |x|_2 \leq A|x|_1$. Prove that a sequence $x_n \in X$ converges in the norm $|\cdot|_1$ if and only if it converges in the norm $|\cdot|_2$. Prove that all norms on the vector space $\mathbf{R}^n$ are equivalent.

8. Prove that the dual space of l^1 is l^∞ and that the dual of l^∞ is strictly bigger than l^1.

9. Let X be a Hilbert space and l—a bounded linear functional on a subspace Y of X (not necessarily closed). Describe all extensions of l to a bounded functional on X.

10*. Construct a bounded linear functional on some subspace of some L^1 space which has infinitely many distinct linear extensions to L^1 of the same norm as the original functional.

11*. Prove that a Banach space X is reflexife if and only if its dual space X' is reflexive.

12*. Let S be a closed subspace of $L^1[0, 1]$ such that each $f \in S$ belongs to some $L^p[0, 1]$ with $p > 1$. Prove that there exists a $p_0 > 1$ such that all S is contained in L^{p_0}.

13*. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Given a vector (a random variable) $X \in L^2(\Omega, \mathcal{F}, P)$, prove that there exists a unique element $Y \in L^2(\Omega, \mathcal{G}, P)$ such that $\int_E X \, dP = \int_E Y \, dP$ for every $E \in \mathcal{G}$. Y is called the conditional expectation of X with respect to the σ-algebra $\mathcal{G}$.

14*. Let C be the space of all continuous functions on $[0, 1]$ with the supremum norm. Let M consist of all $f \in C$ for which

$$\int_0^{\frac{1}{2}} f(t) \, dt - \int_{\frac{1}{2}}^1 dt = 1.$$

Prove that M is a closed convex subset of C which contains no element of minimal norm.