HOMEWORK 1, due September 26

1. **Polarization identity.** Prove that in any complex inner product space

$$(x, y) = 1/4[(\|x + y\|^2 - \|x - y\|^2) + i(\|x + iy\|^2 - \|x - iy\|^2)].$$

2. **Jordan-von Neumann theorem.** In an arbitrary normed vector space the formula of problem 1 does not necessarily define an inner product. Prove that it does if and only if the norm satisfies the parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

3. **Projection onto a finite-dimensional subspace.** Let X be an inner product space and let $x_1, \dots x_N \in X$ be an orthonormal set. Prove that the expression

$$\|x - \sum_{n=1}^{N} c_n x_n\|$$

attains its minimum for $c_n = (x, x_n)$. Interpret this result as an orthogonal decomposition. Do we need to assume X is complete? Why or why not?

4. **Finding the closed linear span.** Let α be a complex number with $0 < |\alpha| < 1$. In the space l^2 we consider the vectors

$$x_n = (1, \alpha^n, \alpha^{2n}, \dots \alpha^{kn}, \dots).$$

Find the closed linear span of the x_n, $n = 1, 2, \dots$.

5. **Fourier series of square integrable functions.**

a)Prove that the family of harmonics $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$, $n = 0, \pm 1, \pm 2, \dots$ is an orthonormal basis in $L^2[-\pi, \pi]$. One way to do it is to use Weierstrass approximation theorem.

b) Use the Fourier expansion of the function x^2 to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

6. **Gram-Schmidt orthonormalization, Legendre polynomials.**

a) Apply the Gram-Schmidt process to the functions $1, x, x^2, x^3, \dots$ on the interval $[-1, 1]$ with the L^2 inner product. The resulting polynomials are the Legendre polynomials, $\phi_n(x)$. Calculate the first four of them.

b) Prove that

$$\phi_n(x) = \sqrt{\frac{2n+1}{2}} \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

for all n.

c)* Prove that ϕ_n, $n = 0, 1, \dots$ is a basis in $L^2[-1, 1]$.

7. **Hermite functions.** In the space $L^2(\mathbf{R})$ (with the Lebesgue measure) consider vectors $\psi_0, \psi_1, \dots$, where $\psi_0(x) = \pi^{-1\ 4} e^{-\frac{1}{2}x^2}$ and for $n \geq 1$

$$\psi_n(x) = (2^n n!)^{-\frac{1}{2}} (-1)^n \pi^{-\frac{1}{4}} e^{\frac{1}{2}x^2} \frac{d^n (e^{-x^2})}{dx^n}.$$

a) Prove that ψ_n form an orthonormal set.

b)* Prove that this orthonormal set is complete, i.e. that it is an orthonormal basis.

8*. **Stability of orthonormal bases.** Let $\mathcal{H}$ be a Hilbert space with a countable orthonormal basis $x_1, x_2, \ldots$. Let $y_1, y_2, \ldots$ be another orthonormal set, such that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < \infty.$$

Prove that this set is also a basis, i.e. that its linear span is all of $\mathcal{H}$.

9. **Hilbert space geometry.** In a Hilbert space H, let $\|x - x_1\| = R$ and $\|x - x_2\| = R$ be two spheres with $\|x_1\| = \|x_2\|$. Show that the intersection of the two spheres is a sphere. Find its radius and its center. Show that it lies in an affine subspace of the form $v + Y$, where $v \in H$ and Y is the subspace orthogonal to the vector $x_1 - x_2$.

10**. **Are Hilbert spaces the only self-dual Banach spaces?** Let X be a real Banach space. Suppose there exists a linear isomorphism $T : X \to X'$, which is an isometry: $|Tx| = |x|$ for every x, where $|Tx|$ denotes the norm (of the linear functional) in the dual space. Does it follow that X is a Hilbert space, i.e. that the norm in X comes from an inner product? I do not know the answer to this question.