Mathematics 528A, Homework 1. Due September 23

0. Please do the following problems from Chapter 1: 8, 15, 58, 64.

1. **Appolonius sphere** Let $\mathcal{H}$ be a Hilbert space; $u, v \in \mathcal{H}$ and $c > 0$. Describe explicitly the set

$$\{w \in \mathcal{H} : ||w - u|| = c||w - v||\}.$$

2*. **Conditional expectation** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Given a random variable $X \in L^2(\Omega, \mathcal{F}, P)$, prove that there exists a unique random variable $Y \in L^2(\Omega, \mathcal{G}, P)$ such that for every event $A \in \mathcal{G}$

$$\int_A X \, dP = \int_A Y \, dP.$$

Y is called the expectation of X conditioned on $\mathcal{G}$. This is a fundamental definition in modern probability theory, due to Kolmogorov. *Hint*: use the projection theorem.

3*. **Stability of orthonormal bases** Let $\phi_1, \phi_2, \ldots$ be an ON basis for $\mathcal{H}$ and let $\psi_1, \psi_2, \ldots$ be an ON system, such that

$$\sum_j ||\phi_j - \psi_j|| < \infty.$$

Prove that ψ_j also are a basis for $\mathcal{H}$.