Douglas Ulmer
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Algebraic Geometry
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Here are more exercises.

16. Let $X \subseteq \mathbf{P}^3$ be the twisted cubic curve. Find equations for the image of X under the projection from $[0,0,1,0]$ to the plane $x_2 = 0$ and from $[1,0,0,-1]$ to $x_3 = 0$. Are the images in the two $\mathbf{P}^2$s isomorphic?

17. Let $X = V(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbf{P}^2$. Let $Y \subseteq \mathbf{P}^5$ be the image of X under the Veronese map $\mathbf{P}^2 \to \mathbf{P}^5$. Find equations defining Y as a closed subset of $\mathbf{P}^5$.

18. Show that $PGL_2(k)$ acts 3-transitively on $\mathbf{P}^1$ and that the stabilizer of an ordered triple is trivial.

19. Show that the intersection of r hyperplanes in $\mathbf{P}^n$ is a linear subspace of dimension at least $n - r$.

20. If k is the finite field with q elements, how many linear subspaces of dimension r are there in k^n?

21. Let F be any linear homogeneous polynomial in $k[x_0, \ldots, x_n]$. Show that the subgroup of PGL_{n+1} preserving $X = V(F)$ is isomorphic to AGL_n. What is the subgroup of PGL_{n+1} fixing X pointwise?

22. Suppose that $X \subseteq \mathbf{A}^n$ is closed and let $Y \subseteq \mathbf{P}^n$ be the closure of the image of X under the map $f_0^{-1} : \mathbf{A}^n \to \mathbf{P}^n$ defined in class. What are the equations of Y? (Hints: Start with the case of a hypersurface. Be careful about the general case: the most obvious generalization is not correct.)

23. Find equations for the image of the Segre embedding.

24. Prove that any two hypersurfaces in $\mathbf{P}^2$ have a non-empty intersection. Can you say more?

25. Show that if $k = \mathbf{C}$, the complex numbers, and $X \subset \mathbf{P}^n$ is a subvariety of $\mathbf{P}^n$ then $\mathbf{P}^n - X$ is path connected.

26. Compute the homology and cohomology groups of $\mathbf{P}^n$ when $k = \mathbf{R}$ or $k = \mathbf{C}$.

27. Suppose $k = \mathbf{C}$ and let $X = V(xz - y^2) \subseteq \mathbf{P}^2$. Show that X defines a homology class in $H_2(\mathbf{P}^2, \mathbf{Z}) \cong \mathbf{Z}$. Which class is it?

28. Show that to test whether a homogeneous ideal is prime, it is enough to check that for any two *homogeneous* elements f and g, $fg \in I$ implies $f \in I$ or $g \in I$.

29. Show that if $X \subseteq \mathbf{P}^n$ is a quasi-projective variety then the cone over X in $\mathbf{A}^{n+1}$ is birationally isomorphic to $X \times \mathbf{A}^1$.

30. Prove that the set of $n \times n$ matrices of rank $r \leq n$ is a rational variety.