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Transversality of sections on elliptic surfaces with applications to elliptic divisibility sequences and geography of surfaces

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Abstract
We consider elliptic surfaces $\mathcal{E}$ over a field k equipped with zero section O and another section P of infinite order. If k has characteristic zero, we show there are only finitely many points where O is tangent to a multiple of P. Equivalently, there is a finite list of integers such that if n is not divisible by any of them, then nP is not tangent to O. Such tangencies can be interpreted as unlikely intersections. If k has characteristic zero or $p > 3$ and $\mathcal{E}$ is very general, then we show there are no tangencies between O and nP. We apply these results to square-freeness of elliptic divisibility sequences and to geography of surfaces. In particular, we construct mildly singular surfaces of arbitrary fixed geometric genus with K ample and K^2 unbounded.

Keywords Elliptic surfaces · Unlikely intersections · Elliptic divisibility sequences · Stable surfaces · Geography of surfaces

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1 Introduction

Our aim in this paper is to study transversality properties of sections of elliptic surfaces and to deduce consequences for elliptic divisibility sequences and geography of surfaces.

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To state the first result, let k be a field of characteristic zero and let $\mathcal{C}$ be a smooth, projective, geometrically irreducible curve over k. Let $\pi : \mathcal{E} \to \mathcal{C}$ be a relatively minimal Jacobian elliptic surface over k (i.e., a smooth, projective elliptic surface with a section O which will play the role of zero section), and let P be another section. We write nP for the section induced by multiplication by n in the group law of the fibers of $\mathcal{E} \to \mathcal{C}$. Assume that P has infinite order, i.e., $nP \neq O$ for all $n \neq 0$. As we will see below, except in degenerate situations the intersection number $(nP).O$ grows like a constant times n^2. Our first result says that the intersections are usually transverse.

Theorem 1.1 *The set*

$$T = \bigcup_{n \neq 0} \{t \in \mathcal{C} \mid nP \textit{ is tangent to } O \textit{ over } t\}$$

is finite.

Here and in the rest of the paper, we conflate the sections $O : \mathcal{C} \to \mathcal{E}$ and $P : \mathcal{C} \to \mathcal{E}$ with their images $O(\mathcal{C}) \subset \mathcal{E}$ and $P(\mathcal{C}) \subset \mathcal{E}$. Thus we say "$P$ is tangent to O" rather than "the image of P is tangent to the image of O."

In [32], we give an explicit upper bound on the cardinality of T.

Remark 1.2 We note that a tangency between nP and O can be regarded as an "unlikely intersection" as follows: Let $T_{\mathcal{E}}$ be the tangent bundle of $\mathcal{E}$ and let $\mathbb{P}T_{\mathcal{E}}$ be the associated projective bundle. Thus $\mathbb{P}T_{\mathcal{E}} \to \mathcal{E}$ is a $\mathbb{P}^1$-bundle, and the total space $\mathbb{P}T_{\mathcal{E}}$ is a smooth, projective threefold. If $C \subset \mathcal{E}$ is a smooth curve, then there is a canonical lift of C to $\tilde{C} \subset \mathbb{P}T_{\mathcal{E}}$ defined by sending a point $t \in C$ to the class of its tangent line $T_{C,t} \subset T_{\mathcal{E},t}$ in $\mathbb{P}T_{\mathcal{E}}$. Two curves C_1 and C_2 in $\mathcal{E}$ that meet at $y \in \mathcal{E}$ are tangent there if and only if their lifts meet at a point of $\mathbb{P}T_{\mathcal{E}}$ over y. Thus a tangency between C_1 and C_2 is equivalent to the "unlikely" intersection of the two curves $\tilde{C}_1$ and $\tilde{C}_2$ in the threefold $\mathbb{P}T_{\mathcal{E}}$. We refer to [33] for a comprehensive account of work on unlikely intersections up to 2012.

Remarks 1.3 (1) A result very similar to our Theorem 1.1 was communicated to us by Corvaja, Demeio, Masser, and Zannier after we posted the first version of this paper. Their methods are rather different, see [11]. They show more generally that finiteness holds when the cyclic group $\{nP | n \in \mathbb{Z}\}$ is replaced by a finitely generated, torsion-free group of sections.

(2) On the other hand, our methods lead to non-trivial results in families, and in particular we show that for "generic" data, the set T above is empty. (See Theorems 1.7 and 1.8 below.) This is crucial for our application to geography of surfaces.

(3) In the first version of this paper, we used a trivialization essentially equivalent to the Betti foliation discussed in Sect. 3 of this version. Later, we learned of the "Betti" terminology used by several authors, including in [12], and adopted it in this paper.

We next reformulate Theorem 1.1 in analogy with the "elliptic divisibility sequence" associated to an elliptic curve and a point. (See [28, Exers. 3.34-36, 9.4, 9.12] for definitions and examples, and [18] for more on the function field case.) Define a sequence of effective divisors on $\mathcal{C}$ for $n \geq 1$ by

$$D_n := O^*(nP),$$

i.e., D_n is the pull-back along the zero section of the divisor nP on $\mathcal{E}$. (We will give several other equivalent definitions in Sect. 2.)

The sequence D_n is a natural analogue of an elliptic divisibility sequence. In particular, we will see below that if m divides n, then D_m divides D_n (i.e., $D_n - D_m$ is effective), and that Möbius inversion gives a sequence of effective divisors D'_m such that

$$D_n = \sum_{m|n} D'_m.$$

We say that a divisor on $\mathcal{C}$ is *reduced* if it has the form

$$D = \sum_i t_i$$

where the t_i are distinct closed points of $\mathcal{C}$ (i.e., each non-zero coefficient of D equals 1). This is an analogue of an integer being square-free.

Theorem 1.4 *Given $\mathcal{E}$ and P as above, there is a finite set of integers $M = \{m_1, \ldots, m_k\}$ such that*

(1) O and nP intersect transversally if and only if n is not divisible by any element of M.
(2) D_n is reduced if and only if n is not divisible by any element of M.
(3) D'_m is reduced if and only if $m \notin M$.

Remark 1.5 Thereom 1.4 is much stronger than what one might predict from standard conjectures. For simplicity, assume that $\mathcal{C} = \mathbb{P}^1$, let $F = \mathbb{C}(\mathcal{C})$, and let E/F be the generic fiber of $\mathcal{E} \to \mathcal{C}$. Choosing a coordinate t on $\mathbb{P}^1$ so that none of the D_n involve the place at infinity, we may identify each D_n with a monic polynomial f_n in t, and to say that D_n is reduced is to say that $\gcd(f_n, df_n/dt) = 1$. Arguments similar to those in [27] applied to a certain Buium jet space of E/F together with a function field analogue of Vojta's conjecture suggest that

$$\deg \gcd(f_n, df_n/dt) \stackrel{?}{=} o(n^2) = o(h(nP))$$

where $h(nP)$ is the canonical height of nP. (See Sect. 2.7 for definitions.) But Theorem 1.4 shows that $\deg \gcd(f_n, df_n/dt)$ is in fact bounded!

Remark 1.6 We have no reason to believe that the analogues of Theorems 1.1 and 1.4 (with n restricted to be prime to the characteristic) are false in positive characteristic. However, our proof uses analytic techniques and does not obviously carry over to the arithmetic situation.

The next two results hold for k a field of characteristic zero or sufficiently large p. As before, $\mathcal{C}$ is a smooth, projective, geometrically irreducible curve over k. The next result says roughly that if $\mathcal{E} \to \mathcal{C}$ is a very general Jacobian elliptic surface with an additional section P, there are no tangencies between nP and O for $n \neq 0$. Recall that a line bundle L on $\mathcal{C}$ is said to be globally generated (or base point free) if for every $t \in \mathcal{C}$, there is a global section of L which does not vanish at t.

Theorem 1.7 *Suppose k is a field of characteristic zero or of characteristic $p > 3$, and let $\mathcal{C}$ be as above. Let L be a globally generated line bundle on $\mathcal{C}$ of degree d and set*

$$V = H^0(L^2 \oplus L^3 \oplus L^4).$$

Then for a very general $a = (a_2, a_3, a_4) \in V$, the elliptic surface $\mathcal{E} \to \mathcal{C}$ associated to

$$E : y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x$$

equipped with the usual zero section O and the section $P = (0, 0)$ has the following properties:

(1) P has infinite order.
(2) The singular fibers of $\mathcal{E} \to \mathcal{C}$ are nodal cubics (i.e., Kodaira type I_1).
(3) P meets each singular fiber in a non-torsion point.
(4) If n is not a multiple of the characteristic of k, then nP meets O transversally in $d(n^2 - 1)$ points.

(Here and elsewhere in the paper, L^i means $L^{\otimes i}$ and not $L^{\oplus i}$.) We will explain the construction of the elliptic surface attached to a in Sect. 5.4 and the meaning of "very general" in Sect. 6.

As with many results about "very general" points, Theorem 1.7 does not allow one to deduce the existence of examples over "small" (countable) fields such as number fields or global function fields. However, after relaxing condition (2) above, we can write down such examples explicitly, at least when L is the square of a globally generated line bundle.

Theorem 1.8 *Let k be a field of characteristic 0 or a field of characteristic $p > 2$ which is not algebraic over the prime field $\mathbb{F}_p$. Let $\mathcal{C}$ be a smooth, projective, geometrically irreducible curve over k with a non-trivial line bundle L which is the square of a globally generated line bundle F. Then there exist infinitely many pairs $(\mathcal{E}, P)$ where $\mathcal{E}$ is a Jacobian elliptic surface $\mathcal{E} \to \mathcal{C}$ equipped with a section P such that:*

(1) P has infinite order.
(2) The singular fibers of $\mathcal{E} \to \mathcal{C}$ are of Kodaira type I_0^*.
(3) P meets each singular fiber in a non-torsion point.

(4) *If n is not a multiple of the characteristic of k, then nP meets O transversally in*

$$\begin{cases} \dfrac{n^2-1}{2}d & \text{if } n \text{ is odd,} \\ \dfrac{n^2-4}{2}d & \text{if } n \text{ is even} \end{cases}$$

points, where $\deg L = 2d$.

(5) $O^*(\Omega^1_{\mathcal{E}/\mathcal{C}}) \cong L$.

The starting point for our collaboration was a remarkable application of Theorem 1.7 to the geography of surfaces due to the second-named author. We give some background before stating the result: There has been a great deal of interest in volumes of stable surfaces, i.e., in the set of values of K_X^2 where X runs through stable surfaces in the sense of Kollár, Shepherd-Barron, and Alexeev (KSBA). A key ingredient in the construction of the KSBA compactifications of moduli spaces of surfaces of general type (see [21]) is a descending chain condition on the set of K_X^2, namely that it admits no strictly decreasing sequences. The famous article [4] establishes this condition. (See also [5].) As this is a set of positive rational numbers, it must have a minimum, whose value is still unknown. Aleexev and Liu have recently found various other special properties around accumulation points and bounds [1–3,22]. For example, [22] shows that when the geometric genus is positive then the volume can be optimally bounded from below. This naturally raises the question: Are there *upper* bounds for K^2 of stable surfaces with a fixed geometric genus?

Recall [8, I.5.5, VI.1] that a smooth, projective, non-ruled surface X over a field of characteristic zero has $c_2(X) \geq 0$, so Noether's formula shows that $K_X^2 \leq 12(1 + p_g)$, i.e., the self-intersection of the canonical bundle K_X is bounded in terms of the geometric genus p_g. The question is whether such bounds continue to hold for mildly singular surfaces.

Any geometric approach to a negative answer to this question via surfaces with rational singularities requires a family of minimal resolution surfaces for any given $p_g > 0$ with an unbounded number of special rational curves. Our work supplies such families and moreover allows for good control on the singularities involved:

Theorem 1.9 *Given integers $g \geq 0$ and N, there exists a normal projective surface X over $\mathbb{C}$ with the following properties:*

(1) *X has geometric genus $p_g = g$.*
(2) *X has only one singular point, which is log-terminal.*
(3) *K_X is $\mathbb{Q}$-Cartier and ample.*
(4) *$K_X^2 > N$.*

1.10 Plan of the paper

In Sect. 2 we present foundational material on torsion points and intersections on elliptic surfaces, including a discussion of basic properties of our elliptic divisibility sequences. We then reformulate Theorems 1.1 and 1.4 as Theorem 2.5. We prove

Theorem 2.5 in Sect. 3. Section 4 discusses two moduli spaces which play a key role in the proof of Theorem 1.7. Section 5 discusses a construction of elliptic surfaces equipped with extra structure associated to elements in certain Riemann-Roch spaces. We then prove Theorem 1.7 in Sect. 6. In Sect. 7, we give an explicit construction of surfaces satisfying the requirements of Theorem 1.8. Finally, in Sect. 8, we prove Theorem 1.9.

2 Preliminaries on torsion and intersections

In this section we gather various foundational results on torsion, intersections, heights, and elliptic divisibility sequences. Some of this material also appears in [18], although our point of view is more geometric. Throughout, $\pi : \mathcal{E} \to \mathcal{C}$ will be a relatively minimal Jacobian elliptic surface over a field k with zero section O.

2.1 Multiplication by *n*

Let $\mathcal{E}^{sm}$ denote the locus where π is smooth (i.e., the complement of the singular points in the bad fibers). Then by [15, Prop. II.2.7], $\mathcal{E}^{sm}$ is a commutative group scheme over $\mathcal{C}$. Let n be an integer not divisible by the characteristic of k. Consider the homomorphism of group schemes given multiplication by n: $[n] : \mathcal{E}^{sm} \to \mathcal{E}^{sm}$.

Clearly, $[n]$ fixes the zero section O pointwise. If $x \in O$, the tangent space to $\mathcal{E}^{sm}$ at x splits canonically into the sum of two lines, namely the tangent space to O at x and the tangent space to the fiber of π through x. Since $[n]$ fixes O, $[n]$ acts as the identity on the former. A calculation in the formal group of $\mathcal{E}^{sm}$ [28, Ch. IV] shows that $[n]$ acts as multiplication by n on the tangent space to the fiber of π through x. It follows that $[n]$ is étale at every point of O, and since $[n]$ is a group scheme homomorphism it is étale everywhere.

The morphism $[n]$ is also quasi-finite: it has degree n^2 on the smooth geometric fibers of π, and degree dividing n^2 on all geometric fibers [28, Ch. III and §VII.6]. It is not in general finite if π has singular fibers.

If $P : \mathcal{C} \to \mathcal{E}$ is a section of π, P necessarily lands in the smooth locus $\mathcal{E}^{sm}$ and we may define a new section nP as the composition $[n] \circ P$. This is the meaning of the notation nP used in the introduction.

2.2 Torsion

With $\mathcal{E}$ as above and $n > 0$ and relatively prime to the characteristic of k, we define

$$\mathcal{E}[n] = [n]^{-1}(O),$$

i.e, $\mathcal{E}[n]$ is the inverse image of the zero section under $[n]$. Since $[n]$ is étale, $\mathcal{E}[n]$ is a reduced, closed subscheme of $\mathcal{E}^{sm}$ of dimension 1, and in particular, locally closed in $\mathcal{E}$. Since $[n]$ is quasi-finite, $\mathcal{E}[n]$ is étale and quasi-finite over $\mathcal{C}$ of generic degree n^2. It is in general not finite over $\mathcal{C}$.

The fiber of $\mathcal{E}[n]$ over a geometric point t of $\mathcal{C}$ consists of the points of $\pi^{-1}(t)$ with order divisible by n. We define

$$\mathcal{E}[n]' \subset \mathcal{E}[n]$$

to be the subscheme whose fiber over t consists of the points of $\pi^{-1}(t)$ of order exactly n. If m divides n, then $\mathcal{E}[m]'$ is a closed subscheme of $\mathcal{E}[n]$, and we have a disjoint union

$$\mathcal{E}[n] = \cup_{m|n}\mathcal{E}[m]' \tag{2.1}$$

where m runs over the positive divisors of n. Each $\mathcal{E}[m]'$ is a union of irreducible components of $\mathcal{E}[n]$ and is étale and quasi-finite over $\mathcal{C}$. Note that $\mathcal{E}[1] = \mathcal{E}[1]' = O$.

We refer to the unions of irreducible components of $\mathcal{E}[n]$ as "torsion multisections".

2.3 Divisibility sequences

In the introduction, we defined divisors D_n for $n \geq 1$ by

$$D_n = O^*(nP).$$

In this section we examine alternative definitions and properties of these divisors always assuming that n is relatively prime to the characteristic of k.

For two smooth curves C_1 and C_2 on $\mathcal{E}$ with no irreducible components in common, write $C_1 \cap C_2$ for the intersection zero-cycle. This is a zero-dimensional closed subscheme of $\mathcal{E}$. With this notation,

$$D_n = \pi_* \left(nP \cap O\right) = \pi_* \left(nP \cap \mathcal{E}[1]\right).$$

Note that nP meets $O = \mathcal{E}[1]$ over t if and only if P meets $\mathcal{E}[n]$ over t, and since $[n]$ is étale, the intersection multiplicity of nP and $\mathcal{E}[1]$ over t equals the intersection multiplicity of P and $\mathcal{E}[n]$ over t. In other words, we have

$$D_n = \pi_* \left(P \cap \mathcal{E}[n]\right). \tag{2.2}$$

Define

$$D_n' = \pi_* \left(P \cap \mathcal{E}[n]'\right).$$

Then the disjoint union (2.1) yields a decomposition of D_n into effective divisors:

$$D_n = \sum_{m|n} D_m' \tag{2.3}$$

where the sum is over positive divisors of n.

Note in particular that if t is a closed point of $\mathcal{C}$ and $P(t)$ is a torsion point, say of order exactly m, then t appears in D_n if and only if m divides n, and the multiplicity of t in such D_n is equal to the multiplicity of t in D'_m.

Remark 2.4 A section P can meet at most one torsion point over a given $t \in \mathcal{C}$. This implies that if $m_1 \neq m_2$, then D'_{m_1} and D'_{m_2} have disjoint support. In particular, as soon as $D'_n \neq 0$, D_n has "primitive divisors' i.e., points in its support which are not in the support of D_m for $m < n$. The existence of primitive divisors for all sufficiently large n is established in [18, §5] by showing that $D'_n \neq 0$ for all sufficiently large n. The key idea is an estimation of intersection numbers using heights as in Sect. 2.7 below. Another simple proof of the existence of primitive divisors (suggested by a referee) can be given using the fact that the "Betti coordinates" (as in Sect. 3) of a non-torsion section P give rise to a locally defined, open map from $\mathcal{C}$ to $\mathbb{R}^2$. The existence and openness of this map can also be viewed as the key point in the simplest case (I_0) of the proof of Theorem 2.5.

We now state a result which implies Theorems 1.1 and 1.4:

Theorem 2.5 *Let $\mathcal{E} \to \mathcal{C}$ be a relatively minimal Jacobian elliptic surface over the complex numbers $\mathbb{C}$, with zero section O and another section P which is not torsion. Then the set*

$$T_{tor} := \bigcup_{n \neq 0} \{t \in \mathcal{C} \mid P \text{ is tangent to } \mathcal{E}[n] \text{ over } t\}$$

is finite.

2.6 Proof that Theorem 2.5 implies Theorems 1.1 and 1.4

First we note that the general cases of Theorems 1.1 and 1.4 follow from the case $k = \mathbb{C}$. Indeed, since the hypotheses and conclusion of Theorems 1.1 and 1.4 are insensitive to the ground field, we may replace k with a subfield k' which is finitely generated over $\mathbb{Q}$ (take the field generated by the coefficients defining $\mathcal{C}$, $\mathcal{E}$, π, and P), then embed k' in $\mathbb{C}$. Thus it suffices to treat the case $k = \mathbb{C}$.

Next, by the definition of D_n, to say that nP is tangent to O over t is to say that t appears in D_n with multiplicity greater than 1. By the equality (2.2), to say that t appears in D_n with multiplicity greater than 1 is to say that P is tangent to $\mathcal{E}[n]$ over t. Thus the set T_{tor} of Theorem 2.5 is equal to the set T of Theorem 1.1, and Theorem 2.5 is equivalent to the case $k = \mathbb{C}$ of Theorem 1.1.

To finish, we show that Theorems 1.1 and 1.4 are equivalent. First note that points (1) and (2) of Theorem 1.4 are equivalent by the definition of D_n. Moreover, D'_m is non-reduced if and only if D_n is non-reduced for all multiples n of m. Thus point (3) of Theorem 1.4 implies points (1) and (2), and Theorem 1.4 is equivalent to the statement that the set of m such that D'_m is non-reduced is finite.

Now consider the "incidence correspondence"

$$I := \left\{(t, m) \,\middle|\, m > 0 \text{ and } P \text{ is tangent to } \mathcal{E}[m]' \text{ over } t\right\} \subset \mathcal{C} \times \mathbb{Z}_{>0}.$$

The set T of Theorem 1.1 is the image of the projection $I \to \mathcal{C}$ and the set M of Theorem 1.4 is the image of the projection $I \to \mathbb{Z}_{>0}$. The fibers of $I \to \mathcal{C}$ are finite (and in fact empty or singletons) because P meets $\mathcal{E}[m]'$ for at most one value of m and *a fortiori* can be tangent to at most one $\mathcal{E}[m]'$. The fibers of $I \to \mathbb{Z}_{>0}$ are finite because for a fixed m, P meets $\mathcal{E}[m]'$ at only finitely many points, so *a fortiori* can be tangent to $\mathcal{E}[m]'$ at only finitely many points. This establishes that Theorem 1.1 and Theorem 1.4 are equivalent, and it completes the proof that Theorem 2.5 implies Theorems 1.1 and 1.4 □

We will prove Theorem 2.5 in Sect. 3. First, we review material on heights used later in the paper.

2.7 Heights

We refer to [13] or [25] or [26] or [31] for the basic assertions on heights in this section. As usual, $\pi : \mathcal{E} \to \mathcal{C}$ is a relatively minimal Jacobian elliptic surface over a field k.

Given a section P of π, there is a unique $\mathbb{Q}$-divisor C_P supported on the non-identity components of the fibers of π with the property that $P - O + C_P$ has zero intersection multiplicity with every irreducible component of every fiber of π. There is a simple recipe for C_P that depends only on the components of the reducible fibers met by P, and in particular, for a fixed $\mathcal{E}$, there are only finitely many possibilities for C_P as P varies over all sections. If π has irreducible fibers, or more generally, if P passes through the identity component of every fiber, then $C_P = 0$.

There is a canonical $\mathbb{Q}$-valued symmetric bilinear form on the group of sections of $\mathcal{E}$ defined by

$$\langle P, Q\rangle := -(P - O + C_P).(Q - O) \tag{2.4}$$

where the dot refers to the intersection number on $\mathcal{E}$. If $\mathcal{E} \to \mathcal{C}$ is non-constant (i.e., is not isomorphic over $\overline{k}$ to a product $E_0 \times \mathcal{C}$), then this pairing is non-degenerate modulo torsion and positive definite. We define $ht(P) = \langle P, P\rangle$. (Note that this is twice the height considered in [18].)

Lemma 2.8 *For $\mathcal{E}$ and P as above,*

$$(nP).O = \frac{ht(P)}{2}n^2 + O(1).$$

If π has irreducible fibers and $d = \deg O^*(\Omega^1_{\mathcal{E}/\mathcal{C}})$, then

$$(nP).O = \frac{ht(P)}{2}n^2 - d.$$

In the first display, $O(1)$ depends only on $\mathcal{E}$ and P, not on n.

Proof It follows from the canonical bundle formula for elliptic surfaces, adjunction, and the definition of d that every section P of π satisfies $P^2 = -d$. From the height formula, we find

$$n^2 ht(P) = ht(nP) = -(nP - O + C_{nP}).(nP - O),$$

and so

$$(nP).O = \frac{ht(P)}{2} n^2 - d + C_{nP}.(nP - O) = \frac{ht(P)}{2} n^2 + O(1).$$

If π has irreducible fibers, then $C_Q = 0$ for all Q and we deduce the stated exact formula. □

In the following lemma we use square brackets to indicate the class of a curve in $\mathrm{NS}(\mathcal{E})$, the Néron-Severi group of $\mathcal{E}$. This allows us to distinguish between $n[P]$ (n times the class of P) and $[nP]$ (the class of nP).

Lemma 2.9 *Suppose that $\pi : \mathcal{E} \to \mathcal{C}$ has irreducible fibers, $d = \deg O^*(\Omega^1_{\mathcal{E}/\mathcal{C}})$, and P is a section of π which does not meet O. Let F be a fiber of π. Then we have an equality*

$$[nP] = n[P] + (1-n)[O] + d(n^2 - n)[F]$$

in $\mathrm{NS}(\mathcal{E})$.

Proof We have an equality $[nP] - [O] = n([P] - [O])$ in the Picard group of the generic fiber of $\mathcal{E}$, so there is an equality of the form

$$[nP] = n[P] + (1-n)[O] + c[F]$$

in $\mathrm{NS}(\mathcal{E})$, and we just need to determine the coefficient of $[F]$. We do this by intersecting with $[O]$. By assumption $[P].[O] = 0$, so $ht(P) = 2d$. By the previous lemma, $[nP].[O] = d(n^2 - 1)$ and solving for c yields $c = d(n^2 - n)$. □

3 Proof of Theorem 2.5

We first note that Theorem 2.5 is a statement about intersections on an elliptic surface over the complex numbers. To prove it, we may replace $\mathcal{E}$ and $\mathcal{C}$ with the corresponding complex manifolds and make use of the classical topology, i.e., the topology induced by the metric topology on $\mathbb{C}$. For the rest of this section, we make this replacement, although we will not change the notation.

Our strategy will be to show that the subset $T_{tor} \subset \mathcal{C}$ is closed and discrete (in the classical topology). This implies Theorem 2.5 since $\mathcal{C}$ is compact. We note in passing that the set of points of intersection of P and $\mathcal{E}[n]$ for varying n is usually everywhere classically dense in P, so the discreteness that lies at the heart of the theorem is not evident.

In fact, will consider a more general set of tangencies and prove that a certain subset $T_{Betti} \subset \mathcal{C}$ is closed and discrete (and thus finite) and contains all points of T_{tor} over which $\mathcal{E}$ has good reduction. Since the set of points of bad reduction is finite, this will establish that T_{tor} is also finite.

We will establish the desired discreteness by using the complex analytic description of $\mathcal{E} \to \mathcal{C}$ given by Kodaira in [20, §8]. Let $\mathcal{C}^0 \subset \mathcal{C}$ be the maximal open subset over which $\mathcal{E}$ has good reduction, and let $\mathcal{E}^0 = \pi^{-1}(\mathcal{C}^0)$.

Let $t \in \mathcal{C}^0$. Write $\mathcal{H}$ for the upper half plane. Then there is a neighborhood U of t biholomorphic to a disk Δ and holomorphic functions $\tau : \Delta \to \mathcal{H}$ and $w : \Delta \to \mathbb{C}$ such that $\pi^{-1}(U) \to U$ sits in a diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d] & (\Delta \times \mathbb{C})/(\mathbb{Z}\tau + \mathbb{Z}) \arrow[d] \\ U \arrow[r] \arrow[u, bend left, "P_{|U}"] & \Delta \arrow[u, bend right, "{[w]}"'] \end{tikzcd} \tag{3.1}$$

where the horizontal maps are biholomorphic, and $(\Delta \times \mathbb{C})/(\mathbb{Z}\tau + \mathbb{Z})$ means the quotient of $\Delta \times \mathbb{C}$ by $\mathbb{Z}^2$ acting as

$$(a, b)(z, w) = (z, w + a\tau(z) + b)\,.$$

For $z \in \Delta$, corresponding to $u \in U$, $P(u)$ corresponds to $[w](z)$, which is the class of $w(z)$ in $\{z\} \times \mathbb{C}/(\mathbb{Z}\tau(z) + \mathbb{Z})$. We also assume $t \in U$ corresponds to $0 \in \Delta$.

Next, we consider a trivialization of $\pi^{-1}(U) \to U$ as a real analytic manifold. Introduce real coordinates as follows: $z = x + iy$ on the base Δ, $\tau = \rho + i\sigma$ on the upper half plane $\mathcal{H}$, and $w = u + iv$ in the $\mathbb{C}$ which uniformizes the fibers of $\pi^{-1}(U) \to U$. Let (r, s) be coordinates on $\mathbb{R}^2$, and note that $w = r\tau + s$ if and only if $r = v/\sigma$ and $s = (u\sigma - v\rho)/\sigma$.

Consider the diagram

$$\begin{tikzcd} \Delta \times \mathbb{C} \arrow[rr] \arrow[d] & & \Delta \times \mathbb{R}^2 \arrow[d] \\ (\Delta \times \mathbb{C})/(\mathbb{Z}\tau + \mathbb{Z}) \arrow[rr] \arrow[dr] & & \Delta \times (\mathbb{R}/\mathbb{Z})^2 \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{3.2}$$

where the upper horizontal map is

$$(z, w) = (x + iy, u + iv) \mapsto (z, r, s) = \left(z, \frac{v}{\sigma}, \frac{u\sigma - v\rho}{\sigma}\right),$$

with inverse

$$(z, w) = (z, r\tau(z) + s) \leftarrow\!\shortmid (z, r, s).$$

The two vertical maps are the natural quotients, the middle horizontal map is induced by the upper horizontal map, and the diagonal maps are the projections to the first factor.

The top horizontal map is a real-analytic isomorphism which is $\mathbb{R}$-linear on each fiber of the projection to Δ. The choice of this map is motivated by the fact that torsion sections of $\mathcal{E}$ over U correspond the surfaces $\Delta \times (r, s) \subset \Delta \times (\mathbb{R}/\mathbb{Z})^2$ where r and s are rational numbers. In other words, we have changed coordinates so that every torsion section becomes a constant section. It will be of interest to consider all the horizontal sections $\Delta \times (r, s)$ for arbitrary real numbers r and s.

Definition 3.1 With notation as above and $(r, s) \in \mathbb{R}^2$, define the *local Betti leaf* $\mathcal{L}_{r,s} \subset \pi^{-1}(U)$ as the image of the map $U \to (\Delta \times \mathbb{C})/(\mathbb{Z}\tau + \mathbb{Z}) \cong \pi^{-1}(U)$ sending z to the class of $(z, r\tau(z) + s)$.

This terminology is inspired by [12], where r and s are called "Betti coordinates". Clearly the assignment $(r, s) \mapsto \mathcal{L}_{r,s}$ factors through $(\mathbb{R}/\mathbb{Z})^2$. Each $\mathcal{L}_{r,s}$ is a closed holomorphic submanifold of $\pi^{-1}(U)$, and the set of $\mathcal{L}_{r,s}$ as (r, s) runs through $(\mathbb{R}/\mathbb{Z})^2$ is a foliation of $\pi^{-1}(U)$. Although the indexing of $\mathcal{L}_{r,s}$ by (r, s) depends on the choice of period map τ, the submanifolds $\mathcal{L}_{r,s}$ themselves are intrinsic (i.e., independent of τ). There is a corresponding global foliation of $\mathcal{E}^0$ which we will not consider in this paper, except implicitly in the following remark: If for some non-empty open $U \subset \mathcal{C}^0$ and some (r, s), $P(U) = \mathcal{L}_{r,s}$ (i.e., P lands in a leaf of the local foliation), then by analytic continuation, the same holds over every open. In this case, we say "P lies in the Betti foliation".

Note that if $(r, s) \in \mathbb{Q}^2$, then each point of $\mathcal{L}_{r,s}$ is a torsion point in its fiber. More precisely, if n is the smallest positive integer such that $(nr, ns) \in \mathbb{Z}^2$, then $\mathcal{L}_{r,s}$ is a connected component of $\mathcal{E}[n]' \cap \pi^{-1}(U)$. On the other hand, if $(r, s) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, then $\mathcal{L}_{r,s}$ is disjoint from every $\mathcal{E}[n]$. Thus, if P lies in the Betti foliation, it is a torsion section (which we have ruled out by hypothesis) or it meets no torsion sections over $\mathcal{C}^0$ and so is not tangent to any torsion section over $\mathcal{C}^0$. In the latter case, Theorem 2.5 is obviously true, so we may assume from now on that P does not lie in the Betti foliation.

We now define

$$T_{Betti} := \left\{ t \in \mathcal{C}^0 | P \text{ is tangent to some } \mathcal{L}_{r,s} \text{ over } t \right\}.$$

The preceding paragraph shows that $T_{tor} \cap \mathcal{C}^0 \subset T_{Betti}$ and so, as explained above, to prove Theorem 2.5 it will suffice to prove that T_{Betti} is a closed, discrete subset of $\mathcal{C}$. More formally:

Claim 3.2 *For every $t \in \mathcal{C}$, there is a classical open neighborhood U_t of t in $\mathcal{C}$ such that $(U_t \setminus \{t\}) \cap T_{Betti} = \emptyset$. In other words, for every $(r, s) \in \mathbb{R}^2$, P is not tangent to $\mathcal{L}_{r,s}$ over $U_t \setminus \{t\}$.*

To establish Claim 3.2, we will consider cases according to the reduction type of $\mathcal{E}$ at t. We use the standard Kodaira notation $(I_n, I_n^*, \ldots)$ to index the cases.

The case of I_0 reduction: Using diagrams (3.1) and (3.2), we identify the section P over U with the graph of a function $\phi : \Delta \to \mathbb{R}^2$. Write $(r_0, s_0) = \phi(0)$ for the image of P over t. Since P is assumed not to be contained in $\mathcal{L}_{r_0,s_0}$, we may shrink U so that P meets $\mathcal{L}_{r_0,s_0}$ only over t, in other words, so that the only value of z with $\phi(z) = (r_0, s_0)$ is $z = 0$.

It is clear that P is tangent to some $\mathcal{L}_{r,s}$ over t' if and only if the derivative of $(x, y) \to (r, s)$ (as a map of 2-manifolds) vanishes at the z corresponding to t'.

To finish, we claim that after possibly shrinking U and Δ, the derivative of ϕ does not vanish away from $0 \in \Delta$. To see this, apply the Lojasiewicz gradient inequality ([10,24]) to the components of (ϕ_1, ϕ_2) of ϕ: That result says that after shrinking Δ, there are constants $C > 0$ and $0 < \theta < 1$ such that

$$|\nabla\phi_i(z)| \geq C\,|\phi_i(z) - \phi_i(0)|^\theta$$

for all $z \in \Delta$. But if $z \in \Delta \setminus \{0\}$, $\phi(z) \neq \phi(0)$ so one of the $\phi_i(z) \neq \phi_i(0)$ which implies that $\nabla\phi_i(z) \neq 0$ and so the derivative of ϕ is also non-zero.

This establishes Claim 3.2 at points of good reduction: if t is such a point, there is an open neighborhood U_t of t in $\mathcal{C}$ such that P is not tangent to any $\mathcal{L}_{r,s}$ over any $t' \in U_t \setminus \{t\}$. To finish the proof, we deal with tangencies near places of bad reduction.

The case of I_1 reduction: We next consider the case of multiplicative reduction with an irreducible special fiber. I.e., assume that $\mathcal{E}$ has reduction type I_1 over $t \in \mathcal{C}$. Let $\Delta \to \mathcal{C}$ be a holomorphic parameterization of a neighborhood of t, where Δ is the unit disk and $0 \in \Delta$ maps to t. Again, over the course of the proof we will reduce the radius of Δ but not change the notation. Let $\mathcal{X} \to \Delta$ be the pull-back of $\mathcal{E} \to \mathcal{C}$ to Δ, let $\Delta' = \Delta \setminus \{0\}$, and let $\mathcal{X}' \to \Delta'$ be the restriction of $\mathcal{X} \to \Delta$ to Δ'. Note that the special fiber

$$\mathcal{X} \setminus \mathcal{X}' = \text{nodal cubic} \cong \mathbb{C}^\times \cup \{q\}$$

where q is the node of the cubic.

According to Kodaira [20, pp. 596ff], we may shrink Δ and choose $\Delta \to \mathcal{C}$ so that $\mathcal{X}'$ has the form

$$\mathcal{X}' \cong \left(\Delta' \times \mathbb{C}^\times\right)/\mathbb{Z}$$

where the action of $\mathbb{Z}$ on $\Delta' \times \mathbb{C}^\times$ is

$$m \cdot (z, w) = (z, z^m w).$$

Moreover, as explained starting in the last paragraph of [20, p. 597], there is a holomorphic map

$$\phi : \Delta \times \mathbb{C}^\times \to \mathcal{X}$$

such that $\{0\} \times \mathbb{C}^\times$ maps biholomorphically to the complement of q in the special fiber, and $\Delta' \times \mathbb{C}^\times \to \mathcal{X}' \subset \mathcal{X}$ is the natural quotient map. We may thus identify the section P with a holomorphic map $f : \Delta \to \mathbb{C}^\times$.

Now let $V \subset \Delta'$ be a non-empty, connected and simply connected subset, and choose a branch of the logarithm $\log : V \to \mathbb{C}$. Then the local Betti leaf $\mathcal{L}_{r,s}$ over V (computed with respect to the period $\tau(z) = (1/2\pi i)\log z$) is the image of the map $V \to \mathcal{X}'$ which sends z to the class of $(z, e^{r\log z}e^{2\pi i s})$. The local Betti leaf $\mathcal{L}_{r,s}$ through (z, w) has

$$r = \frac{\log|w|}{\log|z|} \qquad \text{and} \qquad s = \frac{1}{2\pi i}\log\left(\frac{w}{|w|}\right).$$

(Note that the logarithms appearing in the expression for r are evaluated at real numbers, so we use the standard real logarithm, and the class of s in $\mathbb{R}/\mathbb{Z}$ is independent of the choice of logarithm.)

Then we calculate that P is tangent to $\mathcal{L}_{r,s}$ at $z \in V$ if and only if

$$f'(z) = \left(z \mapsto e^{r\log z}e^{2\pi i s}\right)'(z) = \frac{f(z)}{z}\frac{\log|f(z)|}{\log|z|}. \tag{3.3}$$

Note that the expression on the right is well defined independently of the choice of V and the logarithm.

Now we assume that there is a sequence of tangencies accumulating at t and derive a contradiction. More precisely, assume that there is a sequence $z_i \in \Delta'$ tending to 0 such that for all i,

$$f'(z_i) = \frac{f(z_i)}{z_i}\frac{\log|f(z_i)|}{\log|z_i|}. \tag{3.4}$$

We will show that there is no holomorphic function satisfying such equalities.

If f' is identically zero, then f is a non-zero constant. Equation (3.3) shows that the constant value of f must have absolute value 1, so $f(z) = e^{2\pi i s}$ for some real s, and we find that P lies in a local Betti leaf $\mathcal{L}_{0,s}$, in contradiction to our assumption.

Now assume that f' is not identically zero, so f takes its value at $z = 0$ to finite order $N = \mathrm{ord}_{z=0}(f(z) - f(0)) \geq 1$. Then $g(z) = zf'(z)/f(z)$ is holomorphic on Δ, and Eq. (3.4) says that $g(z_i) = (\log|f(z_i)|)/(\log|z_i|)$ for all i. Shrinking Δ if necessary, we have estimates

$$B_1|z|^N < |g(z)| < B_2|z|^N \tag{3.5}$$

for some positive constants B_1 and B_2 and all $z \in \Delta'$. Shrinking Δ again if necessary, we may write $f(z) = w_0(1 + h(z))$ with

$$C_1|z|^N < |h(z)| < C_2|z|^N$$

for some positive constants C_1 and C_2 and all $z \in \Delta'$. We have

$$\log|1 + h(z)| \le \log(1 + |h(z)|) \le |h(z)| < C_2|z|^N$$

for $z \in \Delta'$. If $|w_0| = 1$ (so $\log|w_0| = 0$), we have

$$\left|\frac{\log|f(z)|}{\log|z|}\right| = \left|\frac{\log|1 + h(z)|}{\log|z|}\right| \le \left|\frac{C_2|z|^N}{\log|z|}\right|.$$

Taking z_i close to zero and noting that $g(z_i) = (\log|f(z_i)|)/(\log|z_i|)$ we get a contradiction to the lower bound in Equation (3.5).

To finish, assume that $|w_0| \ne 1$. Then

$$\left|\frac{\log|f(z)|}{\log|z|}\right| \ge \left|\frac{\log|w_0|}{\log|z|}\right| - \left|\frac{\log|1 + h(z)|}{\log|z|}\right| \ge \left|\frac{\log|w_0|}{\log|z|}\right| - \left|\frac{C_2|z|^N}{\log|z|}\right|.$$

Taking z_i close to zero and noting that $g(z_i) = (\log|f(z_i)|)/(\log|z_i|)$ we get a contradiction to the upper bound in Equation (3.5).

This establishes that there is no accumulation of tangencies between P and local Betti leaves at t when $\mathcal{E}$ has reduction of type I_1 at t.

The case of I_b reduction: Now consider the case of multiplicative reduction of type I_b over $t \in \mathcal{C}$. This case is very similar to the I_1 case, with some notational complications.

Let $\Delta \to \mathcal{C}$ be a holomorphic parameterization of a neighborhood of t, where Δ is the unit disk and $0 \in \Delta$ maps to t. Again, over the course of the proof we will reduce the radius of Δ but not change the notation. Let $\mathcal{X} \to \Delta$ be the pull-back of $\mathcal{E} \to \mathcal{C}$ to Δ, let $\Delta' = \Delta \setminus \{0\}$, and let $\mathcal{X}' \to \Delta'$ be the restriction of $\mathcal{X} \to \Delta$ to Δ'. Then the special fiber $\mathcal{X} \setminus \mathcal{X}'$ has the form

$$\mathcal{X} \setminus \mathcal{X}' = \text{chain of } b \text{ copies of } \mathbb{P}^1 \cong \bigcup_{i \in \mathbb{Z}/b\mathbb{Z}} \mathbb{C}_i^\times \cup \{q_i\}$$

where the q_i are the nodes of the chain. Let

$$\mathcal{X}^{sm} = \mathcal{X} \setminus \{q_1, \ldots, q_b\}$$

be the smooth locus of $\mathcal{X} \to \Delta$.

Kodaira [20, pp. 599ff], gives a covering of $\mathcal{X}^{sm}$ by b open sets as follows: for $i \in \mathbb{Z}/b\mathbb{Z}$, let

$$W_i = W_i' \cup \mathbb{C}_i^\times, \qquad W_i' = \left(\Delta' \times \mathbb{C}^\times\right)/\mathbb{Z}$$

where the action of $\mathbb{Z}$ on $\Delta' \times \mathbb{C}^\times$ is

$$m \cdot (z, w) = (z, z^{bm} w).$$

For $z \in \Delta'$ and $w \in \mathbb{C}^\times$, write $(z, w)_i$ for the class of (z, w) in W_i'. Then $\mathcal{X}^{sm}$ is obtained by glueing the W_i according to the rule

$$(z, w)_i = (z, z^{j-i} w)_j$$

for all $z \in \Delta'$, $w \in \mathbb{C}^\times$, and $i, j \in \mathbb{Z}/b\mathbb{Z}$. Thus $\mathcal{X}_z$, the fiber of $\mathcal{X}^{sm} \to \Delta$ over $z \neq 0$, is the elliptic curve $\mathbb{C}^\times / z^{b\mathbb{Z}}$ and the fiber over $z = 0$ is a disjoint union of b copies of $\mathbb{C}^\times$, one appearing in each open set W_i.

Now assume that the section P meets the special fiber at $w_0 \in \mathbb{C}_i^\times$. Then we may choose a small disk D around w_0 in $\mathbb{C}_i^\times$ and shrink Δ so that the image of

$$\Delta \times D \hookrightarrow W_i \hookrightarrow \mathcal{X}$$

contains the image of P over Δ. We may then identify P with a function $f : \Delta \to D$, and the conditions on f for P to be tangent to a local Betti leaf are the same as they are in the I_1 case. Thus the rest of the argument is essentially identical to that in the I_1 case, and we will omit the rest of the details.

The case of I_b^* reduction: Now consider the case where $\mathcal{E}$ has reduction of type I_b^* at t. Choose as usual a parameterization $\Delta \to \mathcal{C}$ of a neighborhood of t and let $\mathcal{X} \to \Delta$ be the pull-back of $\mathcal{E} \to \mathcal{C}$. Let $\tilde{\Delta} \to \Delta$ be a double cover ramified at $0 \in \tilde{\Delta}$ and let $\tilde{\Delta}' = \tilde{\Delta} \setminus \{0\}$, so that $\tilde{\Delta}' \to \Delta'$ is an unramified double cover. Then it is well known that $\tilde{\mathcal{X}}'$, the pull-back of $\mathcal{X} \to \Delta$ to $\tilde{\Delta}'$ has an extension to $\tilde{\mathcal{X}} \to \tilde{\Delta}$ whose fiber over 0 is of type I_{2b}. Moreover, the section P of $\mathcal{X} \to \Delta$ induces a section $\tilde{P}$ of $\tilde{\mathcal{X}} \to \tilde{\Delta}$. We apply the argument of the previous section to conclude that after shrinking $\tilde{\Delta}$, there are no points of $\tilde{\Delta}'$ over which $\tilde{P}$ is tangent to a local Betti leaf. Since $\tilde{\mathcal{X}}' \to \mathcal{X}'$ is étale, the same must be true after shrinking Δ, i.e., P is not tangent to a local Betti leaf over Δ'. (It is clear from the definition that local Betti leaves are preserved under an étale base change.) This proves the desired discreteness near a point where $\mathcal{E}$ has I_b^* reduction.

The cases of II, II^*, III, III^*, IV, and IV^* reduction: Finally, consider the cases where $\mathcal{E}$ has additive and potentially good reduction. Then by an argument parallel to that of the previous case, we may focus attention on a disk Δ near t, pull-back to a ramified cover $\tilde{\Delta} \to \Delta$ of order 2, 3, 4, or 6, and reduce to the case of good reduction. We leave the details as an exercise for the reader.

This completes the proof that the set of points $t \in \mathcal{C}^0$ over which P is tangent to a local Betti leaf is discrete and therefore finite, and it concludes the proof of Theorem 2.5. □

Remark 3.3 This proof gives no bounds on the cardinality of T_{tor} or T_{Betti}, i.e., on the number tangencies. In [32], we give an explicit upper bound on the number of tangencies. In fact, by exploiting the global Betti foliation, we give an *exact formula* for tangencies counted with multipicities and taking into account the behavior of P at the bad fibers. The resulting upper bound depends only on topological properties of $\mathcal{E}$.

Remark 3.4 A referee points out that case I_0 of the argment above also follows directly from Lemma 6.4 in [9].

4 Interlude on moduli of elliptic curves with a differential and a point

In this section, we discuss certain moduli spaces of elliptic curves with additional structure. These spaces will be useful when we consider families of elliptic surfaces in the following section. We work in more generality than needed in this paper, and readers who are so inclined may replace the base ring R below with a field k of characteristic $\neq 2, 3$ or even with $\mathbb{C}$.

We begin by noting that there is a standard model for an elliptic curve E equipped with a non-zero differential ω and a non-trivial point P: Given the data, choose a Weierstrass model of E

$$y'^2 + a_1'x'y' + a_3'y' = x'^3 + a_2'x'^2 + a_4'x' + a_6'$$

such that $\omega = dx'/(2y' + a_1'x' + a_3')$. Then there is a unique change of coordinates $x' = x + r$, $y' = y + sx + t$ such that P has coordinates $(x, y) = (0, 0)$ and $a_1 = 0$. Thus there is a unique triple (a_2, a_3, a_4) such that E is the elliptic curve defined by

$$y^2 + a_3y = x^3 + a_2x^2 + a_4x,$$

the differential is $\omega = dx/(2y + a_3)$, and the point is $P = (0, 0)$.

We want to formalize this observation. Following Deligne [14], we say that a *curve of genus 1* over a base scheme S is a proper, flat, finitely presented morphism

$$\pi : \mathcal{W} \to S$$

whose geometric fibers are reduced and irreducible curves of arithmetic genus 1 equipped with a section $O : S \to \mathcal{W}$ whose image is contained in the locus where π is smooth.

Let $R = \mathbb{Z}[1/6]$ and consider the stack $\mathcal{M}$ over Spec R whose value on an R-scheme S is the set of triples $(\mathcal{W} \to S, \omega, P)$ where $\mathcal{W} \to S$ is a curve of genus 1 over S as defined above, ω is a nowhere vanishing section of $O^*(\Omega^1_{\mathcal{W}/S})$, and $P : S \to \mathcal{W}$ is a section disjoint from O. Two such triples $(\mathcal{W} \to S, \omega, P)$ and $(\mathcal{W}' \to S, \omega', P')$ are isomorphic if there exists an S-isomorphism $\mathcal{W} \to \mathcal{W}'$ carrying ω to ω' and P to P'.

Proposition 4.1 *The stack $\mathcal{M}$ is represented by the affine scheme* Spec $R[a_2, a_3, a_4]$. *The universal object over $\mathcal{M}$ is the projective family of plane cubics $\mathcal{W} \to$* Spec $R[a_2, a_3, a_4]$ *defined by*

$$y^2 + a_3y = x^3 + a_2x^2 + a_4x$$

equipped with the differential $\omega = dx/(2y+a_3)$ and the section P given by $x = y = 0$. The substack of $\mathcal{M}$ where the curve $\mathcal{W} \to S$ has smooth fibers is represented by the open subscheme where $\Delta \neq 0$ and the substack where the fibers of $\mathcal{W} \to S$ are either smooth or nodal is represented by the open subscheme where either $\Delta \neq 0$ or $2^4a_2^2 - 2^43a_4 \neq 0$.

More formally, "the projective family of plane cubics defined by $y^2 + a_3y = x^3 + a_2x^2 + a_4x$" is defined as follows: Let $\mathcal{R}$ be the graded $R[a_2, a_3, a_4]$-algebra

$$\mathcal{R} = R[a_2, a_3, a_4][x, y, z]/(y^2z + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2)$$

where x, y, and z have weight 1. Then $\mathcal{W} = \mathrm{Proj}_{\mathrm{Spec}\, R[a_2,a_3,a_4]}(\mathcal{R})$.

Here and later in the paper, whenever we have elements a_2, a_3, a_4 in some ring, we set

$$\begin{aligned}
c_4(a_2, a_3, a_4) &= 16a_2^2 - 48a_4 \\
&= 2^4a_2^2 - 2^43a_4 \\
c_6(a_2, a_3, a_4) &= 288a_2a_4 - 64a_2^3 - 216a_3^2 \\
&= 2^53^2a_2a_4 - 2^6a_2^3 - 2^33^3a_3^2 \\
\Delta(a_2, a_3, a_4) &= -16a_2^3a_3^2 + 16a_2^2a_4^2 + 72a_2a_3^2a_4 - 27a_3^4 - 64a_4^3 \\
&= -2^4a_2^3a_3^2 + 2^4a_2^2a_4^2 + 2^33^2a_2a_3^2a_4 - 3^3a_3^4 - 2^6a_4^3.
\end{aligned}$$

We often omit the a_i and simply write c_4, c_6, or Δ. However, in the proof just below, we do not omit the a_i, i.e., we distinguish between the elements c_4, c_6 generating a two-variable polynomial ring $R[c_4, c_6]$ and the elements $c_4(a_2, a_3, a_4)$ and $c_6(a_2, a_4, a_6)$ in the ring $R[a_2, a_3, a_4]$.

Proof of Proposition 4.1 By [14, Prop. 2.5], the stack of pairs $(\mathcal{W} \to S, \omega)$ as above is represented by the affine scheme $\mathrm{Spec}\, R[c_4, c_6]$ with universal curve

$$y'^2 = x'^3 - \frac{c_4}{2^43}x' - \frac{c_6}{2^53^3}$$

and universal differential $dx'/2y'$. (Deligne uses the more traditional coordinates $g_2 = c_4/(2^23)$ and $g_3 = c_6/(2^33^3)$, but this is immaterial since $1/6 \in R$.) Define a morphism $\mathrm{Spec}\, R[a_2, a_3, a_4] \to \mathrm{Spec}\, R[c_4, c_6]$ by sending

$$c_4 \mapsto c_4(a_2, a_3, a_4) \quad \text{and} \quad c_6 \mapsto c_6(a_2, a_3, a_4).$$

Then pulling back the universal curve over $\mathrm{Spec}\, R[c_4, c_6]$ to $\mathrm{Spec}\, R[a_2, a_3, a_4]$ and making the change of coordinates $x' = x + a_2/3$, $y' = y + a_3/2$ yields the curve and differential mentioned in the statement of the theorem.

To finish the proof, one checks that the fibers of $\mathrm{Spec}\, R[a_2, a_3, a_4] \to \mathrm{Spec}\, R[c_4, c_6]$ are the *affine* plane curves

$$y'^2 = x'^3 - \frac{c_4}{2^43}x' - \frac{c_6}{2^53^3},$$

i.e., Spec $R[a_2, a_3, a_4]$ is the universal curve over Spec $R[c_4, c_6]$ minus its zero section. Indeed, the fiber over (c_4, c_6) is

$$c_4 = 16a_2^2 - 48a_4$$
$$c_6 = 288a_2a_4 - 64a_2^3 - 216a_3^2.$$

Eliminating a_4 and dividing by $2^5 3^3$, we find

$$\frac{a_3^2}{2^2} = \frac{a_2^3}{3^3} - \frac{c_4a_2}{2^4 3^2} - \frac{c_6}{2^5 3^3}.$$

Thus setting $a_3 = 2y'$ and $a_2 = 3x'$ yields the stated fiber.

This means that to give a morphism to Spec $R[a_2, a_3, a_4]$ is to give a morphism to Spec $R[c_4, c_6]$ (i.e., a family of curves and a differential) together with a non-zero point in each fiber. This completes the proof that Spec $R[a_2, a_3, a_4]$ represents $\mathcal{M}$.

The assertions about the locus where $\mathcal{W}$ has good or nodal fibers follows from [14, Prop. 5.1], and this completes the proof of the proposition. □

In light of the proposition, from now we change notation and let $\mathcal{M}$ be defined as the scheme Spec $R[a_2, a_3, a_4]$. Also, we write $\mathcal{M}^{sm}$ for the locus where $\Delta \neq 0$ and $\mathcal{M}^n$ for the locus where $\Delta = 0$ and $c_4 \neq 0$. Similarly, let $\mathcal{N} =$ Spec $R[c_4, c_6]$, $\mathcal{N}^{sm}$ the locus where $c_4^3 - c_6^2 \neq 0$, and $\mathcal{N}^n$ the locus where $c_4^3 - c_6^2 = 0$ and $c_4 \neq 0$.

4.2 Torsion

Let $\pi : \mathcal{W} \to \mathcal{M}$ be the universal curve. Then the smooth locus of π is a commutative group scheme over $\mathcal{M}$ and we may speak of points of finite order in the fibers. For each $n > 1$, let $\mathcal{M}[n]$ be the locus where P has order dividing n, let $\mathcal{M}[n]'$ be the locus where P has order exactly n, and let $\mathcal{M}^{sm}[n] = \mathcal{M}^{sm} \cap \mathcal{M}[n]$ and $\mathcal{M}^{sm}[n]' = \mathcal{M}^{sm} \cap \mathcal{M}[n]'$.

Let $n > 1$ and let k be a field of characteristic zero or prime to $6n$. For R-schemes, write $- \otimes k$ for the base change along the unique morphism Spec $k \to$ Spec R. Then it follows from [15, I.6 and II.1.18-20] that $\mathcal{M}[n] \otimes k$ is locally closed in $\mathcal{M} \otimes k$, everywhere regular and of codimension 1, and that $\mathcal{M}^{sm}[n] \otimes k$ is a divisor in $\mathcal{M}^{sm}$ which is étale and finite of degree n^2 over $\mathcal{N}^{sm}$.

In fact, there are explicit recursive equations for divisors $\mathcal{D}_n \subset \mathcal{M}$ such that $\mathcal{M}^{sm}[n] = \mathcal{M}^{sm} \cap \mathcal{D}_n$, namely the "division polynomials" evaluated at P [28, Ex. 3.7]. More precisely, for each $n > 1$, there is a homogenous polynomial ψ_n in a_2, a_3, a_4 (where a_i has weight i) of degree $n^2 - 1$ such that $\mathcal{D}_n$ is defined by ψ_n. We have

$$\psi_2 = a_3,$$
$$\psi_3 = a_2a_3^2 - a_4^2,$$
$$\psi_4 = 2a_2a_3^3a_4 - 2a_3a_4^3 - a_3^5,$$

and the higher ψ_n are defined recursively by

$$\psi_{2m+1} = \psi_{m+2}\psi_m^3 - \psi_{m-1}\psi_{m+1}^3 \qquad m \geq 2,$$
$$\psi_2\psi_{2m} = \psi_{m-1}^2\psi_m\psi_{m+2} - \psi_{m-2}\psi_m\psi_{m+1}^2 \qquad m \geq 3.$$

4.3 Nodal cubics with a point

Let k be a field of characteristic zero or $p > 3$, and let $a = (a_2, a_3, a_4)$ be a k-valued point of $\mathcal{M}^n$, i.e., such that $\Delta(a) = 0$ and $c_4(a) \neq 0$. Then by Proposition 4.1, the plane cubic

$$E_a : \; y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x$$

over k is nodal. We further assume that $(a_3, a_4) \neq (0, 0)$ so that $P = (0, 0)$ and the node, call it Q, are distinct. Let $\mathbb{G}_m$ be the multiplicative group over k. Then, possibly after extending k quadratically, there is a group isomorphism

$$E_a \setminus \{Q\} \to \mathbb{G}_m$$

which is unique up to pre-composing with inversion. We want to write down an explicit expression for the image of P under such an isomorphism.

This is a straightforward calculation: The node is defined by the vanishing of $2y+a_3$ and $3x^2 + 2a_2x + a_4$, and one finds that its coordinates are

$$Q = \left(\frac{18a_3^2 - 8a_2a_4}{c_4}, \frac{-a_3}{2} \right)$$

where as usual $c_4 = 16a_2^2 - 48a_4$. Changing coordinates

$$x = x' + \frac{18a_3^2 - 8a_2a_4}{c_4}, \qquad y = y' + \frac{-a_3}{2}$$

brings E_a into the form

$$y'^2 = x'^3 + \frac{-c_6}{4c_4} x'^2$$

where as usual $c_6 = 288a_2a_4 - 64a_2^3 - 216a_3^2$. Letting γ be a square root of $-c_6/(4c_4)$, the map to $\mathbb{G}_m$ is

$$(x', y') \mapsto \frac{y' - \gamma x'}{y' + \gamma x'}$$

and we find that P maps to

$$\frac{a_3c_4 - \gamma(16a_2a_4 - 36a_3^2)}{a_3c_4 + \gamma(16a_2a_4 - 36a_3^2)} \tag{4.1}$$

which (not surprisingly) is an algebraic expression in the original a_2, a_3, a_4.

5 From E/K to $\mathcal{E} \to \mathcal{C}$

We remind the reader how to go from an elliptic curve over a function field to an elliptic surface. Although this is not strictly necessary for our main purposes, it suggests a fruitful point of view on finite-dimensional families of elliptic surfaces parameterized by certain Riemann-Roch spaces.

5.1 General construction

Let k be a field of characteristic 0 or $p > 3$, let $\mathcal{C}$ be a smooth, projective, absolutely irreducible curve over k, and let $K = k(\mathcal{C})$. Let E be an elliptic curve over K equipped with a non-zero rational point $P \in E(K)$.

Choose a non-zero differential ω on E. Then by Proposition 4.1, there is a unique triple $a = (a_2, a_3, a_4)$ of elements of K such that E is isomorphic to

$$y^2 + a_3y = x^3 + a_2x^2 + a_4x,$$

P is $(0, 0)$ and $\omega = dx/(2y + a_3)$. Let D be the smallest divisor on $\mathcal{C}$ such that $\mathrm{div}(a_i) + iD$ is effective for $i = 2, 3, 4$. (Here "smallest" is with respect to the usual partial ordering: $D_1 \geq D_2$ if $D_1 - D_2$ is effective.) Let $L = \mathcal{O}_{\mathcal{C}}(D)$ so that we may regard a_i as a global section of $L^{\otimes i}$.

If $U \subset \mathcal{C}$ is a non-empty Zariski open subset and ϕ is a trivialization of L over U (i.e., a nowhere vanishing section of L), then over U we may regard the a_i as functions, and we get a morphism $U \to \mathcal{M}$. Pulling back the universal curve gives a family

$$\mathcal{W}_U \to U$$

of curves of genus 1 (in the sense used before Proposition 4.1) with a section P_U disjoint from O, and the general fiber of $\mathcal{W}_U \to U$ is E/K equipped with P. If $\{U_j\}$ is an open cover with trivializations ϕ_j of $L_{|U_j}$, there is a unique way to glue over the intersections compatible with the identification of the generic fiber of $\mathcal{W}_{U_j} \to U_j$ with E/K, and the result is a global family $\mathcal{W} \to \mathcal{C}$ of curves of genus 1 equipped with a section which we again denote by P. Writing $\mathcal{P}$ for the $\mathbb{P}^2$ bundle over $\mathcal{C}$ given by

$$\mathcal{P} = \mathbb{P}_{\mathcal{C}}\left(L^2 \oplus L^3 \oplus \mathcal{O}_{\mathcal{C}}\right)$$

(with coordinates $[x, y, z]$ on the fibers), we see that $\mathcal{W}$ is the closed subset of $\mathcal{P}$ defined by the equation

$$y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x$$

and P is the section $[0, 0, 1]$.

The surface $\mathcal{W}$ may have isolated singularities, and if so, we resolve them and then blow down any remaining (-1)-curves in the fibers of the map to $\mathcal{C}$, thus obtaining a smooth, relatively minimal elliptic surface $\mathcal{E} \to \mathcal{C}$ with a section again denoted by P.

5.2 A geometric subtlety

There is a subtle point hiding in the last step of this construction: The section P of $\mathcal{W} \to \mathcal{C}$ is disjoint from O, yet a section of $\mathcal{E} \to \mathcal{C}$ may very well meet O. Therefore, there may be some blowing down in the last step to force such an intersection. We make a few more comments about this situation and then give an example.

The underlying issue is that the local models $\mathcal{W}_U \to U$ are in a sense minimal with respect to pairs "elliptic fibration + nowhere zero section," but they may not be minimal if we forget the section. We can quantify this as follows: Given E/K and P, choosing ω leads to coefficients $a_i \in K$ and to invariants

$$c_4 = 2^4(a_2^2 - 3a_4) \quad \text{and} \quad c_6 = 2^5 3^2 a_2 a_4 - 2^6 a_2^3.$$

Recall that D was defined as the smallest divisor on $\mathcal{C}$ such that $\mathrm{div}(a_i) + iD \geq 0$ for $i = 2, 3, 4$. Similarly, let D' be the smallest divisor on $\mathcal{C}$ such that $\mathrm{div}(c_j) + jD' \geq 0$ for $j = 2, 4$. Then it is clear that $D \geq D'$ and the points entering into $D - D'$ are exactly those where the model $\mathcal{W} \to \mathcal{C}$ is not minimal (in the sense of [28, p. 816]). Moreover, while $\mathcal{W}$ sits naturally as a divisor in

$$\mathcal{P} = \mathbb{P}_{\mathcal{C}}\left(L^2 \oplus L^3 \oplus \mathcal{O}_{\mathcal{C}}\right),$$

the minimal Weierstrass family associated to $\mathcal{W} \to \mathcal{C}$ is naturally a divisor in

$$\mathcal{P}' = \mathbb{P}_{\mathcal{C}}\left(L'^2 \oplus L'^3 \oplus \mathcal{O}_{\mathcal{C}}\right)$$

where $L' = \mathcal{O}_{\mathcal{C}}(D')$. The choice of ω defines (possibly rational) sections of L and L' with divisors D and D' respectively. Since $O^*(\Omega^1_{\mathcal{E}/\mathcal{C}}) = L'$, in some sense L' is more natural than L.

5.3 An example

Let $\mathcal{C} = \mathbb{P}^1$ and $K = k(t)$, and let E/K be defined by

$$w^2 = z^3 + t^2 z - 1$$

with point $P = (t^{-2}, t^{-3})$ and differential $\omega = dz/2w$. The standard model coming from Proposition 4.1 for this data is

$$y^2 + 2t^{-3}y = x^3 + 3t^{-2}x^2 + (3t^{-4} + t^2)x$$

with $P = (0, 0)$ and $\omega = dx/(2y + 2t^{-3})$. Also, $c_4 = -48t^2$ and $c_6 = 864$ and we find that

$$D = 0 + \infty \quad \text{and} \quad D' = \infty.$$

The local model $\mathcal{W}_{\mathbb{A}^1} \to \mathbb{A}^1$ is given by

$$y^2 + 2y = x^3 + 3x^2 + (3 + t^6)x.$$

The fiber over $t = 0$ is a cubic with cusp at $t = 0$, $x = y = -1$, and the surface $\mathcal{W}_{\mathbb{A}^1}$ is singular at this point. Resolving the singularity requires blowing up once and normalizing, and a further blow down removes a (-1)-curve in the fiber. This last blow down brings the section P into contact with the zero section O.

5.4 Starting with the line bundle

We take the following point of view on constructing elliptic surfaces over $\mathcal{C}$: Start with a line bundle L on $\mathcal{C}$. Then for each

$$a = (a_2, a_3, a_4) \in H^0(\mathcal{C}, L^2 \oplus L^3 \oplus L^4)$$

with $\Delta(a_2, a_3, a_4) \neq 0$, we get $\mathcal{W} \to \mathcal{C}$ defined by the vanishing of

$$y^2z + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2$$

in

$$\mathcal{P} = \mathbb{P}_{\mathcal{C}}\left(L^2 \oplus L^3 \oplus \mathcal{O}_{\mathcal{C}}\right).$$

For "most" choices of a, $\mathcal{W} \to \mathcal{C}$ is already minimal and $L' = L$. This holds if $\Delta(a_2, a_3, a_4)$ has order of vanishing < 12 (as a section of L^{12}) at each place of $\mathcal{C}$. If Δ has only simple zeroes, then $\mathcal{W} \to \mathcal{C}$ is minimal and $\mathcal{W}$ is regular, so $\mathcal{E} = \mathcal{W}$. In this way, we get flat families of elliptic surfaces parameterized by open subsets of certain Riemann-Roch spaces. We will justify the claim "most" in the next section.

6 Very general elliptic surfaces with two sections

In this section, k is a field of characteristic zero or $p > 3$ and $\mathcal{C}$ is a smooth, projective, absolutely irreducible curve over k. Let L be a line bundle on $\mathcal{C}$ which is globally generated and write d for the degree of L.

Let $a = (a_2, a_3, a_4)$ be an element of $V = H^0(\mathcal{C}, L^2 \oplus L^3 \oplus L^4)$ with $\Delta(a) \neq 0$. Then as explained in Sect. 5.4 we get a family $\mathcal{W}_a \to \mathcal{C}$ of curves of genus 1 and a relatively minimal elliptic surface $\mathcal{E}_a \to \mathcal{C}$ equipped with a section P. Our aim is to show that for a very general choice of a, P is transverse to $\mathcal{E}_a[n]$ for all n and enjoys other desirable properties.

We first consider the case where $d = 0$, so L is trivial and the a_i are constants. In this case, it is clear that P is transverse to all torsion sections if and only if it is disjoint from all torsion sections, if and only if it is of infinite order. This happens for very general choices of a, but not on a Zariski open. That suggests what to expect in the general case.

We restate Theorem 1.7 (in the case where L is non-trivial) with an additional claim:

Theorem 6.1 *Let L be a globally generated line bundle on $\mathcal{C}$ of degree $d > 0$, and set*

$$V = H^0(L^2 \oplus L^3 \oplus L^4).$$

Then for a very general $a = (a_2, a_3, a_4) \in V$, the elliptic surface $\mathcal{E}_a \to \mathcal{C}$ associated to

$$E: \qquad y^2 + a_3 y = x^3 + a_2 x^2 + a_4 x$$

equipped with the section $P = (0, 0)$ has the following properties:

(1) P has infinite order
(2) The singular fibers of $\mathcal{E}_a \to \mathcal{C}$ are nodal cubics (i.e., Kodaira type I_1).
(3) P meets each singular fiber in a non-torsion point.
(4) If n is not a multiple of the characteristic of k, then P is transverse to $\mathcal{E}_a[n]$.
(5) If n is not a multiple of the characteristic of k, then nP meets O transversally in $d(n^2 - 1)$ points.

Here, as usual, "for a very general a" means that there is a countable collection of non-empty, Zariski open subsets of V such that if a lies in their intersection, then the assertion holds for a. We will prove several lemmas, each asserting that some Zariski open subset is non-empty, and then put them together to prove the theorem at the end of this section. It is no loss of generality to assume that k is algebraically closed, so for convenience we assume this for the rest of the section.

Recall that "L is globally generated" means that for all $t \in \mathcal{C}$, there is a global section of L not vanishing at t. It is a standard exercise to show that L is globally generated and has positive degree if and only if there is a non-constant morphism $f : \mathcal{C} \to \mathbb{P}^1$ such that $L = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Moreover, if L is globally generated, then the set of global sections of L with reduced divisors (i.e., distinct zeroes) is non-empty and Zariski open.

Lemma 6.2 *The subset $V_\Delta \subset V$ consisting of a such that $\Delta(a)$ has $12d$ distinct zeroes (as a section of L^{12}) is Zariski open and not empty. There are $a \in V_\Delta$ whose zeroes are disjoint from any given finite subset of points of $\mathcal{C}$.*

Proof It is clear that the locus of $a \in V$ where $\Delta(a)$ has distinct zeroes is Zariski open. To prove the lemma, we need to check that V_Δ is not empty. We do this constructively. First assume $\mathcal{C} = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Set $a_2 = 0$, $a_3 = c \in k$, and $a_4 = t^4$. Then $\Delta = -27c^4 - 64t^{12}$ which has distinct zeroes as a section of $\mathcal{O}_{\mathbb{P}^1}(12)$ if $c \neq 0$. Moreover, varying c, we can arrange for the zeros to avoid any finite subset of $\mathbb{P}^1$.

In the general case, choose a morphism $f : \mathcal{C} \to \mathbb{P}^1$ such that $L = f^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Let $S \subset \mathbb{P}^1$ be the branch locus of f. Then setting $a_2 = 0$, $a_3 = c$, and $a_4 = f^*(t^4)$, where c is chosen so that the zeroes of $-27c^4 - 64t^{12}$ are disjoint from S, yields an explicit a with the required properties. Varying c allows us to avoid any finite subset of $\mathcal{C}$. □

As noted in Sect. 5.4, if $a \in V_\Delta$, then the corresponding elliptic surface $\mathcal{W}_a$ is smooth (so no resolution of singularities is needed), $\mathcal{W}_a \to \mathcal{C}$ is relatively minimal (so we may set $\mathcal{E}_a = \mathcal{W}_a$), and $L = O^*(\Omega^1_{\mathcal{E}_a/\mathcal{C}})$. Moreover, the bad fibers of $\mathcal{E}_a \to \mathcal{C}$ are all of type I_1. From now on we always choose a from V_Δ.

Lemma 6.3 *For every $n \geq 1$, there is a non-empty, Zariski open subset V_n of V_Δ such that if $a \in V_n$, then the section P of $\mathcal{E}_a \to \mathcal{C}$ does not intersect any singular fiber in a point of order exactly n.*

Proof It is clear that the locus of a where P has the stated property is open, and our task is to show it is non-empty. Since the bad fibers are all of type I_1, if k has characteristic $p > 0$ and n is divisible by p, there are no points of order exactly n in the fiber, so we may take $V_n = V_\Delta$.

Now assume that n is not divisible by the characteristic of k. We check constructively that there is a non-empty set as described in the statement. As in the previous lemma, we may reduce to the case $\mathcal{C} = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Take $a_2 = 0$, $a_3 = c$, $a_4 = t^4$. Then the bad fibers are at the roots of $t^{12} = (-27/64)c^4$ and at each such root, the coordinate in $\mathbb{G}_m$ of P was given at (4.1). For the data we are considering, the coordinate is

$$\frac{4t^4 - 3c\gamma}{4t^4 + 3c\gamma} \qquad \text{where } \gamma = (-9c^3/2)^{1/2}t^{-2}.$$

Then for each n, there are only finitely many values of c such that for some root t of $t^{12} = (-27/64)c^4$, the displayed quantity is an n-th root of unity. This proves that V_n is non-empty for each n. □

Remark 6.4 Over an uncountable field, intersecting the opens in the theorem gives a non-empty set. We can do a bit better over $\mathbb{C}$: There is an everywhere dense classical open set in V_Δ such that P meets each singular fiber away from the unit circle $S^1 \subset \mathbb{C}^\times$.

Lemma 6.5 *If the characteristic of k is $p > 3$, then for all $a \in V_\Delta$ and any n divisible by p, P does not have order exactly n.*

Proof It will suffice to show that when $a \in V_\Delta$, $\mathcal{E}_a \to \mathcal{C}$ has no non-trivial p-torsion sections. First note that since $a \in V_\Delta$, the zeroes of c_4 are disjoint from those of Δ. This implies that $j = c_4^3/\Delta$ has simple poles, so it is not a constant (implying that

$\mathcal{E}_a \to \mathcal{C}$ is non-isotrivial) and not a p-th power. Then [30, Prop I.7.3] implies that $\mathcal{E}_a \to \mathcal{C}$ has no p-torsion. (In [30], the ground field is finite, but the argument there works over any field of positive characteristic.) $\square$

Proposition 6.6 *For every n not divisible by the characteristic of k, there is a non-empty, Zariski open subset $W_n \subset V_\Delta$ such that if $a \in W_n$, then $nP \neq 0$ and P is transverse to $\mathcal{E}_a[n]$.*

Proof Again, it is clear that the set of a with the desired properties is open. Unfortunately, it seems hopeless to give a constructive proof that it is non-empty, so we have to do something more sophisticated.

Recall the moduli space $\mathcal{M}$ of Sect. 4. We write $\mathcal{M}_k$ for

$$\mathcal{M} \otimes_{\mathbb{Z}[1/6]} k = \operatorname{Spec} k[a_2, a_3, a_4]$$

and $\mathcal{M}_k[n]$ for the locally closed, smooth, codimension 1 locus parameterizing triples (E, ω, P) where P has order n.

Recall also that $V = H^0(\mathcal{C}, L^2 \oplus L^3 \oplus L^4)$ and V_Δ is the open subset consisting of a such that $\Delta(a)$ has distinct zeroes. Choose an open subset $U \subset \mathcal{C}$ and a trivialization of L over U. Then for $a = (a_2, a_3, a_4)$ the a_i may be regarded as functions on U, and we get a morphism $f_a : U \to \mathcal{M}_k$. To say that $nP = O$ is to say that $f_a(U)$ is contained in $\mathcal{M}_k[n]$. To say that P is tangent to $\mathcal{E}_a[n]$ over $x \in U$ is to say that $f_a(U)$ is tangent to $\mathcal{M}_k[n]$ at $f_a(x)$. We will show that these conditions do not hold for most a.

Consider the morphism

$$F : V \times U \to \mathcal{M}_k \qquad (a, t) \mapsto F(a, t) := f_a(t)$$

and let

$$D_n := F^{-1}(\mathcal{M}_k[n]) \cap (V_\Delta \times U)\,.$$

We will use the global generation of L to show that D_n is a smooth, locally closed subset of codimension 1 in $V_\Delta \times U$, and that there is a non-empty open subset $W_{U,n} \subset V_\Delta$ such that the projection $D_n \to V_\Delta$ is étale over $W_{U,n}$. This means that if $a \in W_{U,n}$, then $\{a\} \times U$ is transverse to D_n, i.e., that P meets the n-torsion multisection of $\mathcal{E}_a$ transversally over U. Taking a finite cover $\{U_j\}$ of $\mathcal{C}$ and setting $W_n = \cap_j W_{U_j,n}$ will complete the proof.

Since L is globally generated, so are its powers L^i for $i = 2, 3, 4$. This means that for every $t \in U$, there are global sections a_2, a_3, a_4 not vanishing at t, and for all but finitely many t there are global sections s_2, s_3, s_4 which vanish to order 1 at t. (Since L^i is globally generated, there are sections of L^i inducing a morphism $\mathcal{C} \to \mathbb{P}^1$. If t is not in the ramification locus, a section s_i as above can be obtained by pulling back a section of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing simply at the point of $\mathbb{P}^1$ under t.)

For each $t \in U$, the restriction

$$F_t : V \times \{t\} \to \mathcal{M}_k$$

is a linear map, and since L is globally generated, it is surjective. Thus the fibers are all affine spaces of dimension $h-3$ where $h = \dim V$. Therefore, F is surjective and smooth (smooth because it is submersive, i.e., it has a surjective differential at every point). Moreover, the fibers of F are $\mathbb{A}^{h-3}$-bundles over U, and in particular, they are all irreducible of dimension $h-2$. It follows that each irreducible component $D_{n,i}$ of D_n is smooth and locally closed in $V_\Delta \times U$ of codimension 1 and has the form

$$D_{n,i} = F^{-1}(\mathcal{M}_k[n]_i) \cap (V_\Delta \times U)$$

where $\mathcal{M}_k[n]_i$ is an irreducible component of $\mathcal{M}_k[n]$.

Consider an irreducible component $D_{n,i}$ of D_n. We are going to produce a point of $D_{n,i}$ at which the projection $D_{n,i} \to V$ is étale. Start by choosing any point $(a, t) \in D_{n,i}$ and let $m = F(a, t)$. The fiber of F over m is an $\mathbb{A}^{h-3}$ bundle over U, and $F^{-1}(m) \cap (V_\Delta \times U)$ is a non-empty open subset of this bundle, so it projects to a non-empty open subset of U. This means that we may find another point $(a', t') = (a'_2, a'_3, a'_4, t')$ in $D_{n,i}$ such that L^i admits global sections s_i vanishing simply at t' for $i = 2, 3, 4$.

For all triples $(\alpha_2, \alpha_3, \alpha_4) \in k^3$, we have

$$F(a'_2 + \alpha_2 s_2, a'_3 + \alpha_3 s_3, a'_4 + \alpha_4 s_4, t) = m,$$

and for a non-empty open subset of triples $(\alpha_i) \in k^3$, we have that

$$a'' := (a'_2 + \alpha_2 s_2, a'_3 + \alpha_3 s_3, a'_4 + \alpha_4 s_4) \in V_\Delta.$$

By a suitable choice of the α_i we may arrange for the differential of F restricted to $\{a''\} \times U$ to carry the tangent space of U at t to a line in the tangent space of $\mathcal{M}_k$ at m not contained in $T_{\mathcal{M}[n],m}$. For such a choice, we conclude that $\{a''\} \times U$ is transverse to $D_{n,i}$ at (a'', t). This proves that the projection $D_{n,i} \to V_\Delta$ is étale at (a'', t).

It follows that there is a Zariski open subset $D^o_{n,i}$ of $D_{n,i}$ such that $D^o_{n,i} \to V_\Delta$ is étale. The image of $D_{n,i} \setminus D^o_{n,i}$ in V_Δ is contained in a proper closed subset, and removing these subsets for all i yields an open subset $W_{U,n}$ over which $D_n \to V_\Delta$ is étale. Covering $\mathcal{C}$ with finitely many U_j and setting $W_n = \cap_j W_{U_j,n}$ yields an open subset of V_Δ such that if $a \in W_n$, then P does not have order n and is transverse to $\mathcal{E}_a[n]$. This completes the proof of the proposition. $\square$

Proof of Theorem 6.1 Consider the intersection

$$V' = \left(\bigcap_{n \geq 1} V_n \right) \bigcap \left(\bigcap_{p \nmid n} W_n \right) \subset V_\Delta.$$

The preceding lemmas show that if $a \in V'$, then the corresponding $\mathcal{E}_a$ has the properties asserted in the Theorem. Indeed, since $a \in V_\Delta$, $\Delta(a)$ as $12d$ distinct zeroes, and so $\mathcal{E}_a$ has $12d$ bad fibers of type I_1 and no other bad fibers. Since $a \in \cap_n V_n$, Lemma 6.3 shows that P does not meet a bad fiber in a torsion point. Since $a \in \cap_n W_n$, Lemma 6.5

and Proposition 6.6 show that P has infinite order, and Proposition 6.6 shows that if n is prime to the characteristic, then P is transverse to $\mathcal{E}_a[n]$. This establishes points (1) through (4) of the Theorem.

The transversality in point (5) is equivalent to that in (4), so to finish we just need to calculate the intersection multiplicity $(nP).O$. For this, we first note that $P.O = 0$ by construction, and as explained in the proof of Lemma 2.8, $O^2 = P^2 = -d$. Thus $ht(P) = 2d$ and Lemma 2.8 implies that $(nP).O = d(n^2 - 1)$, as required.

This completes the proof of the theorem. □

Remark 6.7 When $\mathcal{C} = \mathbb{P}^1$, every line bundle of non-negative degree is globally generated. Thus, starting from data (E, P) over $\mathbb{P}^1$, we can find a deformation $(\mathcal{E}', P')$ with the same base $\mathcal{C}$ and bundle L such that P' is transverse to all torsion multisections. For a general $\mathcal{C}$, if we do not assume any positivity for $L = O^*(\Omega^1_{\mathcal{E}/\mathcal{C}})$, it may be impossible to produce deformations with fixed $\mathcal{C}$ and L. Here are two alternatives: First, we may embed $L \hookrightarrow L'$ where L' is globally generated, and deform a non-minimal model of $\mathcal{E}$ (lying in $\mathbb{P}_{\mathcal{C}}(L'^2 \oplus L'^3 \oplus \mathcal{O}_{\mathcal{C}})$). Second, it seems likely that the ideas of Moishezon [23], as explained in [16, Thm. I.4.8] would allow one to find a deformation of $\mathcal{E}$ where the base curve is also allowed to vary (i.e., deform to $\mathcal{E}' \to \mathcal{C}'$ and section P') with the desired transversality.

Remark 6.8 Suppose that k has characteristic zero and that $\pi : \mathcal{E} \to \mathcal{C}$ and P satisfy the conclusions of Theorem 6.1. If n_1 and n_2 are two distinct integers, then $n_1 P \cup n_2 P$ is a normal crossings divisor on $\mathcal{E}$. More generally, if $N \subset \mathbb{Z}$ is a non-empty finite set, then

$$D = \bigcup_{n \in N} nP$$

is a curve on $\mathcal{E}$ with only ordinary multiple points. Indeed, it is a union of smooth components which meet pairwise transversally. This is clear from the facts that O and nP meet transversally for all $n \neq 0$ and that $O \cup (n_2 - n_1)P$ is carried isomorphically to $n_1 P \cup n_2 P$ under translation by $n_1 P$.

7 Explicit examples with even height over small fields

In this section, we show by explicit construction that there are pairs $(\mathcal{E}, P)$ with P transverse to torsion multisections over fields k such as number fields and global function fields. The precise statement is Theorem 1.8 in the introduction. For simplicity, we assume throughout that the characteristic of k is not 2. We begin by constructing examples of height 2 over $\mathbb{P}^1$.

Proposition 7.1 *Let k be a field of characteristic $\neq 2$. Then there exist Jacobian elliptic surfaces $\mathcal{E} \to \mathbb{P}^1$ over k equipped with a section P such that*

(1) P has infinite order.
(2) The singular fibers of $\mathcal{E} \to \mathbb{P}^1$ are of Kodaira type I_0^*.
(3) P meets each singular fiber in a non-torsion point.

(4) *If n is not a multiple of the characteristic of k, then nP meets O transversally in*

$$\begin{cases} \dfrac{n^2-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n^2-4}{2} & \text{if } n \text{ is even,} \end{cases}$$

points.

(5) *The height of $\mathcal{E}$ is 2, i.e., $O^*(\Omega^1_{\mathcal{E}/\mathbb{P}^1}) \cong \mathcal{O}_{\mathbb{P}^1}(2)$.*

Proof We will construct one such $\mathcal{E} \to \mathbb{P}^1$ for every elliptic curve E over k. Suppose that $f \in k[x]$ is a monic polynomial of degree 3 such that E is defined by $y^2 = f(x)$. Form the product $E \times_k E$, and let $\{\pm 1\} \subset \mathrm{Aut}(E)$ act diagonally. The quotient $(E \times_k E)/(\pm 1)$ is a singular (Kummer) surface, and projection to the first factor induces a morphism

$$(E \times_k E)/(\pm 1) \to E/(\pm 1) \cong \mathbb{P}^1.$$

Let $\mathcal{E} \to \mathbb{P}^1$ be the regular minimal model of $(E \times_k E)/(\pm 1) \to \mathbb{P}^1$. Thus $\mathcal{E}$ is obtained from $(E \times_k E)/(\pm 1)$ by blowing up the 16 fixed points of ± 1 on $E \times_k E$, and the bad fibers of $\mathcal{E} \to \mathbb{P}^1$ are of type I_0^* and lie over $t = \infty$ and the roots of $f(t)$.

Let $\Gamma_n \subset E \times_k E$ be the graph of multiplication by n, which we may regard as the image of a section to $E \times_k E \to E$. Then Γ_n is preserved by ± 1 and maps with degree 2 to a section of $(E \times_k E)/(\pm 1) \to \mathbb{P}^1$, and this section lifts to a section nP of $\mathcal{E} \to \mathbb{P}^1$. (The notation is consistent in that nP is n times P in the group law of $\mathcal{E}$.)

The following diagram summarizes the data:

$$\begin{tikzcd} E \times_k E \arrow[r] \arrow[d] & (E \times_k E)/(\pm 1) \arrow[d] & \mathcal{E} \arrow[l] \arrow[d] \\ E \arrow[r] \arrow[u, bend left, "\Gamma_n"] & E/(\pm 1) \cong \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1. \arrow[u, bend right, "nP"'] \end{tikzcd}$$

It will be convenient to have a Weierstrass equation for $\mathcal{E} \to \mathbb{P}^1$. If $f(x) = x^3 + ax^2 + bx + c$, then $\mathcal{E}$ is the Néron model of the elliptic curve

$$y^2 = x^3 + af(t)x^2 + bf^2(t)x + cf^3(t)$$

over $k(t)$, and the point P has coordinates $(x, y) = (tf(t), f^2(t))$. Indeed, if the two factors of $E \times E$ are $v^2 = f(u)$ and $s^2 = f(r)$, then the field of invariants of ± 1 is generated by u, r, and $z = vs$, and these satisfy the equation

$$z^2 = f(u)f(r).$$

Setting $u = t$ and $z = y/f(u)$, and $r = x/f(u)$ yields the equation and point above.

We now verify the cases $n = 1$ and $n = 2$ of the proposition. Since P has polynomial coefficients, it does not meet O over any finite value of t, and since its x and y coordinates have degrees 4 and 6, and $\mathcal{E}$ has height 2, P also does not meet O over $t = \infty$. In summary, P meets O nowhere, as claimed. For later use, we note that at the roots of f, P specializes to $(0, 0)$, i.e., to a singular point of the fiber of $(E \times_k E)/(\pm 1)$, so P lands on a non-identity component of the fiber of $\mathcal{E}$. At $t = \infty$, P specializes to $(1, 1)$, a non-singular, finite point of the fiber (i.e., a point not on O).

A tedious but straightforward calculation (or an algebra package …) shows that $2P$ has coordinates $((1/4)t^4 + \cdots, (1/8)t^6 + \cdots)$ where $\cdots$ indicates terms of lower degree in t. The argument of the previous paragraph shows that $2P$ meets O nowhere, as claimed. For later use, we note that $2P$ passes through a finite point of the identity component in each of the bad fibers.

Now consider $n > 2$. It is clear that Γ_n meets $E \times \{0\}$ exactly at the points of E of order n, and each of these intersections is transverse. If $(p, 0)$ is such a point which is not of order 2, then the quotient map

$$E \times_k E \to (E \times_k E)/(\pm 1)$$

is étale in a neighborhood of $(p, 0)$ and it sends Γ_n 2-to-1 to a curve that meets O transversally. Moreover, the map

$$\mathcal{E} \to (E \times_k E)/(\pm 1)$$

is an isomorphism in a neighborhood of such a point. This proves that nP meets O transversally over the values of t such that there is a point (t, v) with $v^2 = f(t)$ which is n-torsion and not 2-torsion. There are

$$\begin{cases} \frac{n^2-1}{2} & \text{if } n \text{ is odd} \\ \frac{n^2-4}{2} & \text{if } n \text{ is even} \end{cases}$$

such values of t.

It remains to consider what happens over the roots of $f(t)$ and $t = \infty$. But we checked above that P meets a non-trivial point of the identity component at $t = \infty$ and such a point is either of infinite order or of order p when k has characteristic p. So, for n prime to the characteristic of k, nP does not meet O over $t = \infty$. Similarly, over the roots of $f(t)$, P passes through the non-identity component and $2P$ passes through a non-trivial point of the identity component, so nP does not meet O when n is prime to the characteristic. We have thus identified all points where nP and O intersect, the intersections are transverse, and their number is as stated in the proposition. This completes the proof of the proposition. □

Remark 7.2 As a check, we compute the intersection number $(nP).O$ using heights as in Lemma 2.8. We have $O^2 = P^2 = -2$ and $P.O = 0$. Since P passes through a non-identity component of the fibers over roots of $f(t)$ and through the identity component at $t = \infty$, the "correction term" is $-C_P.(P - O) = -3$. (See table 1.19 in [13].) Using the formula (2.4) for the height pairing yields $ht(P) = 1$.

Similarly, for any odd n, $-C_{nP}.(nP - O) = -3$ and using that $ht(nP) = n^2$ and calculating as in Lemma 2.8 we find $(nP).O = (n^2 - 1)/2$.

On the other hand, for even n, nP passes through the identity component in all bad fibers, so $-C_{nP}.(nP - O) = 0$ and we find that $(nP).O = (n^2 - 4)/2$.

This confirms that the intersections we saw above are all transverse.

Proof of Theorem 1.8 Proposition 7.1 implies the case of the Theorem where $\mathcal{C} = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(2)$, and we get infinitely many examples because k is infinite. Indeed, for each $j \in k$, there is an elliptic curve E with j-invariant j, and elliptic curves with distinct j-invariants give rise to non-isomorphic $\mathcal{E} \to \mathbb{P}^1$ since the non-singular fibers are twists of the chosen E.

We deduce the general case by a pull-back construction. Write $\mathcal{E}' \to \mathbb{P}^1$ for one of the surfaces constructed in Proposition 7.1. Let $f : \mathcal{C} \to \mathbb{P}^1$ be a non-constant morphism defined by sections of the globally generated line bundle F, so $F = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $L = f^*\mathcal{O}_{\mathbb{P}^1}(2)$. The conclusions of the theorem will hold for $\mathcal{E} := f^*\mathcal{E}' \to \mathcal{C}$ if the branch locus of f is disjoint from the set of points of $\mathbb{P}^1$ over which $\mathcal{E}'$ has bad reduction or nP meets O. From the construction of $\mathcal{E}'$, we see that the set to be avoided is precisely the set of x coordinates of torsion points of the elliptic curve $y^2 = f(x)$ used to construct $\mathcal{E}'$. Although this set is infinite, we will see that it is sparse in k.

We divide into two cases according to the characteristic of k, starting with the case of characteristic zero. Choose an elliptic curve E over $\mathbb{Q}$, and an auxiliary prime ℓ such that equations defining E are ℓ-integral and E has good reduction modulo ℓ. Then [28, VIII.7.1] implies that the x-coordinate of a torsion point Q (defined over some number field K and taken with respect to an ℓ-integral model) is "almost integral," i.e., it satisfies $\ell^2 x(Q)$ is integral at all primes of K over ℓ. Construct $\mathcal{E}' \to \mathbb{P}^1_{\mathbb{Q}}$ using E as in Proposition 7.1. Then choose any non-constant morphism $f : \mathcal{C} \to \mathbb{P}^1_k$ defined by sections of F. Composing ϕ with a linear fractional transformation, we may arrange that the branch locus of f consists of points with finite, non-zero coordinates, and that any of those coordinates which lie in a number field have large denominators at primes over ℓ. They are thus distinct from the x-coordinates of torsion points of E, and $\mathcal{E} = f^*\mathcal{E}'$ satisfies the requirements of the theorem.

When k has characteristic $p > 2$, the argument is similar, but simpler: Choose an embedding $\mathbb{F}_p(t) \hookrightarrow k$, an elliptic curve E over $\mathbb{F}_p(t)$, and a place v of $\mathbb{F}_p(t)$ where E has good reduction. Then by [29, §4], the coordinates of any torsion point Q of E (defined over some algebraic extension K of $\mathbb{F}_p(t)$ and taken with respect to an integral model) are integral at places of K over v. Use E to construct $\mathcal{E}'$ as in Proposition 7.1. Then choose any non-constant morphism $f : \mathcal{C} \to \mathbb{P}^1_k$ defined by sections of F. Composing ϕ with a linear fractional transformation, we may arrange that the branch locus of f consists of points with finite, non-zero coordinates, and that any of those coordinates which are algebraic over $\mathbb{F}_p(t)$ are not integral at places over v. They are thus distinct from the x-coordinates of torsion points of E, and $\mathcal{E} = f^*\mathcal{E}'$ satisfies the requirements of the theorem. □

Remark 7.3 It seems likely that when k is a number field or a global function field, the construction in Proposition 7.1 gives rise to elliptic divisibility sequences D_n whose "new parts" D'_n are often irreducible, i.e., prime divisors.

8 Application to geography of surfaces

In this section, we will prove Theorem 1.9. Let $k = \mathbb{C}$, $\mathcal{C} = \mathbb{P}^1$, and $L = \mathcal{O}_{\mathbb{P}^1}(d)$ where $d = g+1$, which by assumption satisfies $d \geq 1$. Theorem 1.7 guarantees the existence of an elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ of height d (i.e., such that $O^*(\Omega^1_{\mathcal{E}/\mathbb{P}^1}) = L$) with a section P such that for all n, nP meets O transversally in $d(n^2-1)$ points. Moreover, π has irreducible fibers. Let F be the class of a fiber of π. We have $O^2 = P^2 = -d$, $F^2 = 0$, and the canonical divisor of $\mathcal{E}$ is

$$K_{\mathcal{E}} = (d-2)F.$$

Thus the geometric genus of $\mathcal{E}$ is $d-1 = g$.

Fix an integer $n > 1$. Later in the proof, we will need to assume that n is sufficiently large. Let $h : Y \to \mathcal{E}$ be the result of blowing up all but one of the points of intersection of O and nP, let E_i ($i = 1, \dots, d(n^2-1)-1$) be the exceptional divisors, and let C_j be the strict transform the section jP.

Write $\tilde{F}$ for the strict transform of a general fiber of π in Y. We have

$$C_0^2 = C_n^2 = -dn^2 + 1, \quad C_0.C_n = 1, \quad \text{and} \quad K_Y = (d-2)\tilde{F} + \sum_i E_i.$$

It is a simple exercise to check that the intersection pairing on Y is negative definite on the lattice spanned by C_0 and C_n, and that $p_a(Z) \leq 0$ for all effective divisors supported on $C_0 \cup C_n$. Thus by Artin's contractibility theorem [6, Thm. 2.3] or [7, Thm 3.9], we may contract $C_0 \cup C_n$. In other words, there is a proper birational morphism $f : Y \to X$ where X is a normal, projective surface, $f(C_0 \cup C_n) = \{x\}$, and f induces an isomorphism

$$Y \setminus (C_0 \cup C_n) \cong X \setminus \{x\}.$$

Proof that X satisfies the conditions of Theorem 1.9

We have already observed that X is normal and projective. Since the geometric genus is a birational invariant, and $\mathcal{E}$ has geometric genus g, so does X.

It is evident that X has exactly one singular point, namely x, and the minimal resolution of x is the union of two smooth rational curves (C_0 and C_n) meeting at one point and having self-intersection $-a := -dn^2 + 1$. Such a singularity is analytically equivalent to a cyclic quotient singularity of type $1/(a^2-1)(1, a)$, as one sees by considering the Hirzebruch-Jung continued fraction

$$a - \frac{1}{a} = \frac{a^2-1}{a}.$$

(See [8, Ch. 3].) In particular, it follows that $X$is $\mathbb{Q}$-Gorenstein and K_X is $\mathbb{Q}$-Cartier.

We next compute the discrepancy of x (as defined for example in [19]) and verify that x is log-terminal. Since C_j is smooth and rational with self-intersection $-a$, we have $C_j.K_Y = a-2$. Define coefficients $\alpha_0, \alpha_n \in \mathbb{Q}$ by

$$K_Y = f^*K_X + \alpha_0 C_0 + \alpha_n C_n$$

(an equality in $\mathrm{Pic}(Y) \otimes \mathbb{Q}$). Then

$$\begin{aligned} 0 &= (f_* C_0).K_X \\ &= C_0.f^*K_Y \\ &= (a-2) + \alpha_0 a - \alpha_n \end{aligned}$$

and similarly,

$$0 = (a-2) - \alpha_0 + \alpha_n a.$$

We find that

$$\alpha_0 = \alpha_n = -\frac{a-2}{a-1} > -1.$$

This confirms that x is a log-terminal singularity, and we have

$$f^*K_X = (d-2)\tilde{F} + \sum_i E_i + \frac{a-2}{a-1}\left(C_0 + C_n\right).$$

Next, write $b := (a-2)/(a-1)$ and compute

$$\begin{aligned} K_X^2 &= (f^*K_X)^2 \\ &= \left((d-2)\tilde{F} + \sum_i E_i + b(C_0 + C_n)\right)^2 \\ &= dn^2(4b - 2b^2 - 1) + d + 1 + 4b^2 - 12b. \end{aligned}$$

As $n \to \infty$, $a \to \infty$ and $b \to 1$, so K_X^2 grows like dn^2 and in particular is unbounded as n varies.

To finish the proof, it remains to check that K_X is ample, which we do using the Nakai-Moishezon criterion [7, Thm. 1.22]. We have already seen that $K_X^2 > 0$, so it will suffice to check that for every irreducible curve C on X, $C.K_X > 0$. For any such curve C,

$$f^*C = D + m_0 C_0 + m_n C_n$$

where D is an irreducible curve not equal to C_0 or C_n, and $m_j \geq 0$ for $j = 0, n$. It will thus suffice to prove that $D.f^*K_X > 0$ for all irreducible curves not equal to C_0 or C_n and that $C_j.f^*K_X = 0$ for $j = 0, n$. For the latter assertion, one computes that

$$C_j.f^*K_X = f_*(C_j).K_X = 0$$

for $j = 0, n$.

For the former assertion, we make a case by case analysis of the possibilities for D. They are:

- the strict transform $\tilde{F}$ of a general fiber of π, for which we have

$$\tilde{F}.f^*K_X = 2(a-2)/(a-1) > 0;$$

- one of the exceptional curves E_i, for which we have

$$E_i.f^*K_X = -1 + 2(a-2)/(a-1),$$

which is > 0 if $d > 1$ or $n > 2$;
- the strict transform $\tilde{G}_i = \tilde{F} - E_i$ of a fiber of π passing though an intersection point of O and nP, for which we have $\tilde{G}_i.f^*K_X = 1$;
- and the strict transform $\tilde{Q}$ of a multisection Q of π not equal to O or nP. Let e be the degree of $\pi_{|Q} : Q \to \mathbb{P}^1$, assume that $n > 2$ so that $b > 1/2$, and recall that

$$f^*K_X = (d-2)\tilde{F} + \sum_i E_i + b\,(C_0 + C_n)\,.$$

If $d > 2$, we have

$$\tilde{Q}.f^*K_X \geq (d-2)e > 0.$$

If $\tilde{Q}.\sum_i E_i > e$ or $\tilde{Q}.C_0 > 2e$, then again it is clear that $\tilde{Q}.f^*K_X > 0$ as required. To finish, assume that $d \leq 2$, $\tilde{Q}.\sum_i E_i \leq e$, and $\tilde{Q}.C_0 \leq 2e$. Applying h^* to the equality in Lemma 2.9 implies that

$$C_n = nC_1 + (1-n)C_0 - n\sum_i E_i + d(n^2-n)\tilde{F}$$

in $\mathrm{NS}(Y)$. If $Q \neq P$, we find that

$$\tilde{Q}.C_n \geq (1-n)2e - ne + d(n^2-n)e,$$

which is $> e$ for all $n \geq 4$, and this shows that $\tilde{Q}.f^*K_X > 0$. If $Q = P$, then

$$\tilde{Q}.C_n \geq -nd - n + d(n^2-n),$$

and we find that $\tilde{Q}.f^*K_X > 0$ for all $n \geq 5$. (When $Q = P$,we can also calculate directly that $\tilde{Q}.f^*K_X = d - 2 + b(dn^2 - 2dn)$ which goes to infinity with n.) This completes the check that $\tilde{Q}.f^*K_X > 0$ for all irreducible multisections $\tilde{Q}$ not equal to C_0 or C_n.

The itemized list completes the verification that K_X is ample, and this finishes the proof of the theorem. $\square$

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