Curves and Jacobians over Function Fields

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Introduction

These notes originated in a 12-hour course of lectures given at the Centre de Recerca Matemàtica in February 2010. The aim of the course was to explain results on curves and their Jacobians over function fields, with emphasis on the group of rational points of the Jacobian, and to explain various constructions of Jacobians with large Mordell–Weil rank.

More so than the lectures, these notes emphasize foundational results on the arithmetic of curves and Jacobians over function fields, most importantly the breakthrough works of Tate, Artin, and Milne on the conjectures of Tate, Artin–Tate, and Birch and Swinnerton-Dyer. We also discuss more recent results such as those of Kato and Trihan. Constructions leading to high ranks are only briefly reviewed, because they are discussed in detail in other recent and forthcoming publications.

These notes may be viewed as a continuation of my Park City notes [70]. In those notes, the focus was on elliptic curves and finite constant fields, whereas here we discuss curves of high genera and results over more general base fields.

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I welcome all comments, and I plan to maintain a list of corrections and supplements on my web page. Please check there for updates if you find the material in these notes useful.

1 Dramatis personae

1.1 Notation and standing hypotheses

Unless explicitly stated otherwise, all schemes are assumed to be Noetherian and separated and all morphisms of schemes are assumed to be separated and of finite type.

A *curve* over a field F is a scheme over F that is reduced and purely of dimension 1, and a *surface* is similarly a scheme over F which is reduced and purely of dimension 2. Usually our curves and surfaces will be subject to further hypotheses, like irreducibility, projectivity, or smoothness.

We recall that a scheme Z is *regular* if each of its local rings is regular. This means that for each point $z \in Z$, with local ring $\mathcal{O}_{Z,z}$, maximal ideal $\mathfrak{m}_z \subset \mathcal{O}_{Z,z}$, and residue field $\kappa(z) = \mathcal{O}_{Z,z}/\mathfrak{m}_z$, we have

$$\dim_{\kappa(z)} \mathfrak{m}_z/\mathfrak{m}_z^2 = \dim_z Z.$$

Equivalently, $\mathfrak{m}_z$ should be generated by $\dim_z Z$ elements.

A morphism $f\colon Z \to S$ is *smooth* (of relative dimension n) at $z \in Z$ if there exist affine open neighborhoods U of z and V of $f(z)$ such that $f(U) \subset V$ and a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] & \operatorname{Spec} R[x_1, \dots, x_{n+k}]/(f_1, \dots, f_k) \arrow[d] \\ V \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

where the horizontal arrows are open immersions and where the Jacobian matrix $(\partial f_i/\partial x_j(z))_{ij}$ has rank k. Also, f is *smooth* if it is smooth at each $z \in Z$.

Smoothness is preserved under arbitrary change of base. If $f\colon Z \to \operatorname{Spec} F$ with F a field, then f smooth at $z \in Z$ implies that z is a regular point of Z. The converse holds if F is perfect, but fails in general. See Section 2.2 below for one example. A lucid discussion of smoothness, regularity, and the relations between them may be found in [45, Ch. V].

If Y is a scheme over a field F, the notation $\overline{Y}$ will mean $Y \times_F \overline{F}$ where $\overline{F}$ is an algebraic closure of F. Letting $F^{\mathrm{sep}} \subset \overline{F}$ be a separable closure of F, we will occasionally want the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ to act on objects associated to $\overline{Y}$. To do this, we note that the Galois action on F^{sep} extends uniquely to an action on $\overline{F}$. We also note that the étale cohomology groups $H^i(Y \times_F \overline{F}, \mathbb{Q}_\ell)$ and $H^i(Y \times_F F^{\mathrm{sep}}, \mathbb{Q}_\ell)$ are canonically isomorphic [39, VI.2.6].

1.2 Base fields

Throughout, k is a field, $\mathcal{C}$ is a smooth, projective, absolutely irreducible curve over k, and $K = k(\mathcal{C})$ is the field of rational functions on $\mathcal{C}$. Thus K is a regular extension of k (i.e., K/k is separable and k is algebraically closed in K) and the transcendence degree of K over k is 1. The function fields of our title are of the form $K = k(\mathcal{C})$.

We view the base field K as more-or-less fixed, except that we are willing to make a finite extension of k if it simplifies matters. At a few places below (notably in Sections 2.1 and 2.2), we extend k for convenience, to ensure for example that we have a rational point or smoothness.

We now introduce standing notations related to the field K. Places (equivalence classes of valuations, closed points of $\mathcal{C}$) will be denoted v and for each place v we write K_v, $\mathcal{O}_v$, $\mathfrak{m}_v$, and k_v for the completion of K, its ring of integers, its maximal ideal, and its residue field respectively.

Fix once and for all separable closures K^{sep} of K and K_v^{sep} of each K_v and embeddings $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$. Let k^{sep} be the algebraic closure of k in K^{sep}; it is a separable closure of k and the embedding $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$ identifies k^{sep} with the residue field of K_v^{sep}.

We write G_K for $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and similarly for G_{K_v} and G_k. The embeddings $K^{\mathrm{sep}} \hookrightarrow K_v^{\mathrm{sep}}$ identify G_{K_v} with a decomposition group of K at v. We also write $D_v \subset G_K$ for this subgroup, and $I_v \subset D_v$ for the corresponding inertia group.

1.3 The curve X

Throughout, X will be a curve over K which is always assumed to be smooth, projective, and absolutely irreducible. Thus $K(X)$ is a regular extension of K of transcendence degree 1.

The genus of X will be denoted g_X, and since we are mostly interested in the arithmetic of the Jacobian of X, we always assume that $g_X > 0$.

We do not assume that X has a K-rational point. More quantitatively, we let δ denote the *index* of X, i.e., the gcd of the degrees of the residue field extensions $\kappa(x)/K$ as x runs over all closed points of X. Equivalently, δ is the smallest positive degree of a K-rational divisor on X. We write δ' for the *period* of X, i.e., the smallest degree of a K-rational divisor class. (In fancier language, δ' is the order of Pic^1 as an element of the Weil–Châtelet group of J_X. It is clear that δ' divides δ. It is also easy to see (by considering the divisor of a K-rational regular 1-form) that $\delta|(2g_X - 2)$. Lichtenbaum [29, Theorem 8] showed that $\delta|2\delta'^2$, and if $(2g_X - 2)/\delta$ is even, then $\delta|\delta'^2$.

Similarly, for a closed point v of $\mathcal{C}$, we write δ_v and δ'_v for the index and period of $X \times_K K_v$. It is known that $\delta'_v|(g_X - 1)$, the ratio δ/δ' is either 1 or 2, and it is 1 if $(g_X - 1)/\delta'$ is even. (These facts follow from the arguments in [29, Theorem 7] together with the duality results in [40].)

1.4 The surface $\mathcal{X}$

Given X, there is a unique surface $\mathcal{X}$ equipped with a morphism $\pi\colon \mathcal{X} \to \mathcal{C}$ with the following properties: $\mathcal{X}$ is irreducible and regular, π is proper and relatively minimal (defined below), and the generic fiber of $\mathcal{X} \to \mathcal{C}$ is $X \to \mathrm{Spec}\, K$.

Provided that we are willing to replace k with a finite extension, we may also insist that $\mathcal{X} \to \operatorname{Spec} k$ be smooth. We generally make this extension, and also if necessary extend k so that $\mathcal{X}$ has a k-rational point.

The construction of $\mathcal{X}$ starting from X and discussion of its properties (smoothness, cohomological flatness, etc.) will be given in Section 2 below.

We note that any smooth projective surface $\mathcal{S}$ over k is closely related to a surface $\mathcal{X}$ of our type. Indeed, after possibly blowing up a finite number of points, $\mathcal{S}$ admits a morphism to $\mathbb{P}^1$ whose generic fiber is a curve of our type (except that it might have genus 0). Thus an alternative point of view would be to start with the surface $\mathcal{X}$ and construct the curve X. We prefer to start with X because specifying a curve over a field can be done succinctly by specifying its field of functions.

1.5 The Jacobian J_X

We write J_X for the Jacobian of X, a g-dimensional, principally polarized abelian variety over K. It represents the relative Picard functor $\underline{\mathrm{Pic}}^0_{X/K}$. The definition of this functor and results on its representability are reviewed in Section 3.

We denote by (B, τ) the K/k-trace of J_X. By definition, this is the final object in the category of pairs (A, σ) where A is an abelian variety over k and $\sigma\colon A \times_k K \to J_X$ is a K-morphism of abelian varieties. We refer to [11] for a very complete account of the K/k-trace in the language of schemes. We will calculate (B, τ) (completely in a special case, and up to inseparable isogeny in general) in Subsection 4.2.

One of the main aims of these notes is to discuss the arithmetic of J_X, especially the Mordell–Weil group $J_X(K)/\tau B(k)$ and the Tate–Shafarevich group $\text{Ш}(J_X/K)$ as well as their connections with analogous invariants of $\mathcal{X}$.

1.6 The Néron model $\mathcal{J}_X$

We denote the Néron model of J_X over $\mathcal{C}$ by $\mathcal{J}_X \to \mathcal{C}$, so that $\mathcal{J}_X \to \mathcal{C}$ is a smooth group scheme satisfying a universal property. More precisely, for every place v of K, every K_v-valued point of J_X should extend uniquely to a section of $\mathcal{J}_X \times_{\mathcal{C}} \operatorname{Spec} \mathcal{O}_v \to \operatorname{Spec} \mathcal{O}_v$. We refer to [3] for a brief overview and [9] for a thorough treatment of Néron models.

There are many interesting results to discuss about $\mathcal{J}_X$, including a fine study of the component groups of its reduction at places of $\mathcal{C}$, monodromy results, etc. Due to constraints of time and space, we will not be able to discuss any of these, and so we will have nothing more to say about $\mathcal{J}_X$ in these notes.

1.7 Our plan

Our goal is to discuss the connections between the objects X, $\mathcal{X}$, and J_X. Specifically, we will study the arithmetic of J_X (its rational points, Tate–Shafarevich group, BSD conjecture), the arithmetic of $\mathcal{X}$ (its Néron–Severi group, Brauer group, Tate and Artin–Tate conjectures), and connections between them.

In Sections 2 and 3 we discuss the construction and first properties of $\mathcal{X} \to \mathcal{C}$ and J_X. In the following three sections, we work out the connections between the arithmetic of these objects, mostly in the case when k is finite, with the BSD and Tate conjectures as touchstones. In Section 7 we give a few complements on related situations and other ground fields. In Section 8 we review known cases of the Tate conjecture and their consequences for Jacobians, and in Section 9 we review how one may use these results to produce Jacobians with large Mordell–Weil groups.

2 Geometry of $\mathcal{X} \to \mathcal{C}$

2.1 Construction of $\mathcal{X}$

Recall that we have fixed a smooth, projective, absolutely irreducible curve X over $K = k(\mathcal{C})$ of genus $g_X > 0$.

Proposition 2.1. *There exists a regular, irreducible surface $\mathcal{X}$ over k equipped with a morphism $\pi\colon \mathcal{X} \to \mathcal{C}$ which is projective, relatively minimal, and with generic fiber $X \to \operatorname{Spec} K$. The pair $(\mathcal{X}, \pi)$ is uniquely determined by these properties. We have that $\mathcal{X}$ is absolutely irreducible and projective over k and that π is flat, $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{C}}$, and π has connected fibers.*

Proof. To show the existence of $\mathcal{X}$, we argue as follows. Choose a non-empty affine open subset $U \subset X$ and an affine model for U:

$$U = \operatorname{Spec} K\bigl[x_1, \ldots, x_m\bigr]/\bigl(f_1, \ldots, f_n\bigr).$$

Let $\mathcal{C}^0 \subset \mathcal{C}$ be a non-empty affine open where all of the coefficients of the f_i (which are elements of K) are regular functions. Let $R = \mathcal{O}_{\mathcal{C}}(\mathcal{C}^0)$, a k-algebra and Dedekind domain. Let $\mathcal{U}$ be the affine k-scheme

$$\mathcal{U} = \operatorname{Spec} R\bigl[x_1, \ldots, x_m\bigr]/\bigl(f_1, \ldots, f_n\bigr).$$

Then $\mathcal{U}$ is reduced and irreducible, and the inclusion

$$R \longrightarrow R\bigl[x_1, \ldots, x_m\bigr]/\bigl(f_1, \ldots, f_n\bigr)$$

induces a morphism $\mathcal{U} \to \mathcal{C}^0$ whose generic fiber is $U \to \operatorname{Spec} K$. Imbed $\mathcal{U}$ in some affine space $\mathbb{A}^N_k$ and let $\mathcal{X}_0$ be the closure of $\mathcal{U}$ in $\mathbb{P}^N_k$. Thus $\mathcal{X}_0$ is reduced and irreducible, but it may be quite singular.

Lipman's general result on desingularizing two-dimensional schemes (see [32] or [2]) can be used to find a non-singular model of $\mathcal{X}_0$. More precisely, normalizing $\mathcal{X}_0$ results in a scheme with isolated singularities. Let $\mathcal{X}_1$ be the result of blowing up the normalization of $\mathcal{X}_0$ once at each (reduced) closed singular point. Now inductively let $\mathcal{X}_n$ $(n \geq 2)$ be the result of normalizing $\mathcal{X}_{n-1}$ and blowing up each singular point. Lipman's theorem is that the sequence $\mathcal{X}_n$ yields a regular scheme after finitely many steps. The resulting regular projective scheme $\mathcal{X}_n$ is equipped with a rational map to $\mathcal{C}$.

After further blowing up and/or blowing down, we arrive at an irreducible, regular, two-dimensional scheme projective over k with a projective, relatively minimal morphism $\mathcal{X} \to \mathcal{C}$ whose generic fiber is $X \to \operatorname{Spec} K$. Here relatively minimal means that if $\mathcal{X}'$ is regular with a proper morphism $\mathcal{X}' \to \mathcal{C}$, and if there is a factorization

$$\mathcal{X} \xrightarrow{f} \mathcal{X}' \longrightarrow \mathcal{C}$$

with f a birational proper morphism, then f is an isomorphism. This is equivalent to there being no (-1)-curves in the fibers of π. The uniqueness of $\mathcal{X}$ (subject to the properties) follows from [28, Theorem 4.4]. Since $\mathcal{C}$ is a smooth curve, $\mathcal{X}$ is irreducible, and π is dominant, π is flat.

If k' is an extension of k over which $\mathcal{X}$ becomes reducible, then every component of $\mathcal{X} \times_k k'$ dominates $\mathcal{C} \times_k k'$. (This is because π is flat, so $\mathcal{X} \times_k k' \to \mathcal{C} \times_k k'$ is flat.) In this case, X would be reducible over $k'K$. But we assumed that X is absolutely irreducible, so $\mathcal{X}$ must also be absolutely irreducible.

By construction $\mathcal{X}$ is projective over k.

Let $\mathcal{C}'$ be $\operatorname{Spec}_{\mathcal{O}_\mathcal{C}} \pi_* \mathcal{O}_\mathcal{X}$ (global spec) so that the Stein factorization of π is $\mathcal{X} \to \mathcal{C}' \to \mathcal{C}$. Then $\mathcal{C}' \to \mathcal{C}$ is finite and flat. Let $\mathcal{C}'_\eta = \operatorname{Spec} L \to \eta = \operatorname{Spec} K$ be the generic fiber of $\mathcal{C}' \to \mathcal{C}$. Then the algebra L is finite and thus algebraic over K. On the other hand, L is contained in $k(\mathcal{X})$ and K is algebraically closed in $k(\mathcal{X})$, so we have $L = K$. Thus $\mathcal{C}' \to \mathcal{C}$ is finite flat of degree 1, and, since $\mathcal{C}$ is smooth, it must be an isomorphism. This proves that $\pi_* \mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{C}$. It follows (e.g., by [20, III.11.3]) that the fibers of π are all connected.

This completes the proof of the proposition. $\square$

For the rest of the notes, $\pi \colon \mathcal{X} \to \mathcal{C}$ will be the fibration constructed in the proposition. We will typically extend k if necessary so that $\mathcal{X}$ has a k-rational point.

2.2 Smoothness and regularity

If k is perfect, then, since $\mathcal{X}$ is regular, $\mathcal{X} \to \operatorname{Spec} k$ is automatically smooth. However, it need not be smooth if k is not perfect.

Let us consider an example. Since smoothness and regularity are local properties, we give an affine example and leave to the reader the easy task of making it projective. Let $\mathbb{F}$ be a field of characteristic $p > 0$ and let $\tilde{\mathcal{X}}$ be the closed subset of $\mathbb{A}^4_\mathbb{F}$ defined by $y^2 + xy - x^3 - (t^p + u)$. The projection $(x, y, t, u) \mapsto (x, y, t)$ induces an isomorphism $\tilde{\mathcal{X}} \to \mathbb{A}^3$ and so $\tilde{\mathcal{X}}$ is a regular scheme. Let $k = \mathbb{F}(u)$ and let $\mathcal{X} \to \operatorname{Spec} k$ be the generic fiber of the projection $\tilde{\mathcal{X}} \to \mathbb{A}^1$, $(x, y, t, u) \mapsto u$. Since the local rings of $\mathcal{X}$ are also local rings of $\tilde{\mathcal{X}}$, $\mathcal{X}$ is a regular scheme. On the other hand, $\mathcal{X} \to \operatorname{Spec} k$ is not smooth at the point $x = y = t = 0$. Now let X be the generic fiber of the projection $\mathcal{X} \to \mathbb{A}^1$, $(x, y, t) \mapsto t$. Then X is the affine scheme $\operatorname{Spec} K[x, y]/(y^2 + xy - x^3 - (t^p - u))$ and this is easily seen to be smooth over K.

This shows that the minimal regular model $\mathcal{X} \to \mathcal{C}$ of $X \to \operatorname{Spec} K$ need not be smooth over k.

If we are given $X \to \operatorname{Spec} K$ and the regular model $\mathcal{X}$ is not smooth over k, we can remedy the situation by extending k. Indeed, let $\overline{k}$ denote the algebraic closure of k, and let $\overline{\mathcal{X}} \to \overline{\mathcal{C}}$ be the regular model of $X \times_k \overline{k} \to K\overline{k}$. Then since $\overline{k}$ is perfect, $\overline{\mathcal{X}}$ is smooth over $\overline{k}$. It is clear that there is a finite extension k' of k such that $\overline{\mathcal{X}}$ is defined over k' and birational over k' to $\mathcal{X}$. So replacing k by k' we find that the regular model $\mathcal{X} \to \mathcal{C}$ of $X \to \operatorname{Spec} K$ is smooth over k.

We will generally assume below that $\mathcal{X}$ is smooth over k.

2.2.1 Correction to [70]

In [70, Lecture 2, § 2, p. 237], speaking of a two-dimensional, separated, reduced scheme of finite type over a field k, we say "Such a scheme is automatically quasi-projective and is projective if and only if it is complete." This is not correct in general – we should also assume that $\mathcal{X}$ is non-singular. In fact, when the ground field is finite, it suffices to assume that $\mathcal{X}$ is normal: see [15].

2.3 Structure of fibers

We write $\mathcal{X}_v$ for the fiber of π over the closed point v of $\mathcal{C}$. We already noted that the fibers $\mathcal{X}_v$ are connected.

We next define certain multiplicities of components, following [9, Ch. 9]. Let $\mathcal{X}_{v,i}$, $i = 1, \ldots, r$ be the reduced irreducible components of $\mathcal{X}_v$ and let $\eta_{v,i}$ be the corresponding generic points of $\mathcal{X}_v$. Let $\overline{k}_v$ be an algebraic closure of the residue field at v and write $\overline{\mathcal{X}}_v$ for $\mathcal{X}_v \times_{k_v} \overline{k}_v$ and $\overline{\eta}_{v,i}$ for a point of $\overline{\mathcal{X}}_v$ over $\eta_{v,i}$. We define the *multiplicity* of $X_{v,i}$ in X_v to be the length of the Artin local ring $\mathcal{O}_{X_v,\eta_{v,i}}$, and the *geometric multiplicity* of $X_{v,i}$ in X_v to be the length of $\mathcal{O}_{\overline{\mathcal{X}}_v,\overline{\eta}_{v,i}}$. The geometric multiplicity is equal to the multiplicity when the characteristic of k is zero and it is a power of p times the multiplicity if the characteristic of k is $p > 0$.

We write $\mathcal{X}_v = \sum_i m_{v,i}\mathcal{X}_{v,i}$ where $m_{v,i}$ is the multiplicity of $\mathcal{X}_{v,i}$ in $\mathcal{X}_v$. This is an equality of Cartier divisors on $\mathcal{X}$. We define the *multiplicity* m_v of the fiber $\mathcal{X}_v$ to be the gcd of the multiplicities $m_{v,i}$.

The multiplicity m_v divides the gcd of the geometric multiplicities of the components of $\mathcal{X}_v$ which in turn divides the index δ_v of $\mathcal{X}_v$. In particular, if X has a K-rational point (so that $\mathcal{X} \to \mathcal{C}$ has a section) then for every v we have $m_v = 1$.

We now turn to the combinatorial structure of the fiber $\mathcal{X}_v$. A convenient reference for what follows is [33, Ch. 9].

We write $D.D'$ for the intersection multiplicity of two divisors on $\mathcal{X}$. It is known [33, 9.1.23] that the intersection form restricted to the divisors supported in a single fiber $\mathcal{X}_v$ is negative semi-definite, and that its kernel consists exactly of the divisors which are rational multiples of the entire fiber. (Thus if the multiplicity of the fiber $\mathcal{X}_v$ is m_v, then $(1/m_v)\mathcal{X}_v := \sum_i (m_{v,i}/m_v)\mathcal{X}_{v,i}$ generates the kernel of the pairing.)

It is in principle possible to use this result to give a classification of the possible combinatorial types of fibers (genera and multiplicities of components, intersection numbers) for a fixed value of g_X. Up to a suitable equivalence relation, the set of possibilities for a given value of g_X is finite [5]. When X is an elliptic curve and the residue field is assumed perfect, this is the well-known Kodaira–Néron classification. For higher genus, the situation rapidly becomes intractable. We note that the list of possibilities can be strictly longer when one does not assume that the residue field k_v is perfect. See [59] for a complete analysis of the case where X is an elliptic curve.

2.4 Leray spectral sequence

Fix a prime $\ell \neq \mathrm{Char}(k)$. We consider the Leray spectral sequence for $\pi\colon \overline{\mathcal{X}} \to \overline{\mathcal{C}}$ in ℓ-adic cohomology. The E_2 term

$$E_2^{pq} = H^p\big(\overline{\mathcal{C}}, R^q\pi_*\mathbb{Q}_\ell\big)$$

vanishes outside the range $0 \leq p, q \leq 2$ and so the only possibly non-zero differentials are d_2^{01} and d_2^{02}. We will show that these both vanish and so the sequence degenerates at E_2.

The differential d_2^{01} sits in an exact sequence of low degree terms that includes

$$H^0\big(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Q}_\ell\big) \xrightarrow{d_2^{01}} H^2\big(\overline{\mathcal{C}}, \pi_*\mathbb{Q}_\ell\big) \longrightarrow H^2\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell\big).$$

Now $\pi_*\mathbb{Q}_\ell = \mathbb{Q}_\ell$, and the edge morphism $H^2(\overline{\mathcal{C}}, \mathbb{Q}_\ell) \to H^2(\overline{\mathcal{X}}, \mathbb{Q}_\ell)$ is simply the pull-back π^*. If $i\colon \mathcal{D} \hookrightarrow \mathcal{X}$ is a multisection of degree n and $j = \pi i$, then we have a factorization of j^*:

$$H^2\big(\overline{\mathcal{C}}, \mathbb{Q}_\ell\big) \longrightarrow H^2\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell\big) \longrightarrow H^2\big(\overline{\mathcal{D}}, \mathbb{Q}_\ell\big)$$

as well as a trace map

$$H^2\big(\overline{\mathcal{D}}, \mathbb{Q}_\ell\big) \longrightarrow H^2\big(\overline{\mathcal{C}}, \mathbb{Q}_\ell\big)$$

for the finite morphism $j\colon \mathcal{D} \to \mathcal{C}$. The composition j^* followed by the trace map is just multiplication by n, which is injective, and therefore $H^2(\overline{\mathcal{C}}, \mathbb{Q}_\ell) \to H^2(\overline{\mathcal{X}}, \mathbb{Q}_\ell)$ is also injective, which implies that $d_2^{01} = 0$.

Now consider d_2^{02}, which sits in an exact sequence

$$H^2\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell\big) \longrightarrow H^0\big(\overline{\mathcal{C}}, R^2\pi_*\mathbb{Q}_\ell\big) \xrightarrow{d_2^{02}} H^2\big(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Q}_\ell\big).$$

We can deduce that it too is zero by using duality, or by the following argument, which unfortunately uses terminology not introduced until Section 5. The careful reader will have no trouble checking that there is no circularity.

Here is the argument: Away from the reducible fibers of π, the sheaf $R^2\pi_*\mathbb{Q}_\ell$ is locally constant of rank 1. At a closed point $\overline{v}$ of $\overline{\mathcal{C}}$ where the fiber of π is reducible, the stalk of $R^2\pi_*\mathbb{Q}_\ell$ has rank $f_{\overline{v}}$, the number of components in the fiber. The cycle class of a component of a fiber in $H^2(\overline{\mathcal{X}}, \mathbb{Q}_\ell)$ maps onto the corresponding section in $H^0(\overline{\mathcal{C}}, R^2\pi_*\mathbb{Q}_\ell)$ and so $H^2(\overline{\mathcal{X}}, \mathbb{Q}_\ell) \to H^0(\overline{\mathcal{C}}, R^2\pi_*\mathbb{Q}_\ell)$ is surjective. This implies that $d_2^{02} = 0$, as desired.

For later use, we record the exact sequence of low degree terms (where the zero on the right is because $d_2^{01} = 0$):

$$0 \longrightarrow H^1\big(\overline{\mathcal{C}}, \mathbb{Q}_\ell\big) \longrightarrow H^1\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell\big) \longrightarrow H^0\big(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Q}_\ell\big) \longrightarrow 0. \tag{2.1}$$

We end this subsection by noting a useful property of $R^1\pi_*\mathbb{Q}_\ell$ when k is finite.

Proposition 2.2. *Suppose that k is finite and $\mathcal{X}$ is a smooth, proper surface over k equipped with a flat, generically smooth, proper morphism $\pi\colon \mathcal{X} \to \mathcal{C}$. Let ℓ be a prime not equal to the characteristic of k and let $\mathcal{F} = R^1\pi_*\mathbb{Q}_\ell$, a constructible ℓ=adic sheaf on $\mathcal{C}$. If $j\colon \eta \hookrightarrow \mathcal{C}$ is the inclusion of the generic point, then the canonical morphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism.*

Remark 2.3. The same proposition holds if we let $j\colon U \hookrightarrow \mathcal{C}$ be the inclusion of a non-empty open subset over which π is smooth. Sheaves with this property are sometimes called "middle extension" sheaves. It is also useful to note that for $\mathcal{F}$ and j as above, we have an isomorphism

$$H^1\big(\overline{\mathcal{C}}, \mathcal{F}\big) \cong \operatorname{Im}\big(H^1_c\big(\overline{U}, j^*\mathcal{F}\big) \longrightarrow H^1\big(\overline{U}, j^*\mathcal{F}\big)\big)$$

(image of compactly supported cohomology in usual cohomology). This is "well known" but the only reference I know is [67, 7.1.6].

Proof of Proposition 2.2. We will show that for every geometric point $\overline{x}$ over a closed point x of $\mathcal{C}$, the stalks of $\mathcal{F}$ and $j_*j^*\mathcal{F}$ are isomorphic. (Note that the latter is the group of invariants of inertia at x in the Galois module $\mathcal{F}_{\overline{\eta}}$.)

The local invariant cycle theorem (Theorem 3.6.1 of [13]) says that the map of stalks $\mathcal{F}_{\overline{x}} \to (j_*j^*\mathcal{F})_{\overline{x}}$ is surjective. A closer examination of the proof shows that it is also injective in our situation. Indeed, in the diagram (8) on p. 214 of [13], the group on the left $H^0(X_{\overline{\eta}})_I(-1)$ is pure of weight 2, whereas $H^1(X_s)$ is mixed of weight ≤ 1; also the preceding term in the vertical sequence is (with $\mathbb{Q}_\ell$ coefficients) dual to $H^3(X_s)(2)$ and this vanishes for dimension reasons. Thus the diagonal map sp^* is also injective.

(Note the paragraph after the display (8): this argument only works with $\mathbb{Q}_\ell$ coefficients, not necessarily with $\mathbb{Z}_\ell$ coefficients.) $\square$

See [25, Proposition 7.5.2] for a more general result with a similar proof.

We have a slightly weaker result in integral cohomology.

Corollary 2.4. *Under the hypotheses of the proposition, the natural map*

$$R^1\pi_*\mathbb{Z}_\ell \longrightarrow j_*j^*R^1\pi_*\mathbb{Z}_\ell$$

has kernel and cokernel supported at finitely many closed points of $\mathcal{C}$ and with finite stalks. Thus the induced map

$$H^i\big(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell\big) \longrightarrow H^i\big(\overline{\mathcal{C}}, j_*j^*R^1\pi_*\mathbb{Z}_\ell\big)$$

is an isomorphism for $i > 1$ and surjective with finite kernel for $i = 1$.

Proof. The proof of the Proposition shows that the kernel and cokernel are supported at points where π is not smooth, a finite set of closed points. Also, the Proposition shows that the stalks are torsion. Since they are also finitely generated, they must be finite. □

2.5 Cohomological flatness

Since $\pi\colon \mathcal{X} \to \mathcal{C}$ is flat, general results on cohomology and base change (e.g., [20, III.12]) imply that the coherent Euler characteristic of the fibers of π is constant, i.e., the function $v \mapsto \chi(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v})$ is constant on $\mathcal{C}$. Moreover, the dimensions of the individual cohomology groups $h^i(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v})$ are upper semi-continuous. They are not in general locally constant.

To make this more precise, we recall a standard exact sequence from the theory of cohomology and base change:

$$0 \longrightarrow \big(R^i\pi_*\mathcal{O}_{\mathcal{X}}\big) \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(v) \longrightarrow H^i\big(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v}\big) \longrightarrow \big(R^{i+1}\pi_*\mathcal{O}_{\mathcal{X}}\big)\big[\varpi_v\big] \longrightarrow 0.$$

Here $\mathcal{X}_v$ is the fiber of π at v, the left-hand group is the fiber of the coherent sheaf $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ at v, and the right-hand group is the ϖ_v-torsion in $R^{i+1}\pi_*\mathcal{O}_{\mathcal{X}}$ where ϖ_v is a generator of $\mathfrak{m}_v$. Since $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ is coherent, the function

$$v \longmapsto \dim_{\kappa(v)} \big(R^i\pi_*\mathcal{O}_{\mathcal{X}}\big) \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(v)$$

is upper semi-continuous, and it is locally constant if and only if $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ is locally free. Thus the obstruction to $v \mapsto h^i(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v})$ being locally constant is controlled by torsion in $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ and $R^{i+1}\pi_*\mathcal{O}_{\mathcal{X}}$.

We say that π is *cohomologically flat in dimension i* if formation of $R^i\pi_*\mathcal{O}_{\mathcal{X}}$ commutes with arbitrary change of base, i.e., if for all $\phi\colon T \to \mathcal{C}$ the base change morphism

$$\phi^*R^i\pi_*\mathcal{O}_{\mathcal{X}} \longrightarrow R^i\pi_T\mathcal{O}_{\mathcal{T}\times_{\mathcal{C}}\mathcal{X}}$$

is an isomorphism. Because the base $\mathcal{C}$ is a smooth curve, this is equivalent to the same condition where $T \to \mathcal{C}$ runs through inclusions of closed point, i.e., to the condition that

$$\big(R^i\pi_*\mathcal{O}_{\mathcal{X}}\big) \otimes_{\mathcal{O}_{\mathcal{C}}} \kappa(v) \longrightarrow H^i\big(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v}\big)$$

be an isomorphism for all closed points $v \in \mathcal{C}$. By the exact sequence above, this is equivalent to $R^{i+1}\pi_*\mathcal{O}_\mathcal{X}$ being torsion free, and thus locally free.

Since $\pi\colon \mathcal{X} \to \mathcal{C}$ has relative dimension 1, $R^{i+1}\pi_*\mathcal{O}_\mathcal{X} = 0$ for $i \geq 1$ and π is automatically cohomologically flat in dimension $i \geq 1$. It is cohomologically flat in dimension 0 if and only if $R^1\pi_*\mathcal{O}_\mathcal{X}$ is locally free. To lighten terminology, in this case we simply say that π is cohomologically flat.

Since $\pi_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{C}$ is free of rank 1, we have that π is cohomologically flat if and only if $v \mapsto h^i(\mathcal{X}_v, \mathcal{O}_{\mathcal{X}_v})$ is locally constant (and thus constant since $\mathcal{C}$ is connected) for $i = 0, 1$. Obviously the common value is 1 for $i = 0$ and g_X for $i = 1$.

Raynaud gave a criterion for cohomological flatness in [50, 7.2.1]. Under our hypotheses ($\mathcal{X}$ regular, $\mathcal{C}$ a smooth curve over k, and $\pi\colon \mathcal{X} \to \mathcal{C}$ proper and flat with $\pi_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{C}$), π is cohomologically flat if $\mathrm{Char}(k) = 0$ or if k is perfect and the following condition holds: For each closed point v of $\mathcal{C}$, let d_v be the gcd of the geometric multiplicities of the components of $\mathcal{X}_v$. The condition is that d_v is prime to $p = \mathrm{Char}(k)$ for all v.

Raynaud also gave an example of non-cohomological flatness which we will make more explicit below. Namely, let S be a complete DVR with fraction field F and algebraically closed residue field of characteristic $p > 0$, and let $\mathcal{E}$ be an elliptic curve over S with either multiplicative or good supersingular reduction. Let Y be a principal homogeneous space for $E = \mathcal{E} \times \operatorname{Spec} F$ of order p^e $(e > 0)$ and let $\mathcal{Y}$ be a minimal regular model for Y over S. Then $\mathcal{Y} \to \operatorname{Spec} S$ is not cohomologically flat. Moreover, the invariant δ defined as above is p^e. It follows from later work [34, Theorem 6.6] that the special fiber of $\mathcal{Y}$ is like that of $\mathcal{E}$, but with multiplicity p^e. The explicit example below should serve to make the meaning of this clear.

2.6 Example

Let $k = \mathbb{F}_2$ and $\mathcal{C} = \mathbb{P}^1$, so that $K = \mathbb{F}_2(t)$. We give an example of a curve of genus 1 over $\mathbb{A}^1_K$ which is not cohomologically flat at $t = 0$.

Consider the elliptic curve E over K given by

$$y^2 + xy = x^3 + tx.$$

The point $P = (0, 0)$ is 2-torsion. The discriminant of this model is t^2 so E has good reduction away from the place $t = 0$ of K. At $t = 0$, E has split multiplicative reduction with minimal regular model of type I_2.

The quotient of E by the subgroup generated by P is $\phi\colon E \to E'$, where E' is given by

$$s^2 + rs = r^3 + t.$$

One checks that the degree of the conductor of E' is 4 and so, by [70, Lecture 1, Theorem 9.3 and Theorem 12.1(1)], the rank of $E'(K)$ is zero. Also, $E'(K)$ has no 2-torsion.

Therefore, taking cohomology of the sequence

$$0 \longrightarrow E[\phi] \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

we find that the map

$$K/\wp(K) \cong H^1\big(K, \mathbb{Z}/2\mathbb{Z}\big) \cong H^1\big(E, E[\phi]\big) \longrightarrow H^1\big(K, E\big)$$

is injective. For $f \in K$, we write X_f for the torsor for E obtained from the class of f in $K/\wp(K)$ via this map.

Let L be the quadratic extension of K determined by f, i.e.,

$$L = K[u]/\big(\wp(u) - f\big).$$

The action of $G = \mathrm{Gal}(L/K) = \mathbb{Z}/2\mathbb{Z}$ on L is $u \mapsto u+1$. Let G act on $K(E)$ by translation by P. Explicitly, one finds

$$\big(x, y\big) \longmapsto \big(t/x, t\big(x+y\big)/x^2\big)$$

and so $y/x \mapsto y/x + 1$.

The function field of X_f is the field of G-invariants in $L(E)$ where G acts as above. One finds that

$$K(X_f) = \frac{K(r)[s,z]}{(s^2 + rs + r^3 + t, z^2 + z + r + f)}$$

which presents X_f as a double cover of E'.

Let us now specialize to $f = t^{-1}$ and find a minimal regular model of X_f over $R = \mathbb{F}_2[t]$. Let

$$\mathcal{U} = \mathrm{Spec}\,\frac{R[r,s,w]}{(s^2 + rs + r^3 + t, w^2 + tw + t^2 r + t)}.$$

Then $\mathcal{U}$ is regular away from $t = r = s = w = 0$ and its generic fiber is isomorphic (via $w = tz$) to an open subset of X_f. Let

$$\mathcal{V} = \mathrm{Spec}\,\frac{R[r', s', w']}{(s' + r's' + r'^3 + ts'^3, w'^2 + tw' + t^3 r' s' + t)},$$

which is a regular scheme whose generic fiber is another open subset of X_f. Let $\mathcal{Y}_f$ be the result of gluing $\mathcal{U}$ and $\mathcal{V}$ via $(r,s,w) = (r'/s', 1/s', w' + tr'^2/s')$. The generic fiber of $\mathcal{Y}_f$ is X_f and $\mathcal{Y}_f$ is regular away from $r = s = w = t = 0$. Note that the special fiber of $\mathcal{Y}_f$ is isomorphic to the product of the doubled point $\mathrm{Spec}\,\mathbb{F}_2[w]/(w^2)$ with the projective nodal plane cubic

$$\mathrm{Proj}\,\frac{\mathbb{F}_2[r,s,v]}{(s^2 v + rsv + r^3)}.$$

In particular, the special fiber over $t = 0$, call it $\mathcal{Y}_{f,0}$, satisfies $H^0(\mathcal{Y}_{f,0}, \mathcal{O}_{\mathcal{Y}_{f,0}}) = \mathbb{F}_2[w]/(w^2)$. This shows that $\mathcal{Y}_f$ is not cohomologically flat over $\mathcal{C}$ at $t = 0$.

To finish the example, we should blow up $\mathcal{Y}_f$ at its unique non-regular point. The resulting scheme $\mathcal{X}_f$ is regular and flat over R, but it is not cohomologically flat at $t = 0$. The fiber over $t = 0$ is the product of a double point and a Néron configuration of type I_2, and its global regular functions are $\mathbb{F}_2[w]/(w^2)$. This is in agreement with [34, 6.6].

We will re-use parts of this example below.

Exercise 2.5. By the earlier discussion, $R^1\pi_*\mathcal{O}_{\mathcal{X}_f}$ has torsion at $t = 0$. Make this explicit.

3 Properties of J_X

3.1 Review of Picard functors

We quickly review basic material on the relative Picard functor. The original sources [18] and [50] are still very much worth reading. Two excellent modern references with more details and historical comments are [9] and [26].

3.1.1 The relative Picard functor

For any scheme $\mathcal{Y}$, we write $\mathrm{Pic}(\mathcal{Y})$ for the *Picard group* of $\mathcal{Y}$, i.e., for the group of isomorphism classes of invertible sheaves on $\mathcal{Y}$. This group can be calculated cohomologically: $\mathrm{Pic}(\mathcal{Y}) \cong H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^\times)$ (cohomology computed in the Zariski, étale, or finer topologies).

Now fix a morphism of schemes $f\colon \mathcal{Y} \to \mathcal{S}$ (separated and of finite type, as always). If $\mathcal{T} \to \mathcal{S}$ is a morphism of schemes, we write $\mathcal{Y}_\mathcal{T}$ for $\mathcal{Y} \times_\mathcal{S} \mathcal{T}$ and $f_\mathcal{T}$ for the projection $\mathcal{Y}_\mathcal{T} \to \mathcal{T}$. Define a functor $P_{\mathcal{Y}/\mathcal{S}}$ from schemes over $\mathcal{S}$ to abelian groups by the rule

$$\mathcal{T} \longmapsto P_{\mathcal{Y}/\mathcal{S}}(\mathcal{T}) = \frac{\mathrm{Pic}(\mathcal{Y}_\mathcal{T})}{f_\mathcal{T}^* \mathrm{Pic}(\mathcal{T})}.$$

We define the *relative Picard functor* $\underline{\mathrm{Pic}}_{\mathcal{Y}/\mathcal{S}}$ to be the fppf sheaf associated to $P_{\mathcal{Y}/\mathcal{S}}$. (Here "fppf" means "faithfully flat and finitely presented". See [9, 8.1] for details on the process of sheafification.) Explicitly, if $\mathcal{T}$ is affine, an element of $\underline{\mathrm{Pic}}_{\mathcal{Y}/\mathcal{S}}(\mathcal{T})$ is represented by a line bundle $\xi' \in \mathrm{Pic}(\mathcal{Y} \times_\mathcal{S} \mathcal{T}')$ where $\mathcal{T}' \to \mathcal{T}$ is fppf, subject to the condition that there should exist an fppf morphism $\tilde{\mathcal{T}} \to \mathcal{T}' \times_\mathcal{T} \mathcal{T}'$ such that the pull backs of ξ' via the two projections

$$\tilde{\mathcal{T}} \longrightarrow \mathcal{T}' \times_\mathcal{T} \mathcal{T}' \rightrightarrows \mathcal{T}'$$

are isomorphic. Two such elements $\xi_i \in \mathrm{Pic}(\mathcal{Y} \times_\mathcal{S} \mathcal{T}_i')$ represent the same element of $\underline{\mathrm{Pic}}_{\mathcal{Y}/\mathcal{S}}(\mathcal{T})$ if and only if there is an fppf morphism $\tilde{\mathcal{T}} \to \mathcal{T}_1' \times_\mathcal{T} \mathcal{T}_2'$ such that the pull-backs of the ξ_i to $\tilde{\mathcal{T}}$ via the two projections are isomorphic. Fortunately, under mild hypotheses, this can be simplified quite a bit!

Assume that $f_*\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{S}$ universally. This means that for all $\mathcal{T} \to \mathcal{S}$ we have $f_{\mathcal{T}*}\mathcal{O}_{\mathcal{Y}_\mathcal{T}} = \mathcal{O}_\mathcal{T}$. Equivalently, $f_*\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{S}$ and f is cohomologically flat in dimension 0. In this case, for all $\mathcal{T} \to \mathcal{S}$, we have an exact sequence

$$0 \longrightarrow \operatorname{Pic}\big(\mathcal{T}\big) \longrightarrow \operatorname{Pic}\big(\mathcal{Y}_\mathcal{T}\big) \longrightarrow \underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}\big(\mathcal{T}\big) \longrightarrow \operatorname{Br}\big(\mathcal{T}\big) \longrightarrow \operatorname{Br}\big(\mathcal{Y}_\mathcal{T}\big). \tag{3.1}$$

(This is the exact sequence of low-degree terms in the Leray spectral sequence for $f\colon \mathcal{Y}_\mathcal{T} \to \mathcal{T}$, computed with respect to the fppf topology.) Here the groups $\operatorname{Br}(\mathcal{T}) = H^2(\mathcal{T}, \mathcal{O}_\mathcal{T}^\times)$ and $\operatorname{Br}(\mathcal{Y}_\mathcal{T}) = H^2(\mathcal{Y}_\mathcal{T}, \mathcal{O}_{\mathcal{Y}_\mathcal{T}}^\times)$ are the cohomological Brauer groups of $\mathcal{T}$ and $\mathcal{Y}_\mathcal{T}$, again computed with the fppf topology. (It is known that the étale topology gives the same groups.) See [9, 8.1] for the assertions in this paragraph and the next.

In case f has a section, we get a short exact sequence

$$0 \longrightarrow \operatorname{Pic}(\mathcal{T}) \longrightarrow \operatorname{Pic}(\mathcal{Y}_\mathcal{T}) \longrightarrow \underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}(\mathcal{T}) \longrightarrow 0$$

and so in this case

$$\underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}\big(T\big) = P_{\mathcal{Y}/\mathcal{S}}\big(T\big) = \frac{\operatorname{Pic}(\mathcal{Y}_\mathcal{T})}{f_\mathcal{T}^* \operatorname{Pic}(\mathcal{T})}.$$

3.1.2 Representability and $\mathbf{Pic^0}$ over a field

The simplest representability results will be sufficient for many of our purposes.

To say that $\underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}$ is represented by a scheme $\operatorname{Pic}_{\mathcal{Y}/\mathcal{S}}$ means that, for all S-schemes $\mathcal{T}$,

$$\underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}\big(\mathcal{T}\big) = \operatorname{Pic}_{\mathcal{Y}/\mathcal{S}}\big(\mathcal{T}\big) = \operatorname{Mor}_\mathcal{S}\big(\mathcal{T}, \operatorname{Pic}_{\mathcal{Y}/\mathcal{S}}\big).$$

Suppose $\mathcal{S}$ is the spectrum of a field and $\mathcal{Y} \to \mathcal{S}$ is proper. Then $\underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}$ is represented by a scheme $\operatorname{Pic}_{\mathcal{Y}/\mathcal{S}}$ which is locally of finite type over $\mathcal{S}$ [9, 8.2, Theorem 3]. The connected component of this group scheme will be denoted by $\operatorname{Pic}^0_{\mathcal{Y}/\mathcal{S}}$. If $\mathcal{Y} \to \mathcal{S}$ is smooth and geometrically irreducible, then $\operatorname{Pic}^0_{\mathcal{Y}/\mathcal{S}}$ is proper [9, 8.4, Theorem 3].

The results of the previous paragraph apply in particular to X/K and $\mathcal{X}/k$. Moreover, since X/K is a curve, $H^2(X, \mathcal{O}_X) = 0$ and so $\operatorname{Pic}^0_{X/K}$ is smooth and hence an abelian variety [9, 8.4, Proposition 2] (plus our assumption that morphisms are of finite type to convert formal smoothness into smoothness) or [26, 9.5.19].

In general $\operatorname{Pic}_{\mathcal{X}/k}$ need not be reduced and so need not be smooth over k. If k has characteristic zero or $H^2(\mathcal{X}, \mathcal{O}_\mathcal{X}) = 0$, then $\operatorname{Pic}^0_{\mathcal{X}/k}$ is again an abelian variety. We define

$$\operatorname{PicVar}_{\mathcal{X}/k} = \Big(\operatorname{Pic}^0_{\mathcal{X}/k}\Big)_{\mathrm{red}},$$

the *Picard variety* of $\mathcal{X}$, which is an abelian variety. See [53] and [44] for an analysis of non-reduced Picard schemes and [31] for more in the case of surfaces.

Since we have assumed that k is large enough so that $\mathcal{X}$ has a rational point, for all k-schemes $\mathcal{T}$ we have

$$\operatorname{Pic}^0_{\mathcal{X}/k}(\mathcal{T}) = \frac{\operatorname{Pic}^0(\mathcal{X}_{\mathcal{T}})}{\pi_T^* \operatorname{Pic}^0(\mathcal{T})}.$$

3.1.3 More general bases

For any $\mathcal{Y} \to \mathcal{S}$ which is proper, define $\underline{\operatorname{Pic}}^0_{\mathcal{Y}/\mathcal{S}}$ to be the subfunctor of $\underline{\operatorname{Pic}}_{\mathcal{Y}/\mathcal{S}}$ consisting of elements whose restrictions to fibers $\mathcal{Y}_s$, $s \in \mathcal{S}$, lie in $\operatorname{Pic}^0_{\mathcal{Y}_s/\kappa(s)}$.

We need a deeper result to handle $\mathcal{X} \to \mathcal{C}$. Namely, assume that $\mathcal{Y}$ is regular, $\mathcal{S}$ is one-dimensional and regular, $f\colon \mathcal{Y} \to \mathcal{S}$ is flat, and projective of relative dimension 1 such that $f_*\mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathcal{C}}$. Over the open subset of $\mathcal{S}$ where f is smooth, $\underline{\operatorname{Pic}}^0_{\mathcal{Y}/\mathcal{S}}$ is represented by an abelian scheme [9, 9.4, Proposition 4].

Over all of $\mathcal{S}$, we cannot hope for reasonable representability results unless we make further hypotheses on f. If we assume that each fiber of f has the property that the gcd of the geometric multiplicities of its irreducible components is 1, then $\underline{\operatorname{Pic}}^0_{\mathcal{Y}/\mathcal{S}}$ is represented by a separated $\mathcal{S}$-scheme [9, 9.4, Theorem 2].

3.1.4 Example

Here is an explicit example where an element of $\operatorname{Pic}_{X/K}(K)$ is not represented by an element of $\operatorname{Pic}(X)$.

Let E/K be the elliptic curve of Section 2.6 and let X_f be the homogeneous space for E as above, where $f \in K$ will be selected later. Then E is the Jacobian of X and so, by the results quoted in Subsection 3.1.2, E represents the functor $\underline{\operatorname{Pic}}^0_{X/K}$. Let us consider the point $P = (0,0)$ above in $E(K) = \operatorname{Pic}^0_{X/K}(K)$ and show that, for many choices of f, this class is not represented by an invertible sheaf on $X = X_f$. Equivalently, we want to show that P does not go to 0 in $\operatorname{Br}(K)$ in the sequence (3.1) above.

The image of P in $\operatorname{Br}(K)$ is given by a pairing studied by Lichtenbaum [29]. More precisely, consider the image of P under the coboundary

$$E(K) \longrightarrow H^1\big(K, E'[\phi^\vee]\big) \cong H^1\big(K, \mu_2\big) \cong K^\times/K^{\times 2}.$$

By Kramer's results on 2-descent in characteristic 2 [27, 1.1b], the image of P is the class of t. On the other hand, suppose that $X = X_f$ is the torsor for E corresponding to $f \in K/\wp(K)$. Then the local invariant at v of the image of P in $\operatorname{Br}(K)$ is given by

$$\frac{1}{2}\operatorname{Res}_v\left(f\frac{dt}{t}\right) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset \operatorname{Br}\big(K_v\big).$$

Thus, for example, if we take $f = 1/(t-1)$, then we get an element of the Brauer group ramified at 1 and ∞ and so the class of P does not come from $\operatorname{Pic}(X_f)$.

3.2 The Jacobian

By definition, the Jacobian J_X of X is the group scheme $\mathrm{Pic}^0_{X/K}$. Because X is a smooth, projective curve, J_X is an abelian variety of dimension g_X where g_X is the genus of X, and it is equipped with a canonical principal polarization given by the theta divisor. We refer to [41] for more on the basic properties of J_X, including the Albanese property and autoduality.

3.3 The K/k trace of J_X

Recall that (B,τ) denotes the K/k trace of J_X, which by definition is a final object in the category of pairs (A,σ) where A is a k-abelian variety and $\sigma\colon A\times_k K\to J_X$ is a K-morphism of abelian varieties.

We refer to [11] for a discussion in modern language of the existence and basic properties of (B,τ). In particular, Conrad proves that (B,τ) exists, it has good base change properties, and (when K/k is regular, as it is in our case) τ has finite connected kernel with connected Cartier dual. In particular, τ is purely inseparable.

3.4 The Lang–Néron theorem

Define the Mordell–Weil group $\mathrm{MW}(J_X)$ as

$$\mathrm{MW}\left(J_X\right)=\frac{J_X(K)}{\tau B(k)},$$

where as usual (B,τ) is the K/k trace of J_X. (In view of the theorem below, perhaps this would be better called the Lang–Néron group.)

Generalizing the classical Mordell–Weil theorem, we have the following finiteness result, proven independently by Lang and Néron.

Theorem 3.1. $\mathrm{MW}(J_X)$ *is a finitely generated abelian group.*

Note that when k is finitely generated, $B(k)$ is finitely generated as well, and so $J_X(K)$ is itself a finitely generated abelian group. For large fields k, $B(k)$ may not be finitely generated, and so it really is necessary to quotient by $\tau B(k)$ in order to get a finitely generated abelian group.

We refer to [11] for a proof of the theorem in modern language. Roughly speaking, the proof there follows the general lines of the usual proof of the Mordell–Weil theorem for an abelian variety over a global field: One shows that $\mathrm{MW}(J_X)/n$ is finite by embedding it in a suitable cohomology group, and then uses a theory of heights to deduce finite generation of $\mathrm{MW}(J_X)$.

Another proof proceeds by relating $\mathrm{MW}(J_X)$ to the Néron–Severi group $\mathrm{NS}(\mathcal{X})$ (this is the Shioda–Tate isomorphism discussed in the next section) and then proving that $\mathrm{NS}(\mathcal{X})$ is finitely generated. The latter was proven by Kleiman in [55, XIII].

4 Shioda–Tate and heights

4.1 Points and curves

We write $\mathrm{Div}(\mathcal{X})$ for the group of (Cartier or Weil) divisors on $\mathcal{X}$ and similarly with $\mathrm{Div}(X)$. A prime divisor on $\mathcal{X}$ is *horizontal* if it is flat over $\mathcal{C}$ and *vertical* if it is contained in a fiber of π. The group $\mathrm{Div}(\mathcal{X})$ is the direct sum of its subgroups $\mathrm{Div}^{\mathrm{hor}}(\mathcal{X})$ and $\mathrm{Div}^{\mathrm{vert}}(\mathcal{X})$ generated respectively by horizontal and vertical prime divisors.

Restriction of divisors to the generic fiber of π induces a homomorphism

$$\mathrm{Div}(\mathcal{X}) \longrightarrow \mathrm{Div}(X)$$

whose kernel is $\mathrm{Div}^{\mathrm{vert}}(\mathcal{X})$ and which induces an isomorphism

$$\mathrm{Div}^{\mathrm{hor}}(\mathcal{X}) \cong \mathrm{Div}(X).$$

The inverse of this isomorphism sends a closed point of X to its scheme-theoretic closure in $\mathcal{X}$.

We define a filtration of $\mathrm{Div}(\mathcal{X})$ by declaring that $L^1 \mathrm{Div}(\mathcal{X})$ be the subgroup of $\mathrm{Div}(\mathcal{X})$ consisting of divisors whose restriction to X has degree 0, and by declaring that $L^2 \mathrm{Div}(\mathcal{X}) = \mathrm{Div}^{\mathrm{vert}}(\mathcal{X})$.

We define $L^i \mathrm{Pic}(\mathcal{X})$ to be the image of $L^i \mathrm{Div}(\mathcal{X})$ in $\mathrm{Pic}(\mathcal{X})$. Also, recall that $\mathrm{Pic}^0(\mathcal{X}) = \mathrm{Pic}^0_{\mathcal{X}/k}(k)$ is the group of invertible sheaves which are algebraically equivalent to zero (equivalence over $\overline{k}$ as usual).

Recall that the Néron–Severi group $\mathrm{NS}(\overline{\mathcal{X}})$ is $\mathrm{Pic}(\overline{\mathcal{X}})/\mathrm{Pic}^0(\overline{\mathcal{X}})$ and $\mathrm{NS}(\mathcal{X})$ is by definition the image of $\mathrm{Pic}(\mathcal{X})$ in $\mathrm{NS}(\overline{\mathcal{X}})$. We define $L^i \mathrm{NS}(\mathcal{X})$ as the image of $L^i \mathrm{Pic}(\mathcal{X})$ in $\mathrm{NS}(\mathcal{X})$.

It is obvious that $\mathrm{NS}(\mathcal{X})/L^1 \mathrm{NS}(\mathcal{X})$ is an infinite cyclic group; as generator we may take the class of a horizontal divisor of total degree δ over $\mathcal{C}$, where δ is the index of X.

Recall that the intersection pairing restricted to the group generated by the components of a fiber $\mathcal{X}_v$ is negative semi-definite, with kernel $(1/m_v)\mathcal{X}_v$. From this one deduces that $L^2 \mathrm{NS}(\mathcal{X})$ is the group generated by the irreducible components of fibers, with relations $\mathcal{X}_v = \mathcal{X}_{v'}$ for any two closed points v and v'. Note that if for some $v \neq v'$ we have $m = \gcd(m_v, m_{v'}) > 1$, then $L^2 \mathrm{NS}(\mathcal{X})$ has non-trivial torsion: $m((1/m)\mathcal{X}_v - (1/m)\mathcal{X}_{v'}) = 0$ in $\mathrm{NS}(\mathcal{X})$.

Summarizing the above, we have that $\mathrm{NS}(\mathcal{X})/L^1 \mathrm{NS}(\mathcal{X})$ is infinite cyclic, and $L^2 \mathrm{NS}(\mathcal{X})$ is finitely generated of rank $1 + \sum_v (f_v - 1)$, where f_v is the number of irreducible components of $\mathcal{X}_v$. Also, $L^2 \mathrm{NS}(\mathcal{X})$ is torsion free if and only if $\gcd(m_v, m_{v'}) = 1$ for all $v \neq v'$.

The interesting part of $\mathrm{NS}(\mathcal{X})$, namely $L^1 \mathrm{NS}(\mathcal{X})/L^2 \mathrm{NS}(\mathcal{X})$, is the subject of the Shioda–Tate theorem.

4.2 Shioda–Tate theorem

Write $\mathrm{Div}^0(X)$ for the group of divisors on X of degree 0. Restriction to the generic fiber gives a homomorphism $L^1 \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}^0(X)$ which descends to a homomorphism $L^1 \mathrm{Pic}(\mathcal{X}) \to J_X(K) = \mathrm{Pic}^0_{X/K}(K)$. The Shioda–Tate theorem uses this map to describe $L^1 \mathrm{NS}(\mathcal{X})/L^2 \mathrm{NS}(\mathcal{X})$.

Recall that (B, τ) is the K/k-trace of J_X.

Proposition 4.1 (Shioda–Tate)**.** *The map above induces a homomorphism*

$$\frac{L^1 \mathrm{NS}(\mathcal{X})}{L^2 \mathrm{NS}(\mathcal{X})} \longrightarrow \mathrm{MW}(J_X) = \frac{J_X(K)}{\tau B(k)}$$

with finite kernel and cokernel. In particular the two sides have the same rank as finitely generated abelian groups. This homomorphism is an isomorphism if X has a K-rational point and k is either finite or algebraically closed

Taking into account what we know about $\mathrm{NS}(\mathcal{X})/L^1 \mathrm{NS}(\mathcal{X})$ and $L^2 \mathrm{NS}(\mathcal{X})$, we have a formula relating the ranks of $\mathrm{NS}(\mathcal{X})$ and $\mathrm{MW}(J_X)$.

Corollary 4.2. *We have*

$$\operatorname{Rank} \mathrm{NS}(\mathcal{X}) = \operatorname{Rank} \mathrm{MW}(J_X) + 2 + \sum_v (f_v - 1).$$

Various versions of Proposition 4.1 appear in the literature, notably in [16], [21], [58], and [62], and, as Shioda notes [58, p. 359], it was surely known to the ancients.

Proof of Proposition 4.1. Specialized to X/K, the exact sequence of low degree terms (3.1) gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}_{X/K}(K) \longrightarrow \mathrm{Br}(K).$$

If X has a K-rational point, then $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/K}(K)$ is an isomorphism. In any case, X has a rational point over a finite extension K' of K of degree δ (the index of X), and over K' the coboundary in the analogous sequence

$$\mathrm{Pic}_{X/K'}(K') \longrightarrow \mathrm{Br}(K')$$

is zero. This implies that the cokernel of $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/K}(K)$ maps to the kernel of $\mathrm{Br}(K) \to \mathrm{Br}(K')$ and therefore lies in the δ-torsion subgroup of $\mathrm{Br}(K)$ [54, p. 157]. The upshot is that the cokernel of $\mathrm{Pic}(X) \to \mathrm{Pic}_{X/K}(K)$ has exponent dividing δ.

A simple geometric argument as in Shioda [58, p. 363] shows that the kernel of the homomorphism $L^1 \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}^0(X)$ is exactly $L^2 \mathrm{Pic}(\mathcal{X})$, so we have an isomorphism

$$\frac{L^1 \mathrm{Pic}(\mathcal{X})}{L^2 \mathrm{Pic}(\mathcal{X})} \cong \mathrm{Pic}^0(X).$$

Now consider the composite homomorphism

$$\mathrm{Pic}^0(\mathcal{X}) = \mathrm{Pic}^0_{\mathcal{X}/k}(k) \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0_{X/K}(K).$$

There is an underlying homomorphism of algebraic groups

$$\mathrm{Pic}^0_{\mathcal{X}/k} \times_k K \longrightarrow \mathrm{Pic}^0_{X/K}$$

inducing the homomorphism above on points. By the definition of the K/k-trace, this morphism must factor through B, i.e., we have a morphism of algebraic groups over k,

$$\mathrm{Pic}^0_{\mathcal{X}} \longrightarrow B.$$

We are going to argue that this last morphism is surjective. To do so, first note that a similar discussion applies over $\overline{k}$ and yields the following diagram:

$$\begin{tikzcd}
\dfrac{L^1 \mathrm{Pic}(\mathcal{X} \times \overline{k})}{L^2 \mathrm{Pic}(\mathcal{X} \times \overline{k})} \arrow[r] & \mathrm{Pic}^0\left(X \times K\overline{k}\right) \arrow[r] & \mathrm{Pic}\left(X \times K\overline{k}\right) \arrow[r] & \mathrm{Pic}_{X/K\overline{k}}\left(K\overline{k}\right) \\
\mathrm{Pic}^0\left(\mathcal{X} \times \overline{k}\right) \arrow[u] \arrow[rrr] & & & B\left(\overline{k}\right). \arrow[u]
\end{tikzcd}$$

Now the cokernel of the left vertical map is a subquotient of $\mathrm{NS}(\mathcal{X})$, so finitely generated, and the cokernels of the horizontal maps across the top are trivial, $\mathbb{Z}$, and of finite exponent respectively. On the other hand, $B(\overline{k})$ is a divisible group. This implies that the image of $B(\overline{k})$ in $\mathrm{Pic}_{X/K\overline{k}}(K\overline{k})$ is equal to the image of $\mathrm{Pic}^0(\mathcal{X} \times \overline{k})$ in that same group. Since the kernel of $B(\overline{k}) \to \mathrm{Pic}_{X/K\overline{k}}(K\overline{k})$ is finite, this in turn implies that the morphism $\mathrm{Pic}^0(\mathcal{X}) \to B$ is surjective.

An alternative proof of the surjectivity is to use the exact sequence (2.1). The middle term is $V_\ell \mathrm{Pic}^0(\mathcal{X} \times \overline{k})$ whereas the right hand term is $V_\ell B(\overline{k})$. Thus the morphism $\mathrm{Pic}^0_{\mathcal{X}} \to B$ is surjective.

Next we argue that the map of k-points $\mathrm{Pic}(\mathcal{X}) = \mathrm{Pic}_{\mathcal{X}/k}(k) \to B(k)$ has finite cokernel. More generally, if $\phi\colon A \to A'$ is a surjective morphism of abelian varieties over a field k, then we claim that the map of points $A(k) \to A'(k)$ has finite cokernel. If ϕ is an isogeny, then considering the dual isogeny $\phi^\vee$ and the composition $\phi\phi^\vee$ shows that the cokernel is killed by $\deg\phi$, so it is finite. For a general surjection, if $A'' \subset A$ is a complement (up to a finite group) of $\ker\phi$, then by the above $A''(k) \to A'(k)$ has finite cokernel, and *a fortiori* so does $A(k) \to A'(k)$. (Thanks to Marc Hindry for suggesting this argument.)

When k is algebraically closed, $\mathrm{Pic}^0_{\mathcal{X}}(k) \to B(k)$ is obviously surjective. It is also surjective when k is finite and X has a K-rational point. This follows from Lang's theorem because, according to Proposition 4.4 below, the kernel of $\mathrm{Pic}^0_{\mathcal{X}} \to B$ is an abelian variety.

Summing up, the map

$$\frac{L^1 \operatorname{Pic}(\mathcal{X})}{L^2 \operatorname{Pic}(\mathcal{X})} \longrightarrow J_X(K)$$

has cokernel of finite exponent, and

$$\operatorname{Pic}^0(\mathcal{X}) \longrightarrow B(k)$$

has finite cokernel. Thus the map

$$\frac{L^1 \operatorname{NS}(\mathcal{X})}{L^2 \operatorname{NS}(\mathcal{X})} = \frac{L^1 \operatorname{Pic}(\mathcal{X})}{L^2 \operatorname{Pic}(\mathcal{X}) + \operatorname{Pic}^0(\mathcal{X})} \longrightarrow \operatorname{MW}(J_X) = \frac{J_X(K)}{\tau B(k)}$$

has finite kernel and its cokernel has finite exponent. Since the target group is finitely generated (Lang–Néron), the cokernel must be finite. When X has a rational point the first map displayed above is surjective, and when k is finite or algebraically closed, the second map is surjective. Thus under both of these hypotheses, the third map is an isomorphism.

This completes the proof of the theorem. $\square$

Remark 4.3. It is also possible to deduce the theorem in the case when k is finite from the case when k is algebraically closed by taking Galois invariants.

It will be convenient to have an explicit description of B.

Proposition 4.4. *If X has a K-rational point and k is perfect, then we have an exact sequence*

$$0 \longrightarrow \operatorname{Pic}^0_{\mathcal{C}/k} \longrightarrow \operatorname{PicVar}_{\mathcal{X}/k} \longrightarrow B \longrightarrow 0.$$

Sketch of proof. Obviously there are morphisms $\operatorname{Pic}^0_{\mathcal{C}/k} \to \operatorname{PicVar}_{\mathcal{X}/k} \to B$. The first is injective because $\pi\colon \mathcal{X} \to \mathcal{C}$ has a section, and the second was seen to be surjective in the proof of the Shioda–Tate theorem. It is also clear that the composed map is zero.

Again because π has a section, the argument in Section 2.4 shows that the integral ℓ-adic Leray spectral sequence degenerates and we have exact sequences

$$0 \longrightarrow H^1\big(\overline{\mathcal{C}}, \mathbb{Z}_\ell(1)\big) \longrightarrow H^1\big(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\big) \longrightarrow H^0\big(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1)\big) \longrightarrow 0$$

for all ℓ. These cohomology groups can be identified with the Tate modules of $\operatorname{Pic}^0_{\mathcal{C}/k}$, $\operatorname{PicVar}_{\mathcal{X}/k}$, and B respectively. (For $\ell = p$ we should use flat cohomology.) This shows that $\operatorname{PicVar}_{\mathcal{X}/k} / \operatorname{Pic}^0_{\mathcal{C}/k} \to B$ is purely inseparable. But we also have

$$0 \longrightarrow H^1\big(\mathcal{C}, \mathcal{O}_{\mathcal{C}}\big) \longrightarrow H^1\big(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\big) \longrightarrow H^0\big(\mathcal{C}, R^1\pi_*\mathcal{O}_{\mathcal{X}}\big) \longrightarrow 0$$

and (using the existence of a section and thus cohomological flatness) these three groups can be identified with the tangent spaces of $\operatorname{Pic}^0_{\mathcal{C}/k}$, $\operatorname{PicVar}_{\mathcal{X}/k}$, and B respectively. This shows that $\operatorname{PicVar}_{\mathcal{X}/k} / \operatorname{Pic}^0_{\mathcal{C}/k} \to B$ is separable. Thus it must be an isomorphism. $\square$

Remark 4.5. If we no longer assume that k is perfect, but continue to assume that X has a rational point, then the conclusion of the proposition still holds. Indeed, we have a morphism $\operatorname{PicVar}_{\mathcal{X}/k} / \operatorname{Pic}^0_{\mathcal{C}/k} \to B$ which becomes an isomorphism over $\overline{k}$. (Here we use that K/k is regular, so that formation of B commutes with extension of k [11, 6.8] and similarly for the Picard varieties.) It must therefore have already been an isomorphism over k. In general, the morphism $\operatorname{PicVar}_{\mathcal{X}/k} / \operatorname{Pic}^0_{\mathcal{C}/k} \to B$ may be purely inseparable.

4.3 Heights

We use the Shioda–Tate theorem to define a non-degenerate bilinear pairing on $\operatorname{MW}(J_X)$ which is closely related to the Néron–Tate canonical height when k is finite.

To simplify the notation we write

$$L^1 = (L^1 \operatorname{NS}(\mathcal{X})) \otimes \mathbb{Q} \quad \text{and} \quad L^2 = (L^2 \operatorname{NS}(\mathcal{X})) \otimes \mathbb{Q},$$

so that $\operatorname{MW}(J_X) \otimes \mathbb{Q} \cong L^1/L^2$.

It is well known that the intersection form on $\operatorname{NS}(\mathcal{X}) \otimes \mathbb{Q}$ is non-degenerate (cf. [70, p. 29]). Since L^1 has codimension 1 in $\operatorname{NS}(\mathcal{X}) \otimes \mathbb{Q}$, the kernel of the intersection form restricted to L^1 has dimension at most 1, and it is easy to see that it is in fact one-dimensional, generated by F, the class of a fiber of π. Also, our discussion of the structure of the fibers of π shows that the kernel of the intersection form restricted to L^2 is also one-dimensional, generated by F.

It follows that for each class in $D \in L^1/L^2$, there is a representative $\tilde{D} \in L^1$ such that $\tilde{D}$ is orthogonal to all of L^2; moreover, $\tilde{D}$ is determined by D up to the addition of a multiple of F. We thus have a homomorphism $\phi\colon L^1/L^2 \to L^1/\mathbb{Q}F$ defined by $\phi(D) = \tilde{D}$. We define a bilinear pairing on $\operatorname{MW}(J_X) \otimes \mathbb{Q} = L^1/L^2$ by the rule

$$\big(D, D'\big) = -\phi\big(D\big) \cdot \phi\big(D'\big)$$

where the dot signifies the intersection pairing on $\mathcal{X}$, extended by $\mathbb{Q}$-linearity. The right-hand side is well defined because $F \in L^2$.

Proposition 4.6. *The formula above defines a symmetric, $\mathbb{Q}$-valued bilinear form on $\operatorname{MW}(J_X) \otimes \mathbb{Q}$ which is positive definite on $\operatorname{MW}(J_X) \otimes \mathbb{R}$. If k is a finite field of cardinality q, then $(-,-)\log q$ is the Néron–Tate canonical height on $\operatorname{MW}(J_X) \cong J_X(K) \otimes \mathbb{Q}$*

Sketch of proof. It is clear that the pairing is bilinear, symmetric, and $\mathbb{Q}$-valued.

To see that it is positive definite on $\operatorname{MW}(J_X) \otimes \mathbb{Q}$, we use the Hodge index theorem. Given $D \in L^1/L^2$, choose an irreducible multisection P and a representative $\tilde{\phi}(D) \in L^1$ for $\phi(D)$. The intersection number $P \cdot D$ is in $\mathbb{Q}$ and replacing D with nD for suitable n we can assume it is in $\mathbb{Z}$. Then adding a multiple of F to $\tilde{\phi}(D)$ and calling the result again $\tilde{\phi}(D)$, we get a new representative of $\phi(D)$

such that $P \cdot \tilde{\phi}(D) = 0$. On the other hand $(P + mF)^2 > 0$ if m is large enough and $(P+mF)\cdot\tilde{\phi}(D) = 0$. The Hodge index theorem (e.g., [6, 2.4]) then implies that $\tilde{\phi}(D)^2 < 0$ and so $(D, D) > 0$.

Since the pairing is positive definite and $\mathbb{Q}$-valued on $\mathrm{MW}(J_X)\otimes\mathbb{Q}$, it is also positive definite on $\mathrm{MW}(J_X)\otimes\mathbb{R}$.

The connection with Néron–Tate canonical height can be seen by using the local canonical heights. These were constructed purely on the relative curve $\mathcal{X} \to \mathcal{C}$ (as opposed to on the Néron model $\mathcal{J}_X \to \mathcal{C}$) by Gross in [17]. An inspection of his construction shows that summing the local heights over all places of K gives exactly our definition above times $\log q$. □

Exercise 4.7. When X is an elliptic curve, give a global proof of the connection between the Shioda–Tate height and the canonical height as follows: Check from the definition that $\log q$ times the Shioda–Tate height differs from the naive height associated to the origin of X (as a divisor on X) by a bounded amount. Since the Shioda–Tate height is already bilinear, it must thus be the canonical height as constructed by Tate.

5 Cohomology and cycles

5.1 Tate's conjecture for $\mathcal{X}$

Throughout this section, k is a finite field of characteristic p and $\mathcal{X}$ is a smooth and projective surface over k. We write $\overline{\mathcal{X}}$ for $\mathcal{X}\times_k \overline{k}$. Our goal is to discuss Tate's first conjecture on cycles and cohomology classes for $\mathcal{X}$. Since this was already discussed in [70, Lecture 2], we will be brief.

5.1.1 Basic exact sequences

Let $\mathrm{Br}(\mathcal{X}) = H^2(\mathcal{X}, \mathcal{O}_X^\times) = H^2(\mathcal{X}, \mathbb{G}_m)$ be the (cohomological) Brauer group of $\mathcal{X}$. Here we use the étale or flat topologies. It is known that $\mathrm{Br}(\mathcal{X})$ is a torsion group and it is conjectured to be finite in our situation.

For any positive integer n, consider the Kummer sequence on $\overline{\mathcal{X}}$,

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 0.$$

Here if $p|n$ the sequence is exact in the flat topology, but not in the étale topology.

Let ℓ be a prime (with $\ell = p$ allowed). Taking cohomology and noting that $\mathrm{Pic}(\overline{\mathcal{X}})/\ell^n = \mathrm{NS}(\overline{\mathcal{X}})/\ell^n$ (because $\mathrm{Pic}^0(\overline{\mathcal{X}})$ is divisible), we find an exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\overline{\mathcal{X}}\right)/\ell^n \longrightarrow H^2\left(\overline{\mathcal{X}}, \mu_{\ell^n}\right) \longrightarrow \mathrm{Br}\left(\overline{\mathcal{X}}\right)[\ell^n] \longrightarrow 0.$$

Taking the inverse limit over n yields an exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\overline{\mathcal{X}}\right)\otimes\mathbb{Z}_\ell \longrightarrow H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right) \longrightarrow T_\ell\,\mathrm{Br}\left(\overline{\mathcal{X}}\right) \longrightarrow 0.$$

Now $H^1(G_k, \mathrm{NS}(\overline{\mathcal{X}}) \otimes \mathbb{Z}_\ell)$ is a finite group. On the other hand, if H is a group with finite ℓ-torsion, $T_\ell H$ is finitely generated and torsion-free over $\mathbb{Z}_\ell$. This applies in particular to $T_\ell \operatorname{Br}(\overline{\mathcal{X}})$. Thus taking G_k invariants, we get an exact sequence

$$0 \longrightarrow \left(\mathrm{NS}\left(\overline{\mathcal{X}}\right) \otimes \mathbb{Z}_\ell\right)^{G_k} \longrightarrow H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)^{G_k} \longrightarrow \left(T_\ell \operatorname{Br}\left(\overline{\mathcal{X}}\right)\right)^{G_k} \longrightarrow 0.$$

Next note that, since k is finite, $H^1(G_k, \operatorname{Pic}^0(\overline{\mathcal{X}})) = 0$ and so

$$\left(\mathrm{NS}\left(\overline{\mathcal{X}}\right) \otimes \mathbb{Z}_\ell\right)^{G_k} = \mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Z}_\ell$$

since by definition $\mathrm{NS}(\mathcal{X})$ is the image of $\operatorname{Pic}(\mathcal{X})$ in $\mathrm{NS}(\overline{\mathcal{X}})$.

Now, using the Hochschild–Serre spectral sequence, we have a homomorphism $H^2(\mathcal{X}, \mathbb{Z}_\ell(1)) \to H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1))^{G_k}$ which is surjective with finite kernel. Since $T_\ell \operatorname{Br}(\mathcal{X})$ is torsion-free, the commutative diagram

$$\begin{tikzcd}
 & \mathrm{NS}(\mathcal{X}) \otimes \mathbb{Z}_\ell \arrow[r, equal] \arrow[d] & \left(\mathrm{NS}(\overline{\mathcal{X}}) \otimes \mathbb{Z}_\ell\right)^{G_k} \arrow[d] & \\
\text{(finite)} \arrow[r] & H^2(\mathcal{X}, \mathbb{Z}_\ell(1)) \arrow[r] \arrow[d] & H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1))^{G_k} \arrow[r] \arrow[d] & 0 \\
 & T_\ell \operatorname{Br}(\mathcal{X}) \arrow[r] \arrow[d] & T_\ell \operatorname{Br}(\overline{\mathcal{X}})^{G_k} \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

induces an isomorphism $T_\ell \operatorname{Br}(\mathcal{X}) \tilde{\to} T_\ell \operatorname{Br}(\overline{\mathcal{X}})^{G_k}$.

Putting everything together, we get an exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Z}_\ell \longrightarrow H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)^{G_k} \longrightarrow T_\ell \operatorname{Br}\left(\mathcal{X}\right) \longrightarrow 0. \tag{5.1}$$

5.1.2 First Tate conjecture

Tate's first conjecture for a prime ℓ, which we denote $T_1(\mathcal{X}, \ell)$, says that the cycle class map induces an isomorphism

$$\mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Q}_\ell \cong H^2\left(\overline{\mathcal{X}}, \mathbb{Q}_\ell(1)\right)^{G_k}.$$

The exact sequence (5.1) leads immediately to several equivalent forms of this conjecture.

Proposition 5.1. *The following are equivalent:*

1. $T_1(\mathcal{X}, \ell)$.
2. *The cycle class map induces an isomorphism*

$$\mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Z}_\ell \cong H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)^{G_k}.$$

3. $T_\ell \operatorname{Br}(\mathcal{X}) = 0$.
4. *The ℓ-primary component of* $\operatorname{Br}(\mathcal{X})$ *is finite.*

Proof. Tensoring (5.1) with $\mathbb{Q}_\ell$ shows that $T_1(\mathcal{X}, \ell)$ is equivalent to $V_\ell \operatorname{Br}(\mathcal{X}) := T_\ell \operatorname{Br}(\mathcal{X}) \otimes \mathbb{Q}_\ell = 0$ and therefore to $T_\ell \operatorname{Br}(\mathcal{X}) = 0$, which is in turn equivalent to the injection

$$\mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Z}_\ell \longrightarrow H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)$$

being an isomorphism. This establishes the equivalence of 1–3. Since $\operatorname{Br}(\mathcal{X})$ is a torsion group and $\operatorname{Br}(\mathcal{X})[\ell]$ is finite, the ℓ-primary component of $\operatorname{Br}(\mathcal{X})$ is finite if and only if $T_\ell Br(\mathcal{X})$ is finite. This establishes the equivalence of 3 and 4. □

We will see below that $T_1(\mathcal{X}, \ell)$ is equivalent to $T_1(\mathcal{X}, \ell')$ for any two primes ℓ and ℓ'.

We note also that the flat cohomology group $H^2(\overline{\mathcal{X}}, \mathbb{Q}_p(1))$ is isomorphic to the crystalline cohomology group $(H^2(\overline{\mathcal{X}}/W(\overline{k})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\mathrm{Fr}=p}$, where the exponent indicates the subspace upon which the absolute Frobenius Fr acts by multiplication by p. One can thus reformulate the $\ell = p$ part of the Tate conjecture in terms of crystalline cohomology.

5.2 Selmer group conjecture for J_X

We assume throughout this subsection that the ground field k is finite.

By the Lang–Néron theorem discussed in Section 3.4 above, the Mordell–Weil group $J(K)/\tau B(k)$ is finitely generated, and since $B(k)$ is obviously finite, we have that $J_X(K)$ is also finitely generated. In this subsection we quickly review the standard mechanism to bound (and conjecturally capture) the rank of this group via descent.

5.2.1 Selmer and Tate–Shafarevich groups

For each positive integer n, consider the exact sequence of sheaves for the flat topology on $\operatorname{Spec} K$,

$$0 \longrightarrow J_X[n] \longrightarrow J_X \xrightarrow{n} J_X \longrightarrow 0,$$

where $J_X[n]$ denotes the kernel of multiplication by n on J_X. Taking cohomology yields

$$0 \longrightarrow J_X\left(K\right)/nJ_X\left(K\right) \longrightarrow H^1\left(K, J_X[n]\right) \longrightarrow H^1\left(K, J_X\right)[n] \longrightarrow 0.$$

Similarly, for every completion K_v of K we have a similar sequence and restriction maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_X(K)/nJ_X(K) & \longrightarrow & H^1(K, J_X[n]) & \longrightarrow & H^1(K, J_X)[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_X(K_v)/nJ_X(K_v) & \longrightarrow & H^1(K_v, J_X[n]) & \longrightarrow & H^1(K_v, J_X)[n] & \longrightarrow & 0. \end{array}$$

We define $\mathrm{Sel}(J_X, n)$ to be

$$\ker\left(H^1(K, J_X[n]) \longrightarrow \prod_v H^1(K_v, J_X)\right),$$

where the product is over the places of K. Also, for any prime ℓ, we define $\mathrm{Sel}(J_X, \mathbb{Z}_\ell) = \varprojlim_n \mathrm{Sel}(J_X, \ell^n)$. We define the Tate–Shafarevich group by

$$Ш(J_X) = \ker\left(H^1(K, J_X) \longrightarrow \prod_v H^1(K_v, J_X)\right).$$

There are exact sequences

$$0 \longrightarrow J_X(K)/nJ_X(K) \longrightarrow \mathrm{Sel}\,(J_X, n) \longrightarrow Ш(J_X)[n] \longrightarrow 0$$

for each n. All the groups appearing here are finite. (For n prime to p, the classical proof of finiteness of the Selmer group – reducing it to the finiteness of the class group and finite generation of the unit group of Dedekind domains with fraction field K – works in our context. The p part was first proven by Milne [37].) So taking the inverse limit over powers of a prime ℓ yields an exact sequence

$$0 \longrightarrow J_X(K) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Sel}\,(J_X, \mathbb{Z}_\ell) \longrightarrow T_\ell Ш(J_X) \longrightarrow 0. \tag{5.2}$$

5.2.2 Tate–Shafarevich conjecture

Tate and Shafarevich conjectured (independently) that $Ш(J_X)$ is a finite group. We write $TS(J_X)$ for this conjecture, and $TS(J_X, \ell)$ for the *a priori* weaker conjecture that the ℓ-primary part of $Ш(J_X)$ is finite.

For each prime ℓ, we refer to the statement "the homomorphism

$$J_X(K) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Sel}\,(J_X/K, \mathbb{Z}_\ell)$$

is an isomorphism" as the *Selmer group conjecture* $S(J_X, \ell)$. (This is perhaps nonstandard and is meant to suggest that the Selmer group captures rational points.)

The following is obvious from the exact sequence (5.2) above:

Proposition 5.2. *For each prime ℓ, the Selmer group conjecture $S(J_X, \ell)$ holds if and only if the Tate–Shafarevich conjecture $TS(J_X, \ell)$ holds.*

Note that, if the ℓ-primary component of $Ш(J_X)$ is finite, then knowing $\mathrm{Sel}(J_X, \ell^n)$ for sufficiently large n determines the rank of $J_X(K)$. This observation and some input from L-functions leads (conjecturally) to an effective algorithm for computing generators of $J_X(K)$; see [36].

5.3 Comparison of cohomology groups

There is an obvious parallel between the conjectures $T_1(\mathcal{X},\ell)$ and finiteness of $\mathrm{Br}(\mathcal{X})[\ell^\infty]$ on the one hand, and $S(J_X,\ell)$ and $TS(J_X,\ell)$ on the other. The Shioda–Tate isomorphism gives a precise connection between the groups of cycles and points involved. In this subsection, we give connections between the cohomology groups involved. We restrict to the simplest cases here – they are already complicated enough. In Subsection 6.3 we will give a comparison for all ℓ and the most general hypotheses on $\mathcal{X}$ and X using analytic methods.

Thus, for the rest of this subsection, the following hypotheses are in force (in addition to the standing hypotheses): k is finite of characteristic p, ℓ is a prime distinct from p, and X has a K-rational point. This implies that $\mathcal{X}\to\mathcal{C}$ has a section, and so the multiplicities m_v of the fibers of π are all equal to 1 and π is cohomologically flat.

5.3.1 Comparison of $\mathrm{Br}(\mathcal{X})$ and $\text{Ш}(J_X)$

These groups are closely related – as we will see later, under the standing hypotheses and assuming k is finite, they differ by a finite group. Assuming also that X has a K-rational point, even more is true.

Proposition 5.3 (Grothendieck)**.** *In addition to the standing hypotheses, assume that k is finite and X has a K-rational point. Then we have a canonical isomorphism* $\mathrm{Br}(\mathcal{X})\cong\text{Ш}(J_X)$.

This is proven in detail in [19, §4] by an argument "assez long et technique". We sketch the main points in the rest of this subsection.

First of all, since π is cohomologically flat, we have $\pi_*\mathbb{G}_m=\mathbb{G}_m$. Also, we have a vanishing/cohomological dimension result of Artin: $R^q\pi_*\mathbb{G}_m=0$ for $q>1$. Thus the Leray spectral sequence for π and $\mathbb{G}_m$ gives a long exact sequence

$$H^2(\mathcal{C},\mathbb{G}_m)\longrightarrow H^2(\mathcal{X},\mathbb{G}_m)\longrightarrow H^1(\mathcal{C},R^1\pi_*\mathbb{G}_m)\longrightarrow H^3(\mathcal{C},\mathbb{G}_m)\longrightarrow H^3(\mathcal{X},\mathbb{G}_m).$$

Now $H^2(\mathcal{C},\mathbb{G}_m)=\mathrm{Br}(\mathcal{C})=0$ since $\mathcal{C}$ is a smooth complete curve over a finite field. Also, since π has a section, $H^3(\mathcal{C},\mathbb{G}_m)\to H^3(\mathcal{X},\mathbb{G}_m)$ is injective. Thus we have an isomorphism

$$\mathrm{Br}\left(\mathcal{X}\right)=H^2\left(\mathcal{X},\mathbb{G}_m\right)\cong H^1\left(\mathcal{C},R^1\pi_*\mathbb{G}_m\right).$$

Now write $\mathcal{F}=R^1\pi_*\mathbb{G}_m$ and consider the inclusion $\eta=\operatorname{Spec}K\hookrightarrow\mathcal{C}$. Let $\mathcal{G}=j_*j^*\mathcal{F}$, so that we have a canonical morphism $\mathcal{F}\longrightarrow\mathcal{G}$. If $\overline{x}$ is a geometric point over a closed point of $\mathcal{C}$, we write $\tilde{\mathcal{X}}_{\overline{x}}$, $\tilde{\eta}_{\overline{x}}$, and $\tilde{\mathcal{C}}_{\overline{x}}$ for the strict henselizations at $\overline{x}$. Then the stalk of $\mathcal{F}\to\mathcal{G}$ at $\overline{x}$ is

$$\operatorname{Pic}\left(\tilde{\mathcal{X}}_{\overline{x}}/\tilde{\mathcal{C}}_{\overline{x}}\right)\longrightarrow\operatorname{Pic}\left(\tilde{\mathcal{X}}_{\tilde{\eta}_{\overline{x}}}/\tilde{\eta}_{\overline{x}}\right).$$

Since the Brauer group of $\tilde{\eta}_{\overline{x}}$ vanishes, the displayed map is surjective (cf. the exact sequence (3.1)). The kernel is zero at all $\overline{x}$ where the fiber of π is reduced and irreducible. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{K}$ is a skyscraper sheaf supported on the points of $\mathcal{C}$ where the fibers of π are reducible. (See [19, pp. 115–118] for a more complete description of this sheaf and its cohomology.) Under our hypotheses (k finite and X with a K-rational point), every fiber of π has a component of multiplicity 1, and this is enough to ensure that $H^1(\mathcal{C}, \mathcal{K}) = 0$.

Now write $\mathcal{X}_x$ and $\mathcal{C}_x$ for the ordinary henselizations of $\mathcal{X}$ and $\mathcal{C}$ at a closed point x of $\mathcal{C}$. Then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{C},\mathcal{F}) \arrow[r] \arrow[d] & H^1(\mathcal{C},\mathcal{G}) \arrow[r] \arrow[d] & \prod_x H^2(x,\mathcal{K}_x) \arrow[d, equal] \\
0 \arrow[r] & \prod_x H^1(\mathcal{C}_x,\mathcal{F}) \arrow[r] & \prod_x H^1(\mathcal{C}_x,\mathcal{G}) \arrow[r] & \prod_x H^2(x,\mathcal{K}_x).
\end{tikzcd}$$

Lang's theorem implies that $H^1(\mathcal{C}_x, \mathcal{F}) = 0$ for all x. Thus $\mathrm{Br}(\mathcal{X}) = H^1(\mathcal{C}, \mathcal{F})$ is identified with the subgroup of $H^1(\mathcal{C}, \mathcal{G})$ consisting of elements which go to zero in $H^1(\mathcal{C}_x, \mathcal{G})$ for all x. Call this group $Ш(\mathcal{C}, \mathcal{G})$.

Finally, the Leray spectral sequence for $\eta \hookrightarrow \mathcal{C}$ and the definition of $\mathcal{G}$ leads to an exact sequence

$$0 \longrightarrow Ш(\mathcal{C},\mathcal{G}) \longrightarrow H^1(\eta, j^*\mathcal{F}) \longrightarrow \coprod_x H^1(\kappa(\mathcal{C}_x), j^*\mathcal{F})$$

where $\kappa(\mathcal{C}_x)$ is the field of fractions of the henselization of $\mathcal{C}$ at x. This identifies $Ш(\mathcal{C}, \mathcal{G})$ with $Ш(J_X)$ as defined in Subsection 5.2.1.

This completes our sketch of Grothendieck's theorem. The reader is encouraged to consult [19] for a much more general discussion delivered in the inimitable style of the master.

One consequence of the proposition is that (under the hypotheses there) the conjectures on the finiteness of $\mathrm{Br}(\mathcal{X})$ and $Ш(J_X)$ are equivalent, and thus so are $T_1(\mathcal{X}, \ell)$ and $S(J_X, \ell)$ for all ℓ. We will see below that this holds even without the supplementary hypothesis on X.

5.3.2 Comparison of the Selmer group and $H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))^{G_k}$

The arguments in Subsection 2.4 show that under our hypotheses the Leray spectral sequence degenerates at the integral level as well, and so $H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1))$ has a filtration whose graded pieces are the groups $H^i(\overline{\mathcal{C}}, R^j\pi_*\mathbb{Z}_\ell(1))$ where $i + j = 2$.

As we will see below, from the point of view of the Tate conjecture, the interesting part of $H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1))$ is $H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))$. It turns out that its G_k-invariant part is closely related to the ℓ-adic Selmer group of J_X – they differ by a finite group.

Before being more precise, we make the groups $H^i(\overline{\mathcal{C}}, \mathbb{Z}_\ell(1))$ more explicit. Since the fibers of π are connected, $\pi_*\mathbb{Z}_\ell(1) = \mathbb{Z}_\ell(1)$ and so $H^2(\overline{\mathcal{C}}, \pi_*\mathbb{Z}_\ell(1)) = \mathbb{Z}_\ell$. It is easy to see that, under the map $H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)) \to H^2(\overline{\mathcal{C}}, \pi_*\mathbb{Z}_\ell(1))$, the cycle class of a section maps to a generator of $H^2(\overline{\mathcal{C}}, \pi_*\mathbb{Z}_\ell(1))$.

The stalk of $R^2\pi_*\mathbb{Z}_\ell(1)$ at a geometric point $\overline{x}$ over a closed point x of $\mathcal{C}$ is ${\mathbb{Z}_\ell}^{f_{\overline{x}}}$ where $f_{\overline{x}}$ is the number of irreducible components in the geometric fiber. The action of G_k permutes the factors as it permutes the components. If we write the fiber as $\sum_i m_{\overline{x},i}\mathcal{X}_{\overline{x},i}$ as in Subsection 2.3, then the specialization map

$$ {\mathbb{Z}_\ell}^{f_{\overline{x}}} = \left(R^2\pi_*\mathbb{Z}_\ell(1)\right)_{\overline{x}} \longrightarrow \left(R^2\pi_*\mathbb{Z}_\ell(1)\right)_{\overline{\eta}} = \mathbb{Z}_\ell $$

is $(c_i) \mapsto \sum_i m_{\overline{x},i}c_i$. It follows that $H^0(\overline{\mathcal{C}}, R^2\pi_*\mathbb{Z}_\ell(1))^{G_k}$ has rank $1 + \sum_v (f_v - 1)$ where f_v is the number of irreducible components of the fiber $\mathcal{X}_v$ (as a scheme over the residue field k_v). Also, the cycle classes of components of fibers lie in

$$ H^0\left(\overline{\mathcal{C}}, R^2\pi_*\mathbb{Z}_\ell(1)\right)^{G_k} \subset H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)^{G_k} $$

and span this subgroup.

Thus we see that the cycle class map induces a well-defined homomorphism

$$ (L^1 \operatorname{NS}(\mathcal{X})/L^2 \operatorname{NS}(\mathcal{X})) \otimes \mathbb{Z}_\ell \longrightarrow H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))^{G_k} $$

which is surjective if and only if $T_1(\mathcal{X}, \ell)$ holds. Since the source of this map is closely related to $J_X(K)$ – it is exactly $(J_X(K)/\tau B(k)) \otimes \mathbb{Z}_\ell$ by the Shioda–Tate theorem and $B(k)$ is finite – we must have a close connection between the groups $H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))^{G_k}$ and $\operatorname{Sel}(J_X, \mathbb{Z}_\ell)$. We will prove directly that they differ by a finite group.

We will only sketch a proof, since a complete treatment requires more details on bad reduction and Néron models than we have at our disposal. However, the main ideas should be clear.

To state the result, let $K' = K\overline{k}$ and define another Selmer group as follows:

$$ \operatorname{Sel}\left(J_x/K', \mathbb{Z}_\ell\right) = \varprojlim_n \ker\left(H^1\left(K', J_X[\ell^n]\right) \longrightarrow \prod_v H^1\left(K'_v, J_X\right)\right), $$

where the product is over the places of K'.

Proposition 5.4. *There are homomorphisms* $\operatorname{Sel}(J_X, \mathbb{Z}_\ell) \to \operatorname{Sel}(J_X/K', \mathbb{Z}_\ell)^{G_k}$ *and* $H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))^{G_k} \to \operatorname{Sel}(J_X/K', \mathbb{Z}_\ell)^{G_k}$ *with finite kernels and cokernels.*

Sketch of proof of Proposition 5.4. Inflation-restriction for $G_{K'} \subset G_K$ yields sequences

$$ \begin{aligned} 0 \longrightarrow H^1\left(G_k, J_X[\ell^n](K')\right) \longrightarrow H^1\left(G_K, J_X[\ell^n]\right) &\longrightarrow H^1\left(G_{K'}, J[\ell^n]\right) \\ &\longrightarrow H^2\left(G_k, J_X[\ell^n](K')\right). \end{aligned} $$

Now $J[\ell^n](K')$ is an extension of (finite) by $B[\ell^n](\overline{k})$ where (finite) is bounded independently of n, and B is the K/k-trace of J_X. Using Lang's theorem, we have that the kernel and cokernel of $H^1(G_K, T_\ell J) \to H^1(G_{K'}, T_\ell J)$ are finite and therefore so is the kernel of $\mathrm{Sel}(J_X, \mathbb{Z}_\ell) \to \mathrm{Sel}(J_X/K', \mathbb{Z}_\ell)^{G_k}$.

To control the cokernel, consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathrm{Sel}\big(J_X, \ell^n\big) \arrow[r] \arrow[d] & \mathrm{Sel}\big(J_X/K', \ell^n\big)^{G_k} \arrow[d] \\
 & H^1\big(G_K, J_X[\ell^n]\big) \arrow[r] \arrow[d] & H^1\big(G_{K'}, J[\ell^n]\big)^{G_k} \arrow[d] \\
\prod_v H^1\big(G_{k_v}, J_X(\overline{k}K_v)\big) \arrow[r] & \prod_v H^1\big(G_{K_v}, J\big) \arrow[r] & \prod_v \Big(\prod_{\overline{v}|v} H^1\big(G_{K'_{\overline{v}}}, J\big)\Big)^{G_k}.
\end{tikzcd}$$

The columns define the Selmer groups and the bottom row is a product of inflation-restriction sequences. By [40, I.3.8], the group on the left is zero for all but finitely many v and it is finite at all v. A diagram chase then shows that $\mathrm{Sel}(J_X, \mathbb{Z}_\ell) \to \mathrm{Sel}(J_X/K', \mathbb{Z}_\ell)^{G_k}$ has finite cokernel.

Now let $\mathcal{F} = j_* j^* \mathbb{Z}_\ell(1)$ where $j\colon \eta \hookrightarrow \mathcal{C}$ is the inclusion of the generic point. By Corollary 2.4, we may replace $H^1(\overline{\mathcal{C}}, R^1\pi_*\mathbb{Z}_\ell(1))$ with $H^1(\overline{\mathcal{C}}, \mathcal{F})$. The fact that $\mathcal{F}$ is a middle extension sheaf gives a Ш-like description of $H^1(\overline{\mathcal{C}}, \mathcal{F})$: there is an exact sequence

$$0 \longrightarrow H^1\big(\overline{\mathcal{C}}, \mathcal{F}\big) \longrightarrow H^1\big(K', T_\ell J_X\big) \longrightarrow \prod_{\overline{v}} H^1\big(K'_{\overline{v}}, T_\ell J_X\big).$$

On the other hand, $\mathrm{Sel}(J_X/K', \mathbb{Z}_\ell)$ is defined by a similar sequence, except that instead of vanishing in $H^1(K'_{\overline{v}}, T_\ell J_X)$, a class in the Selmer group should land in the image of $J_X(K'_{\overline{v}})\hat{\otimes}\mathbb{Z}_\ell \to H^1(K'_{\overline{v}}, T_\ell J_X)$. But $J(K'_{\overline{v}})$ is an extension of a finite group by an ℓ-divisible group, and for all but finitely many places $\overline{v}$ the finite group is trivial. (This follows from the structure of the Néron model of J_X.) Thus we get a inclusion $H^1(\overline{\mathcal{C}}, \mathcal{F}) \subset \mathrm{Sel}(J_X/K', \mathbb{Z}_\ell)$ with finite cokernel. Taking G_k-invariants finishes the proof. $\square$

6 Zeta and L-functions

6.1 Zeta functions and T_2 for $\mathcal{X}$

We quickly review the zeta function and the second Tate conjecture for $\mathcal{X}$. Most of this material was covered in [70] and so we will be very brief.

Throughout this section, $k=\mathbb{F}_q$ is the finite field with q elements, p is its characteristic, and $\mathcal{X}$ is a surface satisfying our usual hypotheses (as in Subsection 1.4).

6.1.1 Zetas

Let N_n be the number of points on $\mathcal{X}$ rational over $\mathbb{F}_{q^n}$ and form the Z-function

$$Z(\mathcal{X},T)=\prod_{x\in\mathcal{X}^0}\left(1-T^{\deg x}\right)^{-1}=\exp\left(\sum_{n\geq 1}N_n\frac{T^n}{n}\right)$$

and the zeta function $\zeta(\mathcal{X},s)=Z(\mathcal{X},q^{-s})$.

We choose an auxiliary prime $\ell\neq p$. Then the Grothendieck–Lefschetz trace formula

$$N_n=\sum_{i=0}^{4}\left(-1\right)^i\operatorname{Tr}\left(\mathrm{Fr}_q^n\,|H^i\big(\overline{\mathcal{X}},\mathbb{Q}_\ell\big)\right)$$

leads to an expression for $Z(\mathcal{X},T)$ as a rational function of T:

$$Z(\mathcal{X},T)=\prod_{i=0}^{4}P_i\big(\mathcal{X},T\big)^{(-1)^{i+1}}$$

where

$$P_i(\mathcal{X},T)=\det\left(1-T\ \mathrm{Fr}_q\,|H^i(\overline{\mathcal{X}},\mathbb{Q}_\ell)\right)=\prod_j\left(1-\alpha_{ij}T\right).$$

By the Poincaré duality theorem, there is a functional equation relating $\zeta(\mathcal{X},1-s)$ and $\zeta(\mathcal{X},s)$, and therefore relating $P_i(\mathcal{X},T)$ with $P_{4-i}(\mathcal{X},q/T)$.

By Deligne's theorem, the eigenvalues of Frobenius α_{ij} are Weil numbers of size $q^{i/2}$. It follows that the zeroes and poles of $\zeta(\mathcal{X},s)$ have real parts in the sets $\{1/2,3/2\}$ and $\{0,1,2\}$ respectively, and that the order of pole of $\zeta(\mathcal{X},s)$ at $s=1$ is equal to the multiplicity of q as an eigenvalue of Fr_q on $H^2(\overline{\mathcal{X}},\mathbb{Q}_\ell)$.

6.1.2 Tate's second conjecture

The second main conjecture in Tate's article [60] relates the ζ function to the Néron–Severi group.

Conjecture 6.1 ($T_2(\mathcal{X})$)**.** *We have*

$$\operatorname{Rank}\operatorname{NS}\big(\mathcal{X}\big)=-\operatorname{ord}_{s=1}\zeta\big(\mathcal{X},s\big).$$

Recall from (5.1) that the cycle class map induces an injection

$$\operatorname{NS}\big(\mathcal{X}\big)\otimes\mathbb{Z}_\ell\longrightarrow H^2\big(\overline{\mathcal{X}},\mathbb{Z}_\ell(1)\big)^{G_k}$$

and Conjecture $T_1(\mathcal{X},\ell)$ was the statement that this map is an isomorphism. When $\ell\neq p$, we have $H^2(\overline{\mathcal{X}},\mathbb{Z}_\ell(1))^{G_k}\cong H^2(\overline{\mathcal{X}},\mathbb{Z}_\ell)^{\mathrm{Fr}_q=q}$, where the latter is the subspace where the q-power Frobenius acts by multiplication by q. When $\ell = p$, $H^2(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1))^{G_k}$ is isomorphic to $H^2_{\mathrm{cris}}(\mathcal{X}/W(k))^{\mathrm{Fr}=p}$, the subgroup of crystalline cohomology where the absolute Frobenius acts by multiplication by p. Thus, in either case the $\mathbb{Z}_\ell$-rank of the target of the cycle class map (an eigenspace) is bounded above by the order of vanishing of the ζ function (which is the multiplicity of the corresponding eigenvalue). This proves the first two parts of the following.

Theorem 6.2 (Artin–Tate, Milne)**.** *We have:*

1. $\mathrm{Rank}\,\mathrm{NS}(\mathcal{X}) \leq -\,\mathrm{ord}_{s=1}\,\zeta(\mathcal{X}, s)$.
2. $T_2(\mathcal{X})$ *(i.e., equality in* 1*) implies* $T_1(\mathcal{X}, \ell)$ *and thus finiteness of* $\mathrm{Br}(\mathcal{X})[\ell^\infty]$ *for all* ℓ.
3. $T_1(\mathcal{X}, \ell)$ *for any* ℓ *(the case* $\ell = p$ *is allowed) implies* $T_2(\mathcal{X})$.

We sketch the proof of the last part in the next subsection. The prime-to-p was discussed/sketched in [70, Lecture 2, 10.2] and the general case is similar, although it uses more sophisticated cohomology.

6.1.3 Artin–Tate conjecture

We define two more invariants which enter into the Artin–Tate conjecture below.

Recall that there is a symmetric, integral intersection pairing on the Néron–Severi group which gives $\mathrm{NS}(\mathcal{X})/\mathrm{tor}$ the structure of a lattice. We define the regulator $R(\mathcal{X})$ to be the discriminant of this lattice. More precisely,

$$R = R(\mathcal{X}) = \left| \det\left(D_i.D_j\right)_{i,j=1,\ldots,\rho} \right|,$$

where D_i $(i = 1, \ldots, \rho)$ is a basis of $\mathrm{NS}(\mathcal{X})/\mathrm{tor}$ and the dot denotes the intersection product.

We also define a Tamagawa-like factor $q^{-\alpha(\mathcal{X})}$ where

$$\alpha = \alpha(\mathcal{X}) = \chi(\mathcal{X}, \mathcal{O}_\mathcal{X}) - 1 + \dim \mathrm{PicVar}\left(\mathcal{X}\right).$$

The following was inspired by the BSD conjecture for the Jacobian of X.

Conjecture 6.3 $(AT(\mathcal{X}))$**.** *$T_2(\mathcal{X})$ holds,* $\mathrm{Br}(\mathcal{X})$ *is finite, and we have the asymptotic*

$$\zeta(\mathcal{X}, s) \sim \frac{R\,|\mathrm{Br}(\mathcal{X})|\,q^{-\alpha}}{|\mathrm{NS}(\mathcal{X})_{tor}|^2}\left(1 - q^{1-s}\right)^{\rho(\mathcal{X})}$$

as $s \to 1$.

The analysis showing that $T_1(\mathcal{X}, \ell) \Rightarrow T_2(\mathcal{X})$ can be pushed further to show that $T_2(\mathcal{X}) \Rightarrow AT(\mathcal{X})$. Indeed, we have the following spectacular result:

Theorem 6.4 (Artin–Tate, Milne)**.** *If $T_2(\mathcal{X})$ holds, then so does $AT(\mathcal{X})$.*

To prove this, we must show that for all but finitely many ℓ, $\mathrm{Br}(\mathcal{X})[\ell]$ is trivial, and also relate the order of $\mathrm{Br}(\mathcal{X})$ and other invariants to the leading term of the zeta function.

The proofs of the third part of Theorem 6.2 and Theorem 6.4 proceed via a careful consideration of the following big commutative diagram:

$$\begin{tikzcd}
\mathrm{NS}\left(\mathcal{X}\right)\otimes\hat{\mathbb{Z}} \arrow[r, "e"] \arrow[d, "h"'] & \mathrm{Hom}\left(\mathrm{NS}\left(\mathcal{X}\right)\otimes\hat{\mathbb{Z}},\mathbb{Z}_\ell\right) \arrow[r, equal] & \mathrm{Hom}\left(\mathrm{NS}\left(\mathcal{X}\right)\otimes\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}\right) \\
H^2\left(\overline{\mathcal{X}},T\mu\right)^{G_k} \arrow[r, "f"] & H^2\left(\overline{\mathcal{X}},T\mu\right)_{G_k} \arrow[r, "j"] & \mathrm{Hom}\left(H^2\left(\mathcal{X},\mu(\infty)\right)^{G_k},\mathbb{Q}/\mathbb{Z}\right). \arrow[u, "g^*"']
\end{tikzcd}$$

Here $H^2(\overline{\mathcal{X}},T\mu)$ means the inverse limit over n of the flat cohomology groups $H^2(\overline{\mathcal{X}},\mu_n)$, and $H^2(\mathcal{X},\mu(\infty))$ is the direct limit over n of the flat cohomology groups $H^2(\mathcal{X},\mu_n)$. The map e is induced by the intersection form, h is the cycle class map, f is induced by the identity map of $H^2(\overline{\mathcal{X}},T\mu)$, g^* is the transpose of a map

$$g\colon\ \mathrm{NS}(\mathcal{X})\otimes\mathbb{Q}/\mathbb{Z}\longrightarrow H^2(\mathcal{X},\mu(\infty))$$

obtained by taking the direct limit over positive integers n of the Kummer map $\mathrm{NS}(\mathcal{X})\otimes\mathbb{Z}/n\mathbb{Z}\to H^2(\mathcal{X},\mu_n)$, and j is induced by the Hochschild–Serre spectral sequence and Poincaré duality.

We say that a homomorphism $\phi\colon A\to B$ of abelian groups is a *quasi-isomorphism* if it has finite kernel and cokernel. In this case, we define

$$z(\phi)=\frac{\#\ker(\phi)}{\#\operatorname{coker}(\phi)}.$$

It is easy to check that if $\phi_3=\phi_2\phi_1$ (composition) and if two of the maps ϕ_1, ϕ_2, ϕ_3 are quasi-isomorphisms, then so is the third and we have $z(\phi_3)=z(\phi_2)z(\phi_1)$.

In the diagram above, the map e is a quasi-isomorphism and $z(e)$ is the order of the torsion subgroup of $\mathrm{NS}(\mathcal{X})$ divided by the discriminant of the intersection form. The map j is also a quasi-isomorphism and $z(j)$ is the order of the torsion subgroup of Néron–Severi times a power of q determined by a certain p-adic cohomology group.

Now assume $T_1(\mathcal{X},\ell)$ for one ℓ and consider the ℓ-primary part of the diagram above. The assumption $T_1(\mathcal{X},\ell)$ implies that the ℓ parts of the maps h and g^* are quasi-isomorphisms. Thus the ℓ part of f must also be a quasi-isomorphism. This means that Fr_q acts semisimply on the generalized eigenspace of $H^2(\overline{\mathcal{X}},\mathbb{Q}_\ell(1))$ corresponding to the eigenvalue 1, and this implies $T_2(\mathcal{X})$.

Now assume $T_2(\mathcal{X})$, so we have that $T_1(\mathcal{X},\ell)$ holds for all ℓ and that h is an isomorphism. A fairly intricate analysis leads to a calculation of $z(f)$ in terms of the zeta function of $\mathcal{X}$ and shows that $z(g^*)$ (as a "supernatural number") is the order of the Brauer group. Since all the other z's in the diagram are rational numbers, so is $z(g^*)$ and we get finiteness of $\mathrm{Br}(\mathcal{X})$. The product formula

$$z(e) = z(h)\,z(f)\,z(j)\,z(g^*)$$

leads, after some delicate work in p-adic cohomology, to the leading coefficient formula in the conjecture $AT(\mathcal{X})$. We refer to Milne's paper [38] for the details.

6.1.4 The case $p = 2$ and de Rham–Witt cohomology

In Milne's paper [38], the case $\ell = p = 2$ had to be excluded due to the state of p-adic cohomology at the time. More complete results are now available, and so this restriction is no longer needed. This is pointed out by Milne on his web site and was implicit in the more general results in [42].

In slightly more detail, what is needed is a bridge between crystalline cohomology (which calculates the zeta function and receives cycle classes) and flat cohomology (which is closely related to the Brauer group). This bridge is provided by the cohomology of the de Rham–Witt complex, generalizing Serre's Witt vector cohomology. Such a theory was initiated by Bloch, whose approach required $p > 2$, and this is the source of the original restriction in [38]. Soon afterward, Illusie developed a different approach without any restriction on the characteristic, and important further developments were made by Illusie–Raynaud, Ekedahl, and Milne.

Briefly, the theory provides groups $H^j(\mathcal{X}, W\Omega^i)$ with operators F and V and a spectral sequence

$$E_1^{ij} = H^j\bigl(\mathcal{X}, W\Omega^i\bigr) \implies H^{i+j}_{\mathrm{cris}}\bigl(\mathcal{X}/W\bigr)$$

which degenerates at E_1 modulo torsion. There are also variants involving sheaves of cycles $ZW\Omega^i$, boundaries $BW\Omega^i$, and logarithmic differentials $W\Omega^i_{\log}$. The various cohomology groups (modulo torsion) are related by the spectral sequence to pieces of crystalline cohomology defined by "slope" conditions, i.e., by the valuations of eigenvalues of Frobenius. The cohomology of the logarithmic differentials for $i = 1$ is closely related to the flat cohomology

$$H^j\bigl(\mathcal{X}, \mathbb{Z}_p(1)\bigr) = \varprojlim_n H^j\bigl(\mathcal{X}, \mu_{p^n}\bigr).$$

The torsion in the de Rham–Witt cohomology can be "large" and it provides further interesting invariants.

I recommend [22] and [23] for a thorough overview of the theory and [42] for further important developments.

6.2 L-functions and the BSD conjecture for J_X

In this subsection we again assume that $k = \mathbb{F}_q$ and we consider X and J_X satisfying the usual hypotheses. The discussion will be parallel to that of the preceding subsection.

6.2.1 *L*-functions

Choose an auxiliary prime $\ell \neq p$ and consider the ℓ-adic Tate module

$$V_\ell J_X = \left(\varprojlim_n J_X[\ell^n]\big(\overline{K}\big) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

equipped with the natural action of G_K. There are canonical isomorphisms

$$V_\ell J_X \cong H^1\big(\overline{J}_X, \mathbb{Q}_\ell\big)^* \cong H^1\big(\overline{X}, \mathbb{Q}_\ell\big)^* = H^1\big(\overline{X}, \mathbb{Q}_\ell(1)\big)$$

where the $*$ indicates the dual vector space.

We define

$$L\big(J_X, T\big) = \prod_v \det\big(1 - T\ \mathrm{Fr}_v\, |V_\ell^{I_v}\big)$$

where the product runs over the places of K, I_v denotes an inertia group at v, and Fr_v denotes the *arithmetic* Frobenius at v. (Alternatively, we could use $H^1(\overline{X}, \mathbb{Q}_\ell)$ and the geometric Frobenius.) Also, we write $L(J_X, s)$ for the L-function above with $T = q^{-s}$. The product defining $L(J_X, s)$ converges in a half-plane and the cohomological description below shows that it is a rational function of q^{-s} and so it extends meromorphically to all s.

Let $j\colon U \hookrightarrow \mathcal{C}$ be a non-empty open subset over which X has good reduction. Then the representation $G_K \to \mathrm{Aut}(H^1(\overline{X}, \mathbb{Q}_\ell))$ is unramified at all places $v \in U$ and so it defines a lisse sheaf $\mathcal{F}_U$ over U. We set $\mathcal{F} = j_*\mathcal{F}_U$. The resulting constructible sheaf on $\mathcal{C}$ is independent of the choice of U. Its stalk $\mathcal{F}_{\overline{x}}$ at a geometric point $\overline{x}$ over closed point $x \in \mathcal{C}$ is the group of inertial invariants $H^1(\overline{X}, \mathbb{Q}_\ell)^{I_x}$.

The Grothendieck–Lefschetz trace formula for $\mathcal{F}$ reads

$$\sum_{x \in \mathcal{C}(\mathbb{F}_{q^n})} \mathrm{Tr}\left(\mathrm{Fr}_x\, |\mathcal{F}_{\overline{x}}\right) = \sum_i (-1)^i\, \mathrm{Tr}\left(\mathrm{Fr}_{q_n}\, |H^i\big(\overline{\mathcal{C}}, \mathcal{F}\big)\right)$$

and this leads to a cohomological expression for the L-function as a rational function in T:

$$L\big(J_X, T\big) = \prod_{i=0}^{2} Q_i\big(T\big)^{(-1)^{i+1}} \tag{6.1}$$

where

$$Q_i\big(J_X, T\big) = \det\big(1 - T\ \mathrm{Fr}_q\, |H^i\big(\overline{\mathcal{C}}, \mathcal{F}\big)\big) = \prod_j (1 - \beta_{ij} T).$$

By Deligne's theorem, the β_{ij} are Weil integers of size $q^{(i+1)/2}$.

It is convenient to have a criterion for $L(J_X, T)$ to be a polynomial.

Lemma 6.5. *$L(J_X, T)$ is a polynomial in T if and only if $H^0(\overline{\mathcal{C}}, \mathcal{F}) = H^2(\overline{\mathcal{C}}, \mathcal{F}) = 0$ if and only if the K/k-trace $B = \mathrm{Tr}_{K/k} J_X$ is zero.*

Proof. The first equivalence is immediate from the cohomological formula (6.1). For the second equivalence, we use the Lang–Néron theorem which says that if B is zero, then $J_X(\overline{k}K)$ is finitely generated. In this case, its torsion subgroup is finite and so

$$H^0\big(\overline{\mathcal{C}},\mathcal{F}\big) = \varprojlim_n \big(J_X[\ell^n]\big(\overline{k}K\big)\big) = 0.$$

Conversely, if $B \neq 0$, then $V_\ell B \subset V_\ell J_X$ and we see that $H^0(\overline{\mathcal{C}},\mathcal{F})$ contains a subspace of dimension $2\dim B > 0$. Thus $B = 0$ is equivalent to $H^0(\overline{\mathcal{C}},\mathcal{F}) = 0$. That these statements are equivalent to $H^2(\overline{\mathcal{C}},\mathcal{F}) = 0$ follows from duality (using autoduality up to a Tate twist of $\mathcal{F}$). □

Whether or not $L(J_X, s)$ is a polynomial, its zeroes lie on the line $\operatorname{Re} s = 1$, and the order of zero at $s = 1$ is equal to the multiplicity of q as an eigenvalue of Fr_q on $H^1(\overline{\mathcal{C}},\mathcal{F})$.

6.2.2 The basic BSD conjecture

Conjecture 6.6 (BSD)**.** *We have*

$$\operatorname{Rank} J_X\big(K\big) = \operatorname{ord}_{s=1} L\big(J_X, s\big).$$

Parallel to the Tate conjecture case, we have an inequality and implications among the conjectures.

Proposition 6.7. *We have*

1. $\operatorname{Rank} J_X(K) \leq \operatorname{ord}_{s=1} L(J_X, s)$.
2. BSD *(i.e., equality in* 1*) implies the Tate–Shafarevich conjecture* $TS(J_X,\ell)$ *(and thus also the Selmer group conjecture* $S(J_X,\ell)$*) for all* ℓ.
3. $TS(J_X,\ell)$ *for any* ℓ *implies* BSD.

Proof. The discussion in Subsection 5.3.2 shows that

$$\begin{aligned} -\operatorname{ord}_{s=1}\zeta\big(\mathcal{X},s\big) &- \operatorname{ord}_{s=1} L\big(J_X,s\big) \\ &= \dim H^2\big(\overline{\mathcal{X}},\mathbb{Q}_\ell(1)\big)^{G_k} - \dim H^1\big(\overline{\mathcal{C}},\mathbb{R}^1\pi_*\mathbb{Q}_\ell(1)\big)^{G_k} \\ &= 2 + \sum_v \big(f_v - 1\big). \end{aligned}$$

By the Shioda–Tate theorem,

$$\operatorname{Rank}\mathrm{NS}\big(\mathcal{X}\big) - \operatorname{Rank} J_X\big(K\big) = 2 + \sum_v \big(f_v - 1\big).$$

Thus

$$\operatorname{Rank} J_X\big(K\big) - \operatorname{ord}_{s=1} L\big(J_X,s\big) = \operatorname{Rank}\mathrm{NS}(\mathcal{X}) + \operatorname{ord}_{s=1}\zeta\big(\mathcal{X},s\big)$$

and so Rank $J_X(K) \leq \operatorname{ord}_{s=1} L(J_X(K), s)$ with equality if and only if $T_2(\mathcal{X})$ holds. But $T_2(\mathcal{X})$ implies the finiteness of the ℓ-primary part of $\mathrm{Br}(\mathcal{X})$ for all ℓ and thus finiteness of the ℓ-primary part of $\text{Ш}(J_X)$ for all ℓ. By Proposition 5.2, this proves that BSD implies the Selmer group conjecture $S(J_X, \ell)$ for all ℓ.

Conversely, if $S(J_X, \ell)$ holds for some ℓ, then the ℓ-primary part of $\text{Ш}(J_X)$ is finite and so is the ℓ-primary part of $\mathrm{Br}(\mathcal{X})$. By Proposition 5.1 this implies $T_1(\mathcal{X}, \ell)$ and therefore $T_2(\mathcal{X})$. By the previous paragraph, this implies BSD. □

6.2.3 The refined BSD conjecture

We need two more invariants. Recall that we have the canonical height pairing

$$\begin{aligned} J_X(K) \times J_X(K) &\longrightarrow \mathbb{R} \\ (P, Q) &\longmapsto \langle P, Q \rangle, \end{aligned}$$

which is bilinear and symmetric, and which takes values in $\mathbb{Q} \cdot \log q$. It makes $J_X(K)/\mathrm{tor}$ into a Euclidean lattice. We define the regulator of J_X to be the discriminant of the height pairing,

$$R = R(J_X) = \left| \det \left(\langle P_i, P_j \rangle \right)_{i,j=1,\ldots,r} \right|,$$

where $P_1, \ldots, P_r$ is a basis of $J_X(K)/\mathrm{tor}$. The regulator is $(\log q)^r$ times a rational number (with an *a priori* bounded denominator).

We also define the Tamagawa number of J_X as follows. Choose a non-zero, top-degree differential form ω on J_X over K. At each place v, let a_v be the integer such that $\pi_v^{a_v}\omega$ is a Néron differential at v. (Here π_v is a generator of the maximal ideal $\mathfrak{m}_v$ at v.) Also, let c_v be the number of connected components in the special fiber of the Néron model at v. Then we set

$$\tau = \tau(J_X) = q^{g_X(1-g_C)} \prod_v q^{a_v} c_v.$$

(See [62] for a less *ad hoc* version of this definition.)

The refined BSD conjecture ($rBSD$ for short) relates the leading coefficient of the L-function at $s = 1$ to the other invariants attached to J_X.

Conjecture 6.8 ($rBSD$)**.** *BSD holds, $\text{Ш}(J_X)$ is finite, and we have the asymptotic*

$$L(J_X, s) \sim \frac{R|\text{Ш}(J_X)|\tau}{|J_X(K)_{tor}|^2} (s-1)^r$$

as $s \to 1$.

As with the Tate and Artin–Tate conjectures, the basic conjecture implies the refined conjecture.

Theorem 6.9 (Kato–Trihan)**.** *If J_X satisfies the BSD conjecture, then it also satisfies the $rBSD$ conjecture.*

(Kato and Trihan [24] actually treat the general case of abelian varieties, not just Jacobians.)

This is a difficult theorem which completes a long line of research by many authors, including Artin, Tate, Milne, Gordon, Schneider, and Bauer. Its proof is well beyond what can be covered in these lectures, but we sketch a few of the main ideas.

In [51], Schneider generalized the results of Artin–Tate to abelian varieties: $|Ш(A)[\ell^\infty]| < \infty$ for every $\ell \neq p$ implies the prime-to-p part of $rBSD$. In [7], Bauer was able to handle the p-part for abelian varieties with everywhere good reduction. In all of the above, the main thing is to compare a "geometric cohomology" (such as $H^i(\overline{\mathcal{X}}, \mathbb{Z}_\ell)$ or crystalline cohomology) that computes the L-function to an "arithmetic cohomology" (such as the flat cohomology of μ_n) that relates to $A(K)$ and $Ш(A)$. Ultimately, on the geometric side one needs an integral theory which is supple enough to handle degenerating coefficients. Log crystalline cohomology does the job in the context of semistable reduction. The Kato–Trihan paper [24] handles the general case by an elaborate argument involving passing to an extension of K where A achieves semistable reduction, using log-syntomic cohomology upstairs to make a comparison with flat cohomology, and showing that the results can be brought back down to K by a Galois argument. It is a *tour de force* of p-adic cohomology theory.

6.3 Summary of implications

The results of this and the preceding two subsections show that many statements related to the BSD and Tate conjectures are equivalent. In fact, there are many redundancies, so the reader is invited to choose his or her favorite way to organize the implications. The net result is the following.

Theorem 6.10. *Let $\mathcal{X}$ be a smooth, proper, geometrically irreducible surface over a finite field k, equipped with a generically smooth morphism $\mathcal{X} \to \mathcal{C}$ to a smooth, proper, geometrically irreducible curve $\mathcal{C}$. Let K be the function field $k(\mathcal{C})$, let $X \to \operatorname{Spec} K$ be the generic fiber of π, and let J_X be the Jacobian of X. Then*

$$\operatorname{ord}_{s=1} L\big(J_X, s\big) - \operatorname{Rank} J_X\big(K\big) = -\operatorname{ord}_{s=1} \zeta\big(\mathcal{X}, s\big) - \operatorname{Rank} \operatorname{NS}\big(\mathcal{X}\big) \geq 0 \quad (6.2)$$

and the following are equivalent:

- *Equality holds in* (6.2).
- $Ш(J_X)[\ell^\infty]$ *is finite for every prime number* ℓ.
- $\operatorname{Br}(\mathcal{X})[\ell^\infty]$ *is finite for every prime number* ℓ.
- $Ш(J_X)$ *is finite.*
- $\operatorname{Br}(\mathcal{X})$ *is finite.*

If these conditions hold, then the refined BSD conjecture $rBSD(J_X)$ and the Artin–Tate conjecture $AT(\mathcal{X})$ hold as well.

To end the section, we quote one more precise result [35] on the connection between $Ш(J_X)$ and $\operatorname{Br}(\mathcal{X})$:

Proposition 6.11 (Liu–Lorenzini–Raynaud). *Assume that the equivalent conditions of Theorem* 6.10 *hold. Then the order of* $\mathrm{Br}(\mathcal{X})$ *is a square and we have*

$$\left|Ш(J_X)\right|\prod_v \delta_v\delta'_v = \left|\,\mathrm{Br}\left(\mathcal{X}\right)\right|\delta^2.$$

Here δ is the index of X, and δ_v and δ'_v are the index and period of $X \times_K K_v$.

We refer to [35] for an interesting history of misconceptions about when the order of the Brauer group or Tate–Shafarevich group is a square.

7 Complements

7.1 Abelian varieties over K

Assume that k is finite and let A be an abelian variety over K. Then the BSD and refined BSD conjectures make sense for A. (See [24] for the statement, which involves also the dual abelian variety.) Kato–Trihan and predecessors proved in this case too that $\mathrm{Rank}\, A(K) \le \mathrm{ord}_{s=1} L(A, s)$ with equality if and only if $Ш(A)[\ell^\infty]$ is finite for one ℓ if and only if $Ш(A)$ is finite, and that, when these conditions hold, the refined BSD conjecture does too.

We add one simple observation to this story:

Lemma 7.1. *If the BSD conjecture holds for Jacobians over K, then it also holds for abelian varieties over K.*

Proof. It is well known that given an abelian variety A over K, there is another abelian variety A' over K and a Jacobian J over K with an isogeny $J \to A \oplus A'$. If BSD holds for Jacobians, then it also holds for $A \oplus A'$. But since we have an inequality "rank $\le$ ord" for abelian varieties, equality for the direct sum implies equality for the factors. Thus BSD holds for A as well. □

7.2 Finite k, other special value formulas

Special value conjectures for varieties over finite fields can be made more streamlined by stating them in terms of Euler characteristics of cohomology of suitable complexes of sheaves (the so-called "motivic cohomology") and the whole zeta function (as opposed to just the piece corresponding to one part of cohomology) – in contrast with the Artin–Tate conjecture, which relates to H^2. See for example [30], [42], and [51]. See also the note [42] on Milne's web site for a particularly streamlined statement using Weil-étale cohomology.

7.3 Finitely generated k

The analytic and algebraic conjectures discussed in the preceding sections have generalizations to the case where k is finitely generated, rather than finite. In this subsection we give a quick overview of the statements and a few words about some of the comparisons.

Assume that k is finitely generated over its prime field. We assume as always that $\mathcal{X}$ is a smooth, proper, geometrically irreducible surface over k equipped with a flat, projective, relatively minimal morphism $\pi\colon \mathcal{X} \to \mathcal{C}$ where $\mathcal{C}$ is a smooth, geometrically irreducible curve over k. As usual, $K = k(\mathcal{C})$ is the function field of $\mathcal{C}$ and $X \to \operatorname{Spec} K$ is the generic fiber of π.

7.3.1 Algebraic conjectures

Fix a prime ℓ (with $\ell = p$ allowed). The conjectures $T_1(\mathcal{X}, \ell)$ and $S(J_X, \ell)$ have straightforward generalizations to the case where k is finitely generated.

As before, we have an exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\overline{\mathcal{X}}\right) \otimes \mathbb{Q}_\ell \longrightarrow H^2\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell(1)\big) \longrightarrow V_\ell \operatorname{Br}\left(\overline{\mathcal{X}}\right) \longrightarrow 0$$

induced from the Kummer sequence as in Subsection 5.1.1.

Taking G_k invariants, we have an exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Q}_\ell \longrightarrow H^2\big(\overline{\mathcal{X}}, \mathbb{Q}_\ell(1)\big)^{G_k} \longrightarrow \big(V_\ell \operatorname{Br}\left(\overline{\mathcal{X}}\right)\big)^{G_K} \longrightarrow 0.$$

There is a 0 on the right since $H^1(k, \mathrm{NS}(\overline{\mathcal{X}}) \otimes \mathbb{Q}_\ell)$ is both a torsion group and a $\mathbb{Q}_\ell$-vector space. Conjecture $T_1(\mathcal{X}, \ell)$ is the statement that the first map above is an isomorphism.

The exact sequence above shows that the vanishing of a certain Brauer group (namely $(V_\ell \operatorname{Br}(\overline{\mathcal{X}}))^{G_K}$) is equivalent to $T_1(\mathcal{X}, \ell)$. Note that one cannot expect that $\operatorname{Br}(\mathcal{X})$ is finite in general, as it may contain copies of $\operatorname{Br}(k)$.

It is also straightforward to generalize the Selmer group and Tate–Shafarevich conjectures. Define groups $\operatorname{Sel}(J_X, \mathbb{Z}_\ell)$ and $Ш(J_X)$ exactly as in Subsection 5.2.1. Then we have an exact sequence

$$0 \longrightarrow J_X\big(K\big) \otimes \mathbb{Z}_\ell \longrightarrow \operatorname{Sel}\big(J_X, \mathbb{Z}_\ell\big) \longrightarrow T_\ell Ш\big(J_X\big) \longrightarrow 0.$$

The Selmer group conjecture is that the first map is an isomorphism, and this is obviously equivalent to the vanishing of $T_\ell Ш(J_X)$.

7.3.2 Setup for analytic conjectures

Recall that k is assumed to be finitely generated over its prime field. This means there is an irreducible, regular scheme Z of finite type over $\operatorname{Spec} \mathbb{Z}$ whose function field is k. Of course Z is only determined up to birational isomorphism and we may shrink it in the course of the discussion. We write d for the dimension of Z.

We now choose models of the data over Z. That is, we choose schemes $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{C}}$ with morphisms

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[rr, "h"] \arrow[dr, "f"'] & & \tilde{\mathcal{C}} \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

such that the generic fiber is

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \operatorname{Spec} k. & \end{tikzcd}$$

After shrinking Z if necessary, we can and will assume that f, g, and h are smooth and proper. For a closed point $z \in Z$, the residue field at z is finite, and we assume that the fiber over z of the diagram above satisfies our usual hypotheses (in particular h_z should be relatively minimal). Shrinking Z further, we may assume that h is "equisingular" in the sense that as $\overline{z}$ varies through geometric points over closed points $z \in Z$, the fibers $h_{\overline{z}}\colon \tilde{\mathcal{X}}_{\overline{z}} \to \tilde{\mathcal{C}}_{\overline{z}}$ have the same number of singular fibers with the same configurations of components. (More precisely, we may assume that the set of critical values of h is étale over Z.)

7.3.3 Analytic conjectures

For the rest of this subsection, ℓ is a prime $\neq \mathrm{Char}(k)$. For each smooth, projective scheme V of dimension n over a finite field of cardinality q, we write the zeta function of V as

$$\zeta\big(V,s\big) = \frac{P_1(V,s)\cdots P_{2n-1}(V,s)}{P_0(V,s)\cdots P_{2n}(V,s)},$$

where the P_i are the characteristic polynomials of Frobenius at q^{-s},

$$P_i\big(V,s\big) = \det\big(1 - q^{-s}\,\mathrm{Fr}_q\,|H^i\big(\overline{V},\mathbb{Q}_\ell\big)\big)\,.$$

Now define

$$\Phi_2\big(\mathcal{X}/k,s\big) = \prod_{z\in Z} P_2\big(\tilde{\mathcal{X}}_z,s\big)^{-1}.$$

Here the product is over closed points z of Z and $\tilde{\mathcal{X}}_z$ is the fiber of f over z, which by our hypotheses is a smooth projective surface over the residue field at z.

Similarly, define

$$\Phi_1\big(X/K,s\big) = \prod_{c\in\tilde{\mathcal{C}}} P_1\big(\tilde{\mathcal{X}}_c,s\big)^{-1};$$

here the product is now over the closed points of $\mathcal{C}$. In this case, the fibers are projective curves, but in general they will not be smooth or irreducible, and so we need a slight extension of our definition of the polynomials P_1. (For the order of vanishing conjecture to follow, we could shrink $\tilde{\mathcal{C}}$ to avoid this problem, but for the comparison between conjectures to follow, it is more convenient to set up the definitions as we have done.)

With these definitions, we have the following conjectures:

$$T_2(\mathcal{X}): \qquad -\operatorname{ord}_{s=d+1} \Phi_2(\mathcal{X}/k, s) = \operatorname{Rank} \operatorname{NS}(\mathcal{X})$$

and

$$BSD(X): \qquad \operatorname{ord}_{s=d+1} \Phi_1(X/K, s) = \operatorname{Rank} J_X(K).$$

7.3.4 Comparison of conjectures

Comparing the various conjectures for finitely generated k seems to be much more complicated than for finite k.

The Shioda–Tate isomorphism (4.1) works for general k and gives a connection between the finitely generated groups $\operatorname{NS}(\mathcal{X})$, $J_X(K)$, and $B(k)$.

To discuss the zeta function side of the analytic conjectures, we make the assumption that $\operatorname{Char}(k) = p > 0$. (In characteristic 0, very little can be said about zeta functions without automorphic techniques which are far outside the scope of these notes.) So let us assume that Z has characteristic p and that its field of constants has cardinality q.

In this case, the Grothendieck–Lefschetz trace formula shows that the order of the pole of $\Phi_2(\mathcal{X}/k, s)$ at $s = d+1$ is equal to the multiplicity of q^{d+1} as an eigenvalue of Fr_q on $H^{2d}_c(\overline{Z}, R^2 f_* \mathbb{Q}_\ell)$ for any $\ell \neq p$. The Leray spectral sequence for $f = gh$ relates $R^2 f_* \mathbb{Q}_\ell$ to $R^2 g_*(h_* \mathbb{Q}_\ell)$, $R^1 g_*(R^1 h_* \mathbb{Q}_\ell)$, and $g_*(R^2 h_* \mathbb{Q}_\ell)$. It is not hard to work out $H^{2d}_c(\overline{Z}, R^2 g_*(h_* \mathbb{Q}_\ell))$ and $H^{2d}_c(\overline{Z}, g_*(R^2 h_* \mathbb{Q}_\ell))$, and one finds that they contribute exactly $1 + \sum_v (f_v - 1)$ (notation as in Corollary 4.2) to the order of the pole.

It remains to consider $H^{2d}_c(\overline{Z}, R^1 g_*(R^1 h_* \mathbb{Q}_\ell))$. The Leray spectral sequence for h leads to an exact sequence

$$\begin{aligned} 0 \longrightarrow H^{2d}_c\big(\overline{Z}, R^1 g_*\big(R^1 h_* \mathbb{Q}_\ell\big)\big) &\longrightarrow H^{2d+1}_c\Big(\overline{\tilde{\mathcal{C}}}, R^1 h_* \mathbb{Q}_\ell\Big) \\ &\longrightarrow H^{2d-1}_c\big(\overline{Z}, R^2 g_*\big(R^1 h_* \mathbb{Q}_\ell\big)\big) \longrightarrow 0. \end{aligned}$$

The middle group is exactly what controls $\Phi_1(X/K, s)$ (in other words, its order of vanishing at $s = d+1$ is the multiplicity of q^{d+1} as eigenvalue of Fr_q on this group). On the other hand, after some unwinding, one sees that the group $H^{2d-1}_c(\overline{Z}, R^2 g_*(R^1 h_* \mathbb{Q}_\ell))$ is related to B via a BSD conjecture.

Thus it seems possible (after filling in many details) to show that $T_2(\mathcal{X})$ and $BSD(J_X) + BSD(B)$ should be equivalent in positive characteristic. (We remark that Tate's article [60] gives a different analytic comparison, namely between $T_2(\mathcal{X})$ and a BSD conjecture related to the rank of $\operatorname{PicVar}_{\mathcal{X}}$, for general finitely generated k.)

A comparison between the algebraic conjectures along the lines of Proposition 5.4 looks like an interesting project. We remark that it will certainly be more complicated than over a finite k. For example, if $K' = K\overline{k}$, then in the limit

$$\varprojlim_n J_X\big(K'\big)/\ell^n$$

the subgroup $B(\overline{k})$ dies as it is ℓ-divisible. Thus the kernel of a map $\mathrm{Sel}(J_X, \mathbb{Z}_\ell) \to \mathrm{Sel}(J_X/K', \mathbb{Z}_\ell)^{G_k}$ cannot be finite in general.

Finally, a comparison of analytic and algebraic conjectures, along the lines of the results of Artin–Tate and Milne (as always, assuming that k has positive characteristic), also seems to be within the realm of current technology (although the author does not pretend to have worked out the details).

I know of no strong evidence in favor of the conjectures in this subsection beyond the case where k is a global field.

7.4 Large fields k

We simply note that if k is "large" then one cannot expect finiteness of $\mathrm{Br}(\mathcal{X})$ or $\text{Ш}(J_X)$. For example, the exact sequence

$$0 \longrightarrow \mathrm{NS}\left(\overline{\mathcal{X}}\right) \otimes \mathbb{Z}_\ell \longrightarrow H^2\big(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\big) \longrightarrow T_\ell \,\mathrm{Br}\left(\overline{\mathcal{X}}\right) \longrightarrow 0$$

shows that over a separably closed field, when the rank of H^2 is larger than the Picard number, Br has divisible elements.

It may also happen that $\mathrm{Br}(\mathcal{X})$ or $\text{Ш}(J_X)$ has infinite p-torsion. For examples in the case of Ш (already for elliptic curves), see [64, 7.12b], where there is a Selmer group which is in a suitable sense linear as a function of the finite ground field, and is thus infinite when $k = \overline{\mathbb{F}}_p$. This phenomenon is closely related to torsion in the de Rham–Witt cohomology groups. See [65] for some related issues.

8 Known cases of the Tate conjecture and consequences

8.1 Homomorphisms of abelian varieties and T_1 for products

Let k be a field finitely generated over its prime field. In the article where he first conjectured $T_1(\mathcal{X}, \ell)$ for smooth projective varieties $\mathcal{X}$ over k, Tate explained that the case of abelian varieties is equivalent to an attractive statement on homomorphisms of abelian varieties. Namely, if A and B are abelian varieties over k, then for any $\ell \neq \mathrm{Char}(k)$ the natural homomorphism

$$\mathrm{Hom}_k\big(A, B\big) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{Hom}_{G_k}\big(T_\ell A, T_\ell B\big) \tag{8.1}$$

should be an isomorphism.

In [61], Tate gave an axiomatic framework showing that isomorphism in (8.1) follows from a fundamental finiteness statement for abelian varieties over k, which is essentially trivial for finite k. The finiteness statement (and thus isomorphism in (8.1)) was later proven for all finitely generated fields of characteristic $p > 0$ by Zarhin [73] ($p > 2$) and Mori ($p = 2$) – see [43] – and for finitely generated fields of characteristic zero by Faltings [14].

Isomorphism in (8.1) in turn implies $T_1(\mathcal{X}, \ell)$ for products of curves and abelian varieties (and more generally for products of varieties for which T_1 is known). This is explained, for example, in [61] or [70].

Summarizing the part of this most relevant for us, we have the following statement.

Theorem 8.1. *If k is a finitely generated field and $\mathcal{X}$ is a product of curves over k, then for all $\ell \neq \mathrm{Char}(k)$ there is an isomorphism*

$$\mathrm{NS}\left(\mathcal{X}\right) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} H^2\left(\overline{\mathcal{X}}, \mathbb{Z}_\ell(1)\right)^{G_k}.$$

8.2 Descent property of T_1 and domination by a product of curves

The property in question is the following.

Lemma 8.2. *Suppose k is a finitely generated field and $\mathcal{X}$ is smooth, proper variety over k satisfying T_1. If $\mathcal{X} \dashrightarrow \mathcal{Y}$ is a dominant rational map, then $T_1(\mathcal{Y})$ holds as well.*

This is explained in [63] and, with an unnecessary hypothesis on resolution of singularities, in [52]. The latter article points out the utility of combining Theorem 8.1 and Lemma 8.2 to prove the following result.

Corollary 8.3. *If k is a finitely generated field and $\mathcal{X}$ is a variety admitting a dominant rational map from a product of curves $\prod_i \mathcal{C}_i \dashrightarrow \mathcal{X}$, then T_1 holds for $\mathcal{X}$.*

The corollary implies most of the known cases of T_1 for surfaces (some of which were originally proven by other methods). Namely, we have T_1 for: rational surfaces, unirational surfaces, Fermat surfaces, and more generally hypersurfaces of dimension 2 defined by an equation in 4 monomials (with mild hypotheses). We refer to [56] for these last surfaces, which Shioda calls "Delsarte surfaces". See also [66] and [68].

Of course, for finite k, surfaces which satisfy T_1 lead to curves whose Jacobians satisfy BSD. We will explain in the next subsection how this leads to Jacobians of large rank over global function fields.

Remark 8.4. In a letter to Grothendieck [10] dated March 31, 1964, Serre constructed an example of a surface which is not dominated by a product of curves. In a note to this letter, Serre says that Grothendieck had hoped to prove the Weil conjectures by showing that every variety is dominated by a product of curves, thus reducing the problem to the known case of curves. We are thankful to Bruno Kahn for pointing out this letter. See also [52] for other examples of varieties not dominated by products of curves.

8.3 Other known cases of T_1 for surfaces

Assume that k is a finite field. The other main systematic cases of T_1 for surfaces over k are for $K3$ surfaces. Namely, Artin and Swinnerton-Dyer showed in [4] that $Ш(E)$ is finite for an elliptic curve E over $k(t)$ when the corresponding elliptic surface $\mathcal{E} \to \mathbb{P}^1_k$ is a K3 surface. Also, Nygaard and Ogus showed in [46] that if $\mathcal{X}$ is a $K3$ surface of finite height and $\mathrm{Char}(k) \geq 5$, then T_1 holds for $\mathcal{X}$.

In a recent preprint, Lieblich and Maulik show that the Tate conjecture holds for $K3$ surfaces over finite fields of characteristic $p \geq 5$ if and only if there are only finitely many isomorphism classes of $K3$s over each finite field of characteristic p. This is reminiscent of Tate's axiomatization of T_1 for abelian varieties.

It was conjectured by Artin in [1] that a K3 surface of infinite height has Néron–Severi group of rank 22, the maximum possible, and so this conjecture together with [46] would imply the Tate conjecture for $K3$ surfaces over fields of characteristic ≥ 5. Just as we are finishing these notes, Maulik and Madapusi Pera have announced (independently) results that would lead to a proof of the Tate conjecture for $K3$s with a polarization of low degree or of degree prime to $p = \mathrm{Char}(k)$.

Finally, we note the "direct" approach: We have *a priori* inequalities

$$\mathrm{Rank}\,\mathrm{NS}\,\big(\mathcal{X}\big) \leq \dim_{\mathbb{Q}_\ell} H^2\big(\mathcal{X}, \mathbb{Q}_\ell(1)\big)^{G_k} \leq -\operatorname{ord}_{s=1} \zeta\big(\mathcal{X}, s\big)$$

and if equality holds between the ends or between the first two terms, then we have the full Tate conjectures. It is sometimes possible to find, say by a geometric construction, enough cycles to force equality. We will give an example of this in the context of elliptic surfaces in the next section.

9 Ranks of Jacobians

In this final section, we discuss how the preceding results on the BSD and Tate conjectures can be used to find examples of Jacobians with large analytic rank and algebraic rank over function fields over finite fields. In the first four subsections we briefly review results from recent publications, and in the last three subsections we state a few results that will appear in future publications.

9.1 Analytic ranks in the Kummer tower

In this subsection k is a finite field, $K = k(t)$, and $K_d = k(t^{1/d})$ for each positive integer d prime to p.

It turns out that roughly half of all abelian varieties defined over K have unbounded analytic rank in the tower K_d. More precisely, those that satisfy a simple parity condition have unbounded rank.

Theorem 9.1. *Let A be an abelian variety over K with Artin conductor $\mathfrak{n}$ (an effective divisor on $\mathbb{P}^1_k$). Write $\mathfrak{n}'$ for the part of $\mathfrak{n}$ prime to the places 0 and ∞ of K. Let $\mathrm{Swan}_v(A)$ be the exponent of the Swan conductor of A at a place v of K. Suppose that the following integer is odd:*

$$\deg(\mathfrak{n}') + \mathrm{Swan}_0(A) + \mathrm{Swan}_\infty(A).$$

Then there is a constant c depending only on A such that if $d = p^f + 1$ then

$$\mathrm{ord}_{s=1} L\big(A/K_d, s\big) \geq \frac{p^f + 1}{o_d(q)} - c,$$

where $o_q(d)$ is the order of q in $(\mathbb{Z}/d\mathbb{Z})^\times$.

This is a slight variant of [68, Theorem 4.7] applied to the representation of G_K on $V_\ell A$. Note that $o_d(q) \leq 2f$, so the rank tends to infinity with f.

9.2 Jacobians of large rank

Theorem 9.1 gives an abundant supply of abelian varieties with large analytic rank. Using Corollary 8.3, we can use this to produce Jacobians of large algebraic rank.

Theorem 9.2. *Let p be an odd prime number, $K = \mathbb{F}_p(t)$, and $K_d = \mathbb{F}_p(t^{1/d})$. Choose a positive integer g such that $p \not| (2g+2)(2g+1)$ and let X be the hyperelliptic curve over K defined by*

$$y^2 = x^{2g+2} + x^{2g+1} + t.$$

Let J_X be the Jacobian of X, an abelian variety of dimension g over K. Then for all d the BSD conjecture holds for J_X over K_d, and there is a constant depending only on p and g such that for all d of the form $d = p^f + 1$ we have

$$\mathrm{Rank}\, J_X\big(K_d\big) \geq \frac{p^f + 1}{2f} - c.$$

This is one of the main results of [68]. We sketch the key steps in the proof. The analytic rank of J_X is computed using Theorem 9.1. Because X is defined by an equation involving 4 monomials in 3 variables, by Shioda's construction it turns out that the surface $\mathcal{X} \to \mathbb{P}^1$ associated to X/K_d is dominated by a product of curves. Therefore T_2 holds for $\mathcal{X}$ and BSD holds for J_X. In [68] we also checked that J_X is absolutely simple and has trivial $K/\mathbb{F}_q$ trace for all d.

In [68] we gave other examples so that for every p and every $g > 0$ there is an absolutely simple Jacobian of dimension g over $\mathbb{F}_p(t)$ which satisfies BSD and has arbitrarily large rank in the tower of fields K_d.

9.3 Berger's construction

The results of the preceding subsections show that Shioda's 4-monomial construction is a powerful tool for deducing BSD for certain specific Jacobians. However, it has the defect that it is rigid – the property of being dominated by a Fermat surface, or more generally by a product of Fermat curves, does not deform.

In her thesis, Berger gave a more flexible construction which leads to *families* of Jacobians satisfying BSD in each layer of the Kummer tower. We quickly sketch the main idea; see [8] and [71] for more details.

Let k be a general field and fix two smooth, proper, geometrically irreducible curves $\mathcal{C}$ and $\mathcal{D}$ over k. Fix also two separable, non-constant rational functions $f \in k(\mathcal{C})^\times$ and $g \in k(\mathcal{D})^\times$ and form the rational map

$$\mathcal{C} \times \mathcal{D} \dashrightarrow \mathbb{P}^1, \qquad (x, y) \longmapsto f(x)/g(y).$$

Under mild hypotheses on f and g, the generic fiber of this map has a smooth, proper model $X \to \operatorname{Spec} K = k(t)$.

Berger's construction is designed so that the following is true.

Theorem 9.3 (Berger). *With notation as above, for all d prime to $p = \mathrm{Char}(k)$, the surface $\mathcal{X}_d \to \mathbb{P}^1$ associated to X over $K_d = k(t^{1/d}) \cong k(u)$ is dominated by a product of curves. Thus if k is finite, the BSD conjecture holds for J_X over K_d.*

It is convenient to think of the data in Berger's construction as consisting of a discrete part, namely the genera of $\mathcal{C}$ and $\mathcal{D}$ and the combinatorial type of the divisors of f and g, and a continuous part, namely the moduli of the curves and the locations of the zeroes and poles of the functions. From this point of view it is clear that there is enormous flexibility in the choice of data. In [8], Berger used this flexibility to produce families of elliptic curves over $\mathbb{F}_p(t)$ such that for almost all specializations of the parameters to values in $\mathbb{F}_q$, the resulting elliptic curve over $\mathbb{F}_q(t)$ satisfies BSD and has unbounded rank in the tower $K_d = \mathbb{F}_q(t^{1/d})$.

Note that the values of d which give high ranks (via Theorem 9.1) are those of the form $p^f + 1$ and there is a well-known connection between such values and supersingularity of abelian varieties over extensions of $\mathbb{F}_p$. Using Berger's construction and ideas related to those in the next subsection, Occhipinti produced elliptic curves over $\mathbb{F}_p(t)$ which have high rank at *every* layer of the tower $\overline{\mathbb{F}}_p(t^{1/d})$. More precisely, he found curves $E/\mathbb{F}_p(t)$ such that $\operatorname{Rank} E(\overline{\mathbb{F}}_p(t^{1/d}) \geq d$ for all d prime to p. This seems to be a completely new phenomenon. See [47] and the paper based on it for details.

9.4 Geometry of Berger's construction

In [71], we made a study of the geometry of Berger's construction with a view toward more explicit results on ranks and Mordell–Weil groups.

To state one of the main results, let $\mathcal{C}$, $\mathcal{D}$, f, and g be as in Berger's construction. Define covers $\mathcal{C}_d \to \mathcal{C}$ and $\mathcal{D}_d \to \mathcal{D}$ by the equations

$$z^d = f(x), \qquad w^d = g(x).$$

We have an action of μ_d on $\mathcal{C}_d$ and $\mathcal{D}_d$ and their Jacobians. It turns out that the surface $\mathcal{X}_d \to \mathbb{P}^1$ associated to X over $K_d = k(t^{1/d})$ is birational to the quotient of $\mathcal{C}_d \times \mathcal{D}_d$ by the diagonal action of μ_d. This explicit domination by a product of curves makes it possible to compute the rank of J_X over K_d in terms of invariants of $\mathcal{C}_d$ and $\mathcal{D}_d$. More precisely,

Theorem 9.4. *Suppose that k is algebraically closed. Then there exist non-negative integers c_1, c_2, and N such that for all d relatively prime to N we have*

$$\operatorname{Rank} \mathrm{MW}\left(J_X/K_d\right) = \operatorname{Rank} \operatorname{Hom}_{k-av}\left(J_{\mathcal{C}_d}, J_{\mathcal{D}_d}\right)^{\mu_d} - dc_1 + c_2.$$

Here the superscript indicates those homomorphisms which commute with the action of μ_d on the two Jacobians.

The constants c_1 and c_2 are given explicitly in terms of the input data, and the integer N is there just to make the statement simple – there is a formula for the rank for any value of d. See [71, Theorem 6.4] for the details.

The theorem relates ranks of abelian varieties over a function field K_d to homomorphisms of abelian varieties over the constant field k. The latter is sometimes more tractable. For example, when $k = \overline{\mathbb{F}}_p$, the homomorphism groups can be made explicit via Honda–Tate theory.

Here is another example where the endomorphism side of the formula in Theorem 9.4 is tractable: Using it and Zarhin's results on endomorphism rings of superelliptic Jacobians, we gave examples in [72] of Jacobians over function fields of characteristic zero with bounded ranks in certain towers. More precisely:

Theorem 9.5. *Let g_X be an integer ≥ 2 and let X be the smooth, proper curve of genus g_X over $\mathbb{Q}(t)$ with affine plane model*

$$ty^2 = x^{2g_X+1} - x + t - 1.$$

Then for every prime number p and every integer $n \geq 0$ we have

$$\operatorname{Rank} J_X\left(\overline{\mathbb{Q}}\left(t^{1/p^n}\right)\right) = 2g_X.$$

Another dividend of the geometric analysis is that it can sometimes be used to find explicit points in high rank situations. (In principle this is also true in the context of Shioda's 4-monomial construction, but I know of few cases where it has been successfully carried out, a notable exception being [57], which relates to an isotrivial elliptic curve.)

The following example is [71, Theorem 8.1]. The points which appear here are related to the graphs of Frobenius endomorphisms of the curves $\mathcal{C}_d$ and $\mathcal{D}_d$ for a suitable choice of data in Berger's construction.

Theorem 9.6. *Fix an odd prime number p, let $k = \overline{\mathbb{F}}_p$, and let $K = k(t)$. Let X be the elliptic curve over K defined by*

$$y^2 + xy + ty = x^3 + tx^2.$$

For each d prime to p, let $K_d = k(t^{1/d}) \cong k(u)$. We write ζ_d for a fixed primitive dth root of unity in k. Let $d = p^n + 1$, let $q = p^n$, and let

$$P(u) = \left(\frac{u^q(u^q - u)}{(1+4u)^q}, \ \frac{u^{2q}(1 + 2u + 2u^q)}{2(1+4u)^{(3q-1)/2}} - \frac{u^{2q}}{2(1+4u)^{q-1}} \right).$$

Then the points $P_i = P(\zeta_d^i u)$ for $i = 0, \dots, d-1$ lie in $X(K_d)$ and they generate a finite index subgroup of $X(K_d)$, which has rank $d-2$. The relations among them are that $\sum_{i=0}^{d-1} P_i$ and $\sum_{i=0}^{d-1} (-1)^i P_i$ are torsion.

The main result of [71] also has implications for elliptic curves over $\mathbb{C}(t)$. We refer the reader to the last section of that paper for details.

9.5 Artin–Schreier analogues

The results of the preceding four subsections are all related to the arithmetic of Jacobians in the Kummer tower $K_d = k(t^{1/d})$. It turns out that all the results – high analytic ranks, a Berger-style construction of Jacobians satisfying BSD, a rank formula, and explicit points – have analogues for the Artin–Schreier tower, that is, for extensions $k(u)/k(t)$ where $u^q - u = f(t)$. See a forthcoming paper with Rachel Pries [49] for more details.

9.6 Explicit points on the Legendre curve

As we mentioned before, if one can write down enough points to fill out a finite index subgroup of the Mordell–Weil group of an abelian variety, then the full BSD conjecture follows. This plays out in a very satisfying way for the Legendre curve.

More precisely, let p be an odd prime, let $K = \mathbb{F}_p(t)$, and let

$$K_d = \mathbb{F}_p(\mu_d)(t^{1/d}) \cong \mathbb{F}_p(\mu_d)(u).$$

Consider the Legendre curve

$$E: \qquad y^2 = x(x+1)(x+t)$$

over K. (The signs are not the traditional ones, and E is a twist of the usual Legendre curve. It turns out that the + signs are more convenient, somewhat analogously to the situation with signs of Gauss sums.)

If we take d of the form $d = p^f + 1$, then there is an obvious point on $E(K_d)$, namely $P(u) = (u, u(u+1)^{d/2})$. Translating this point by $\mathrm{Gal}(K_d/K)$ yields more points: $P_i = P(\zeta_d^i u)$ where $\zeta_d \in \mathbb{F}_p(\mu_d)$ is a primitive dth root of unity and $i = 0, \dots, d-1$.

In [69] we give an elementary proof of the following.

Theorem 9.7. *The points P_i generate a subgroup of $E(K_d)$ of finite index and rank $d-2$. The relations among them are that $\sum_{i=0}^{d-1} P_i$ and $\sum_{i=0}^{d-1}(-1)^i P_i$ are torsion.*

It is easy to see that the L-function of E/K_d has order of zero at $s=1$ *a priori* bounded by $d-2$, so we have the equality $\operatorname{ord}_{s=1} L(E/K_d, s) = \operatorname{Rank} E(K_d)$, i.e., the BSD conjecture. Thus we also have the refined BSD conjecture. Examining this leads to a beautiful "analytic class number formula": Let V_d be the subgroup of $E(K_d)$ generated by the P_i and $E(K_d)_{\mathrm{tor}}$. (The latter is easily seen to be of order 8.) Then we have

$$\big[E(K_d) : V_d\big]^2 = \big|Ш\big(E/K_d\big)\big|$$

and these numbers are powers of p.

It then becomes a very interesting question to make the displayed quantity more explicit. It turns out that E is closely related to the X of the previous section, and this brings the geometry of "domination by a product of curves" into play. Using that, we are able to give a complete description of $Ш(E/K_d)$ and $E(K_d)/V_d$ as modules over the group ring $\mathbb{Z}_p[\mathrm{Gal}(K_d/K)]$. These results will appear in a paper in preparation.

It also turns out that we can control the rank of $E(K_d)$ for general d, not just those of the form p^f+1. Surprisingly, we find that there is high rank for many other values of d, and in a precise quantitative sense, among the values of d where $E(K_d)$ has large rank, those prime to all p^f+1 are more numerous than those not prime to p^f+1. The results of this paragraph will appear in publications of the author and collaborators, including [48] and works cited there.

9.7 Characteristic 0

Despite significant effort, I have not been able to exploit Berger's construction to produce elliptic curves (or Jacobians of any fixed dimension) over $\mathbb{C}(t)$ with unbounded rank. Roughly speaking, efforts to increase the $\mathrm{Hom}(\dots)^{\mu_d}$ term, say by forcing symmetry, seem also to increase the value of c_1. (Although one benefit of this effort came from examination of an example that led to the explicit points in Theorem 9.6 above.)

At the workshop I explained a heuristic which suggests that ranks might be bounded in a Kummer tower in characteristic zero. Theorem 9.5 above is an example in this direction. I do not know how to prove anything that strong, but we do have the following negative result.

Recall that an elliptic surface $\pi\colon \mathcal{E} \to \mathcal{C}$ has a height, which can be defined as the degree of the invertible sheaf $(R^1\pi_*\mathcal{O}_{\mathcal{E}})^{-1}$. When $\mathcal{C} = \mathbb{P}^1$, the height is ≥ 3 if and only if the Kodaira dimension of $\mathcal{E}$ is 1. Recall also that there is a reasonable moduli space for elliptic surfaces of height d over $\mathbb{C}$ and it has dimension $10d-2$.

In what follows, “very general” means “belongs to the complement of countably many divisors in the moduli space”.

Theorem 9.8. *Suppose that $E/\mathbb{C}(t)$ is the generic fiber of $\mathcal{E} \to \mathbb{P}^1$ over $\mathbb{C}$ where $\mathcal{E}$ is a very general elliptic surface of height $d \geq 3$. Then for every finite extension $L/\mathbb{C}(t)$ where L is a rational field (i.e., $L \cong \mathbb{C}(u)$) we have $E(L) = 0$.*

The theorem is proven by controlling the collection of rational curves on $\mathcal{E}$ and is a significant strengthening of a Noether–Lefschetz type result (cf. [12]) according to which the Néron–Severi group of a very general elliptic surface is generated by the zero section and a fiber. More details on Theorem 9.8 and the heuristic that suggested it will appear in a paper currently in preparation.

Bibliography

[1] M. Artin. Supersingular $K3$ surfaces. *Ann. Sci. École Norm. Sup.* (4), 7:543–567 (1975), 1974.

[2] M. Artin. Lipman's proof of resolution of singularities for surfaces. In *Arithmetic Geometry* (*Storrs, Conn.,* 1984), pages 267–287. Springer, New York, 1986.

[3] M. Artin. Néron models. In *Arithmetic Geometry* (*Storrs, Conn.,* 1984), pages 213–230. Springer, New York, 1986.

[4] M. Artin and H.P.F. Swinnerton-Dyer. The Shafarevich–Tate conjecture for pencils of elliptic curves on $K3$ surfaces. *Invent. Math.*, 20:249–266, 1973.

[5] M. Artin and G. Winters. Degenerate fibres and stable reduction of curves. *Topology*, 10:373–383, 1971.

[6] L. Bădescu. *Algebraic Surfaces*. Universitext. Springer-Verlag, New York, 2001. Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[7] W. Bauer. On the conjecture of Birch and Swinnerton-Dyer for abelian varieties over function fields in characteristic $p > 0$. *Invent. Math.*, 108:263–287, 1992.

[8] L. Berger. Towers of surfaces dominated by products of curves and elliptic curves of large rank over function fields. *J. Number Theory*, 128:3013–3030, 2008.

[9] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*, vol. 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) [*Results in Mathematics and Related Areas* (3)]. Springer-Verlag, Berlin, 1990.

[10] P. Colmez and J.-P. Serre, eds. *Correspondance Grothendieck–Serre*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 2. Société Mathématique de France, Paris, 2001.

[11] B. Conrad. Chow's K/k-image and K/k-trace, and the Lang–Néron theorem. *Enseign. Math.* (2), 52:37–108, 2006.

[12] D.A. Cox. The Noether–Lefschetz locus of regular elliptic surfaces with section and $p_g \geq 2$. *Amer. J. Math.*, 112:289–329, 1990.

[13] P. Deligne. La conjecture de Weil, II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[14] G. Faltings. Finiteness theorems for abelian varieties over number fields. In *Arithmetic Geometry* (*Storrs, Conn.,* 1984), pages 9–27. Springer, New York, 1986. Translated from the German original [Invent. Math. **73** (1983), no. 3, 349–366; ibid. **75** (1984), no. 2, 381; MR 85g:11026ab] by Edward Shipz.

[15] M. Fontana. Sur le plongement projectif des surfaces définies sur un corps de base fini. *Compositio Math.*, 31:1–22, 1975.

[16] W.J. Gordon. Linking the conjectures of Artin–Tate and Birch–Swinnerton-Dyer. *Compositio Math.*, 38:163–199, 1979.

[17] B.H. Gross. Local heights on curves. In *Arithmetic Geometry* (*Storrs, Conn.,* 1984), pages 327–339. Springer, New York, 1986.

[18] A. Grothendieck. *Fondements de la géométrie algébrique.* [*Extraits du Séminaire Bourbaki,* 1957–1962.]. Secrétariat mathématique, Paris, 1962.

[19] A. Grothendieck. Le groupe de Brauer, III. Exemples et compléments. In *Dix exposés sur la cohomologie des schémas*, pages 88–188. North-Holland, Amsterdam, 1968.

[20] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics, 52. Springer-Verlag, New York, 1977.

[21] M. Hindry and A. Pacheco. Sur le rang des jacobiennes sur un corps de fonctions. *Bull. Soc. Math. France*, 133:275–295, 2005.

[22] L. Illusie. Complexe de de Rham–Witt. In *Journées de Géométrie Algébrique de Rennes* (*Rennes,* 1978)*, Vol. I*, volume 63 of *Astérisque*, pages 83–112. Soc. Math. France, Paris, 1979.

[23] L. Illusie. Finiteness, duality, and Künneth theorems in the cohomology of the de Rham–Witt complex. In *Algebraic Geometry* (*Tokyo/Kyoto,* 1982), volume 1016 of *Lecture Notes in Math.*, pages 20–72. Springer, Berlin, 1983.

[24] K. Kato and F. Trihan. On the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$. *Invent. Math.*, 153:537–592, 2003.

[25] N.M. Katz. *Moments, Monodromy, and Perversity: a Diophantine Perspective*, volume 159 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2005.

[26] S.L. Kleiman. The Picard scheme. In *Fundamental Algebraic Geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[27] K. Kramer. Two-descent for elliptic curves in characteristic two. *Trans. Amer. Math. Soc.*, 232:279–295, 1977.

[28] S. Lichtenbaum. Curves over discrete valuation rings. *Amer. J. Math.*, 90:380–405, 1968.

[29] S. Lichtenbaum. Duality theorems for curves over p-adic fields. *Invent. Math.*, 7:120–136, 1969.

[30] S. Lichtenbaum. Zeta functions of varieties over finite fields at $s = 1$. In *Arithmetic and Geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 173–194. Birkhäuser Boston, Boston, MA, 1983.

[31] C. Liedtke. A note on non-reduced Picard schemes. *J. Pure Appl. Algebra*, 213:737–741, 2009.

[32] J. Lipman. Desingularization of two-dimensional schemes. *Ann. of Math.* (2), 107:151–207, 1978.

[33] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[34] Q. Liu, D. Lorenzini, and M. Raynaud. Néron models, Lie algebras, and reduction of curves of genus one. *Invent. Math.*, 157:455–518, 2004.

[35] Q. Liu, D. Lorenzini, and M. Raynaud. On the Brauer group of a surface. *Invent. Math.*, 159:673–676, 2005.

[36] Ju.I. Manin. Cyclotomic fields and modular curves. *Uspekhi Mat. Nauk*, 26:7–71, 1971.

[37] J.S. Milne. Elements of order p in the Tate–Šafarevič group. *Bull. London Math. Soc.*, 2:293–296, 1970.

[38] J.S. Milne. On a conjecture of Artin and Tate. *Ann. of Math.* (2), 102:517–533, 1975.

[39] J.S. Milne. *Étale Cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980.

[40] J.S. Milne. *Arithmetic Duality Theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1986.

[41] J.S. Milne. Jacobian varieties. In *Arithmetic Geometry* (*Storrs, Conn.,* 1984), pages 167–212. Springer, New York, 1986.

[42] J.S. Milne. Values of zeta functions of varieties over finite fields. *Amer. J. Math.*, 108:297–360, 1986.

[43] L. Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque*, 129:266, 1985.

[44] D. Mumford. *Lectures on Curves on an Algebraic Surface*. With a section by G.M. Bergman. Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[45] D. Mumford and T. Oda. *Algebraic Geometry II*. In preparation.

[46] N. Nygaard and A. Ogus. Tate's conjecture for $K3$ surfaces of finite height. *Ann. of Math.* (2), 122:461–507, 1985.

[47] T. Occhipinti. *Mordell–Weil groups of large rank in towers*. ProQuest LLC, Ann Arbor, MI, 2010. Ph.D. Thesis, The University of Arizona.

[48] C. Pomerance and D. Ulmer. *On balanced subgroups of the multiplicative group*, in *Number theory and related fields*, volume 43 of Springer Proc. Math. Stat., pages 253–270, Springer, New York, 2013.

[49] R. Pries and D. Ulmer. Arithmetic of abelian varieties in Artin–Schreier extensions, preprint, 2013, arxiv:1305.5247

[50] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, 38:27–76, 1970.

[51] P. Schneider. Zur Vermutung von Birch und Swinnerton-Dyer über globalen Funktionenkörpern. *Math. Ann.*, 260:495–510, 1982.

[52] C. Schoen. Varieties dominated by product varieties. *Internat. J. Math.*, 7:541–571, 1996.

[53] J.-P. Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica, International Symposium on Algebraic Topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[54] J.-P. Serre. *Local Fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[55] *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6). Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J.P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J.-P. Serre.

[56] T. Shioda. An explicit algorithm for computing the Picard number of certain algebraic surfaces. *Amer. J. Math.*, 108:415–432, 1986.

[57] T. Shioda. Mordell–Weil lattices and sphere packings. *Amer. J. Math.*, 113:931–948, 1991.

[58] T. Shioda. Mordell--Weil lattices for higher genus fibration over a curve. In *New Trends in Algebraic Geometry* (*Warwick,* 1996), volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 359–373. Cambridge Univ. Press, Cambridge, 1999.

[59] M. Szydlo. Elliptic fibers over non-perfect residue fields. *J. Number Theory*, 104:75–99, 2004.

[60] J.T. Tate. Algebraic cycles and poles of zeta functions. In *Arithmetical Algebraic Geometry* (*Proc. Conf. Purdue Univ.,* 1963), pages 93–110. Harper & Row, New York, 1965.

[61] J.T. Tate. Endomorphisms of abelian varieties over finite fields. *Invent. Math.*, 2:134–144, 1966.

[62] J.T. Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. In *Séminaire Bourbaki, Vol.* 9, Exp. No. 306, pages 415–440. Soc. Math. France, Paris, 1966.

[63] J.T. Tate. Conjectures on algebraic cycles in ℓ-adic cohomology. In *Motives* (*Seattle, WA,* 1991), volume 55 of *Proc. Sympos. Pure Math.*, pages 71–83. Amer. Math. Soc., Providence, RI, 1994.

[64] D. Ulmer. p-descent in characteristic p. *Duke Math. J.*, 62:237–265, 1991.

[65] D. Ulmer. On the Fourier coefficients of modular forms, II. *Math. Ann.*, 304:363–422, 1996.

[66] D. Ulmer. Elliptic curves with large rank over function fields. *Ann. of Math.* (2), 155:295–315, 2002.

[67] D. Ulmer. Geometric non-vanishing. *Invent. Math.*, 159:133–186, 2005.

[68] D. Ulmer. L-functions with large analytic rank and abelian varieties with large algebraic rank over function fields. *Invent. Math.*, 167:379–408, 2007.

[69] D. Ulmer. Explicit points on the Legendre curve, *J. Number Theory*, 136:165–194, 2014.

[70] D. Ulmer. Elliptic curves over function fields. In *Arithmetic of L-functions* (*Park City, UT,* 2009), volume 18 of *IAS/Park City Math. Ser.*, pages 211–280. Amer. Math. Soc., Providence, RI, 2011.

[71] D. Ulmer. On Mordell–Weil groups of Jacobians over function fields, *J. Inst. Math. Jussieu*, 12:1–29, 2013.

[72] D. Ulmer and Y.G. Zarhin. Ranks of Jacobians in towers of function fields. *Math. Res. Lett.*, 17:637–645, 2010.

[73] Ju.G. Zarhin. Abelian varieties in characteristic p. *Mat. Zametki*, 19:393–400, 1976.