Slopes of modular forms and congruences

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Our aim in this paper is to prove congruences between on the one hand certain eigenforms of level pN and weight greater than 2 and on the other hand twists of eigenforms of level pN and weight 2. One knows *a priori* that such congruences exist; the novelty here is the we determine the character of the form of weight 2 and the twist in terms of the slope of the higher weight form, i.e., in terms of the valuation of its eigenvalue for U_p. Curiously, we also find a relation between the leading terms of the p-adic expansions of the eigenvalues for U_p of the two forms. This allows us to determine the restriction to the decomposition group at p of the Galois representation modulo p attached to the higher weight form.

1. Slopes and congruences Fix a prime number p, embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$, and let v be the induced valuation of $\overline{\mathbf{Q}}$, normalized so that $v(p) = 1$. We denote by $\wp$ the corresponding maximal ideal of $\mathcal{O}_{\overline{\mathbf{Q}}}$, the ring of algebraic integers, and we identify $\mathcal{O}_{\overline{\mathbf{Q}}}/\wp$ with $\overline{\mathbf{F}_p}$, an algebraic closure of the field of p elements. Let $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to {\mathbf{Q}_p}^\times$ be the Teichmüller character; using the embeddings we identify it with a complex valued character also denoted χ. Let N be a positive integer relatively prime to p. Any Dirichlet character $\psi : (\mathbf{Z}/pN\mathbf{Z})^\times \to \mathbf{C}^\times$ can be written uniquely as $\chi^a\epsilon$ with ϵ a character modulo N and $0 \le a < p-1$. If ϵ and δ are two Dirichlet characters, we write $\epsilon \equiv \delta \pmod{\wp}$ if $\epsilon(n) \equiv \delta(n) \pmod{\wp}$ for all integers n.

Let $M_w(\Gamma_0(pN), \chi^a\epsilon)$ and $S_w(\Gamma_0(pN), \chi^a\epsilon)$ be the complex vector spaces of modular forms and cusp forms of weight w and character $\chi^a\epsilon$ for $\Gamma_0(pN)$. Acting on these spaces we have Hecke operators T_ℓ for $\ell \not| pN$, U_ℓ for $\ell | pN$, $\langle d\rangle_p$ for $d \in (\mathbf{Z}/p\mathbf{Z})^\times$, and $\langle d\rangle_N$ for $d \in (\mathbf{Z}/N\mathbf{Z})^\times$. An eigenform $f = \sum a_n q^n$ will be called normalized if $a_1 = 1$.

Suppose that f is an eigenform for the U_p operator, so $U_p f = \alpha f$. Then α is an algebraic integer and we define the *slope* of f to be the rational number $v(\alpha)$. It is known that if $a \neq 0$ or f lies in the subspace of forms which are "old at p" (i.e., come from level

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N), then the slope of f lies in the interval $[0, w-1]$; on the other hand, if $a = 0$ and f is new at p, then the slope of f is $(w-2)/2$.

We define two Eisenstein series, $E_{p-1}^{(N)}$ of weight $p-1$ and level N and $E_{2,\chi^{-2}}^{(N)}$ of weight 2, level pN, and character χ^{-2}, as follows:

$$E_{p-1}^{(N)} = \frac{\zeta(2-p)}{2} \prod_{\ell | N} (1 - \ell^{p-2}) + \sum_{n \geq 1} \left(\sum_{\substack{d|n \\ (N,d)=1}} d^{p-2} \right) q^n$$

$$E_{2,\chi^{-2}}^{(N)} = \frac{L(-1, \chi^{-2})}{2} \prod_{\ell | N} (1 - \chi^{-2}(\ell)\ell) + \sum_{n \geq 1} \left(\sum_{\substack{d|n \\ (N,d)=1}} \chi^{-2}(d) d \right) q^n.$$

Here the products extend over all primes ℓ dividing N. Both of these Eisenstein series are normalized eigenforms; they play a special role because of their connection with $E_{p-1}^{(1)}$ and $E_{2,\chi^{-2}}^{(1)}$, which are Eisenstein series whose constant terms are not integral at p.

Theorem 1.1. *Let p be an odd prime number, N a positive integer relatively prime to p, k a positive integer, and a an integer with $0 < a < p-1$.*

a) Let i be an integer satisfying $1 \leq i \leq k$, $i \leq a$, and $k + 1 - i \leq p - 1 - a$ and set

$$b = a + k - 2i, \qquad c = \binom{a+k+1-i}{i} \Big/ \binom{a}{i}.$$

Suppose $f = \sum a_n q^n$ is a normalized eigenform in $S_{k+2}(\Gamma_0(pN), \chi^a \epsilon)$ of slope i. Then either there exists a normalized eigenform $g = \sum b_n q^n$ in $S_2(\Gamma_0(pN), \chi^{b+2}\delta)$ of slope 1 such that

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ a_n &\equiv n^{i-1} b_n \pmod{\wp} \text{ for all } n \geq 1 \\ p^{-i} a_p &\equiv c p^{-1} b_p \pmod{\wp}; \end{aligned} \tag{1.2}$$

or there exists a normalized eigenform $h = \sum c_n q^n$ in $M_2(\Gamma_0(pN), \chi^b \delta)$ of slope 0 such that

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ a_n &\equiv n^i c_n \pmod{\wp} \text{ for all } n \geq 1 \\ p^{-i} a_p &\equiv c c_p \pmod{\wp}. \end{aligned} \tag{1.3}$$

We can assume $h \neq E^{(N)}_{2,\chi^{-2}}$ *and if* $b+2 \equiv 0 \pmod{p-1}$ *(resp.* $b \equiv 0 \pmod{p-1}$*) then we can assume* g *(resp.*h*) is old at* p*. Conversely, if* $g = \sum b_n q^n$ *is a normalized eigenform in* $S_2(\Gamma_0(pN), \chi^{b+2}\delta)$ *of slope 1 which is old at* p *if* $b+2 \equiv 0 \pmod{p-1}$ *(resp.* $h = \sum c_n q^n$ *is a normalized eigenform in* $M_2(\Gamma_0(pN), \chi^b\delta)$ *of slope 0 with* $h \neq E^{(N)}_{2,\chi^{-2}}$ *which is old at* p *if* $b \equiv 0 \pmod{p-1}$*) then there exists a normalized eigenform* $f = \sum a_n q^n$ *in* $S_{k+2}(\Gamma_0(pN), \chi^a\epsilon)$ *of slope* i *such that the congruences 1.2 (resp. 1.3) hold.*

b) Let i *be an integer* $0 \le i \le k$ *and suppose that either:* $i+1 \le a$ *and* $k+1-i \le p-1-a$*; or* $i = 0$ *and* $p-1-a \ge k$*; or* $i = k$ *and* $a \ge k$*. Let* $b = a+k-2i$*. Suppose that* $f = \sum a_n q^n$ *is a normalized eigenform in* $S_{k+2}(\Gamma_0(pN), \chi^a\epsilon)$ *whose slope lies in the open interval* $(i, i+1)$*. Then there exists a normalized eigenform* $h = \sum c_n q^n$ *in* $S_2(\Gamma_0(pN), \chi^b\delta)$ *whose slope lies in* $(0,1)$ *such that*

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ a_n &\equiv n^i c_n \pmod{\wp} \text{ for all } n \ge 1. \end{aligned} \tag{1.4}$$

If $b \equiv 0 \pmod{p-1}$ *we can assume* h *is old at* p*. Conversely, for every normalized eigenform* h *in* $S_2(\Gamma_0(pN), \chi^b\delta)$ *whose slope lies in* $(0,1)$ *and which is old at* p *if* $b \equiv 0$ $\pmod{p-1}$*, there exists a normalized eigenform* f *in* $S_{k+2}(\Gamma_0(pN), \chi^a\epsilon)$ *whose slope lies in the open interval* $(i, i+1)$ *such that the congruences 1.4 hold.*

Remarks: 1) According to a recent result of Diamond which improves a Lemma of Carayol (cf. [Di], Lemma 2.2), we can frequently choose the Dirichlet character δ arbitrarily among those with $\delta \equiv \epsilon \pmod{\wp}$. This is the case, for example, when $p > 3$ and the Galois representation modulo p attached to f is irreducible.

2) In part b), it is natural to ask whether we can take the slope of f to be precisely i plus the slope of g.

As an example, let us verify the theorem directly for $p = 3$, $N \le 4$. In case b), the hypotheses force $k = a = 1$ and $i = 0$ or $i = 1$. Computation with Pari reveals that there are no forms with $N \le 4$ of the relevant slopes. In case a), the hypotheses force $k = a = i = 1$. Again there are no relevant forms if $N < 4$; for $N = 4$, there is a unique normalized eigenform in $S_3(\Gamma_0(12), \chi)$ and it has slope 1. Indeed, define a Hecke character

ϕ of $\mathbf{Q}(\sqrt{-3})$ with conductor (2) by setting $\phi((\alpha)) = \alpha^2$ where α is a generator of (α) congruent to 1 modulo 2. Then by a theorem of Shimura, the q-expansion

$$f = \sum a_n q^n = \sum_{(\alpha) \subseteq \mathcal{O}_{\mathbf{Q}(\sqrt{-3})}} \phi((\alpha)) q^{\mathbf{N}(\alpha)}$$

is a normalized eigenform of weight 3, level 12, and character $\chi = \left(\frac{}{-3}\right)$. Visibly $a_3 = -3$ and $a_\ell = 0$ if ℓ is a prime $\equiv -1 \pmod 3$ (so f is a form "with complex multiplication by $\mathbf{Q}(\sqrt{-3})$"). It is an easy exercise to check that $a_\ell \equiv 2 \pmod 3$ if ℓ is a prime $\equiv 1 \pmod 3$.

On the other hand, there are two normalized eigenforms in $M_2(\Gamma_0(12))$ which have slope 0 and are old at 3 and neither is a cusp form. One of the forms is $E^{(4)}_{2,\chi^{-2}}$ and the other is

$$F = \sum b_n q^n = \sum_{\substack{n \geq 1 \\ (2,n)=1}} \left(\sum_{\substack{d|n \\ (3,d)=1}} d \right) q^n.$$

We have $b_\ell \equiv 2 \pmod 3$ if ℓ is a prime $\equiv 1 \pmod 3$, $b_\ell \equiv 0$ if ℓ is a prime $\equiv -1 \pmod 3$, and $b_3 = 1$. Since the constant $c = 2$, we see that the congruences 1.3 do indeed hold. This checks that the theorem is correct for $p = 3$ and $N \leq 4$.

Before giving some consequences of the Theorem, we sketch how it is proven in the general case. There are three key points to the proof: First of all, there is a pair $M = (\tilde{X}, \Pi)$, where $\tilde{X}$ is a smooth complete variety over $\mathbf{F}_p$ and Π is a projector in $\mathbf{Z}_p[\mathrm{Aut}\,\tilde{X}]$ (i.e., M is a motive), such that the crystalline cohomology of M is a Hecke module with the same eigenvalue packages as $S_{k+2}(\Gamma_0(pN), \chi^a\epsilon)$. This is of course a standard idea by now (cf. [D] and [Sc]), but an important point here is that the coefficients of Π lie in $\mathbf{Z}_p$, not just $\mathbf{Q}_p$, so it makes sense to apply Π to certain characteristic p vector spaces, such as coherent cohomology groups of $\tilde{X}$.

Secondly, for "good" slopes i, namely those figuring in Theorem 1.1, there is a canonical direct factor of the integral crystalline cohomology of M on which Frobenius acts with slope i. This direct factor contains a canonical $\mathbf{Z}_p$-lattice and the reduction of this lattice modulo p can be identified with the cohomology of M with coefficients in a sheaf of logarithmic differentials. For good ranges of slope $(i, i+1)$, there is a similar connection between a direct factor of the integral crystalline cohomology of M and the cohomology of M with coefficients in a sheaf of exact differentials. The logarithmic and exact cohomology groups thus capture Hecke eigenvalues modulo p, and the point then is that they are relatively calculable. (The relation between crystalline cohomology and logarithmic or exact cohomology follows from general results of Illusie and Raynaud (cf. [I]), the crucial input being the finiteness of the logarithmic groups. We remark that this finiteness definitely fails for slopes not satisfying the inequalities in Theorem 1.1, so these strange inequalities are crucial hypotheses.)

Thirdly, there is a remarkable connection between logarithmic or exact cohomology groups for the M related to weight $k+2$ and the M related to weight 2. Roughly speaking, the logarithmic group for weight $k+2$ and slope i is isomorphic to an extension of the logarithmic group for weight 2 and slope 0 by the logarithmic group for weight 2 and slope 1. (This is why there are two possible types of forms of weight 2 to which a form of weight $k+2$ is related.) For the exact groups, the situation is simpler: the group for weight $k+2$ and slopes in $(i, i+1)$ is isomorphic to the group for weight 2 and slopes $(0, 1)$. For both logarithmic and exact groups, the isomorphisms just mentioned introduce a twist in the Hecke action. The factors n^{i-1}, n^i, and the funny constant c in Theorem 1.1 are a manifestation of this twist and much of the work in the paper is related to keeping track of it.

Here is the plan of the rest of the paper: It will be convenient to use the language of Hecke algebras, so in Section 2 we briefly review the connection between Hecke algebras attached to cusp forms and to various cohomology groups, such as the crystalline cohomology of M. In Section 4 we relate $\mathbf{T}_{\mathrm{cris}}$, the Hecke algebra attached to the crystalline cohomology of M, to Hecke algebras $\mathbf{T}_{\log}$ and $\mathbf{T}_{\mathrm{exact}}$ attached to the logarithmic and exact cohomology groups of M and we introduce other Hecke algebras associated to the Eisenstein series appearing in Theorem 1.1. In Section 5 we relate the $\mathbf{T}_{\log}$ and $\mathbf{T}_{\mathrm{exact}}$ algebras for weight $k+2$ to their analogues for weight 2. Keeping track of the twist in the Hecke action here requires some information on how Hecke operators act on sections of various line bundles on the Igusa curve; this information is recorded in Section 3. Finally, in Section 6 we assemble the pieces into a proof of Theorem 1.1.

In the rest of this section we outline some corollaries of Theorem 1.1. If $g = \sum b_n q^n$ is a (formal) q-expansion, we write ϑg for the "twisted" series $g = \sum n b_n q^n$. With this notation, part of the congruences congruence 1.2 (resp. 1.3 and 1.4) could be written $f \equiv \vartheta^{i-1} g \pmod{\wp}$ (resp. $f \equiv \vartheta^i h \pmod{\wp}$). Let us say that two normalized eigenforms f and g are *congruent after a twist* if there exists an integer t so that $f \equiv \vartheta^t g \pmod{\wp}$. Well-known results of Serre, Tate, and others (cf. [Ri] for a survey) say that every normalized eigenform f of weight $w \geq 2$ and level $p^e N$ $(e \geq 0)$ is congruent after a twist to a normalized eigenform of weight 2 and level pN. Theorem 1.1 determines, under suitable hypotheses, the correct twisting integer t and the character of the form of weight 2.

On the other hand, every normalized eigenform of weight 2 and level pN is congruent to a form of level N and weight w with $2 \leq w \leq p+1$; we can take $w \leq p$ if the form has a non-trivial power of χ in its character, or is old at p. This gives a reformulation of the theorem in terms of level N forms. In the corollary below, we say that an eigenform of level N is *ordinary* if its eigenvalue for T_p is a unit at $\wp$. (This differs slightly from Hida's usage in that he always takes ordinary forms to have level divisible by p.)

Corollary 1.5.

a) Hypotheses as in 1.1a). Suppose that $f = \sum a_n q^n$ is a normalized eigenform in $S_{k+2}(\Gamma_0(pN), \chi^a \epsilon)$ of slope i. Then either there exists an ordinary normalized eigenform $g = \sum b_n q^n$ in $S_{p-1-b}(\Gamma_0(N), \delta)$ such that

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ f &\equiv \vartheta^{a+k+1-i} g \pmod{\wp} \\ p^{-i} a_p &\equiv \epsilon(p) c b_p^{-1} \pmod{\wp}; \end{aligned} \tag{1.6}$$

or there exists an ordinary normalized eigenform $h = \sum c_n q^n$ in $M_{b+2}(\Gamma_0(N), \delta)$ such that

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ f &\equiv \vartheta^i h \pmod{\wp} \\ p^{-i} a_p &\equiv c c_p \pmod{\wp}. \end{aligned} \tag{1.7}$$

We can assume $h \neq E_{p-1}^{(N)}$. Conversely, if $g \in S_{p-1-b}(\Gamma_0(N), \delta)$ (resp. $h \in M_{b+2}(\Gamma_0(N), \delta)$ with $h \neq E_{p-1}^{(N)}$) is an ordinary normalized eigenform then there exists a normalized eigenform f in $S_{k+2}(\Gamma_0(pN), \chi^a \epsilon)$ of slope i such that the congruences 1.6 (resp. 1.7) hold.

b) Hypotheses as in 1.1b). Suppose that f is a normalized eigenform in $S_{k+2}(\Gamma_0(pN), \chi^a \epsilon)$ whose slope lies in the open interval $(i, i+1)$. Then there exist non-ordinary normalized eigenforms $g \in S_{p+1-b}(\Gamma_0(N), \delta)$ and $h \in S_{b+2}(\Gamma_0(N), \delta)$ such that

$$\begin{aligned} \epsilon &\equiv \delta \pmod{\wp} \\ f &\equiv \vartheta^{a+k-i} g \pmod{\wp} \\ &\equiv \vartheta^i h \pmod{\wp}. \end{aligned} \tag{1.8}$$

Conversely, if $g \in S_{p+1-b}(\Gamma_0(N), \delta)$ or $h \in S_{b+2}(\Gamma_0(N), \delta)$, is a non-ordinary normalized eigenform, then there exists a normalized eigenform f in $S_{k+2}(\Gamma_0(pN), \chi^a \epsilon)$ whose slope lies in the open interval $(i, i+1)$ such that the congruences 1.8 hold. □

Let us say that an eigenform is of type (k, a, i) if it has weight $k+2$, level pN, character $\chi^a \epsilon$ for some ϵ modulo N, and slope i. Suppose $p > 3$. Hida has shown that if f is a normalized eigenform of type $(k, a, 0)$ with $k \geq 0$, then there exists a normalized eigenform form of type $(k+1, a-1, 0)$ congruent to f modulo $\wp$. Using the w-operator, one can show that if f is a normalized eigenform of type $(k, a, k+1)$ with $k \geq 0$, then there exists a normalized eigenform of type $(k+1, a+1, k+2)$ congruent to ϑf modulo $\wp$. The following result generalizes both of these statements to certain forms of intermediate slope.

Corollary 1.9. *Suppose p is odd and let k be a non-negative integer.*

a) If i and a are integers with $1 \leq i \leq k$, $2 \leq a \leq p-2$, $i \leq a-1$, and $k+1-i \leq p-1-a$ and if f is a normalized eigenform of type (k, a, i), then there exists a normalized eigenform g of type $(k+1, a-1, i)$ with $f \equiv g \pmod{\wp}$.

b) If i and a are integers with $1 \leq i \leq k$, $1 \leq a \leq p-3$, $i \leq a$, and $k+2-i \leq p-1-a$ and if f is a normalized eigenform of type (k, a, i), then there exists a normalized eigenform g of type $(k+1, a+1, i+1)$ with $\vartheta f \equiv g \pmod{\wp}$. □

Remark: It would be interesting to have a statement which integrated this corollary with the conjectures of Gouvêa and Mazur.

If f is a normalized eigenform, write ρ_f for the mod p Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}_p})$$

attached to f. Deligne, Serre, and Fontaine have obtained detailed information on the representation ρ_f attached to an eigenform of weight 2 and level pN (cf. [G] 12.1 and [E] 2.6 for proofs). Theorem 1.1 allows us to translate this into information about ρ_f for f of higher weight.

We use χ also to denote also the character of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ giving its action on the p-th roots of unity. Let $D_p \subseteq \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the decomposition group at $\wp$, $I_p \subseteq D_p$ the inertia group, and $I_p^w \subseteq I_p$ the wild inertia group. By local class field theory, the tame inertia group I_p/I_p^w is isomorphic to $\varprojlim_n \mathbf{F}_{p^n}{}^\times$. By definition, an $\overline{\mathbf{F}_p}$-valued character of I_p has level n if it factors through $\mathbf{F}_{p^n}{}^\times$; the fundamental characters of level n are the ones induced by the n embeddings of $\mathbf{F}_{p^n}$ in $\overline{\mathbf{F}_p}$. For any $x \in \overline{\mathbf{F}_p}$, let $\lambda(x)$ be the unramified character of D_p which sends the Frobenius to x.

Corollary 1.10. *Let p be an odd prime number, N a positive integer relatively prime to p, k a positive integer, ϵ a Dirichlet character modulo N, and a an integer with $0 < a < p-1$. Suppose that $f \in S_{k+2}(\Gamma_0(pN), \chi^a\epsilon)$ is a normalized eigenform and let ρ_f be the mod p Galois representation attached to f.*

a) Suppose f has slope i where i is an integer satisfying $1 \le i \le k$, $i \le a$, and $k+1-i \le p-1-a$. Write the eigenvalue of U_p on f as $p^i u$ and let

$$c = \binom{a+k+1-i}{i} \Big/ \binom{a}{i}.$$

Then

$$\rho_f|_{D_p} \cong \begin{pmatrix} \phi_1 & * \\ 0 & \phi_2 \end{pmatrix}$$

where $\{\phi_1, \phi_2\} = \{\chi^i \lambda(c^{-1}u), \chi^{a+k+1-i}\lambda(\epsilon(p)cu^{-1})\}$.

b) Suppose that the slope of f lies in the open interval $(i, i+1)$ where i is an integer $0 \le i \le k$ and either: $i+1 \le a$ and $k+1-i \le p-1-a$; or $i = 0$ and $a \ge k$; or $i = k$ and $p-1-a \ge k$. Then $\rho_f|_{D_p}$ is irreducible and

$$\rho_f|_{I_p} \cong \begin{pmatrix} \psi^{a+k+1-i}\psi'^i & 0 \\ 0 & \psi^i \psi'^{a+k+1-i} \end{pmatrix}$$

where ψ and ψ' are the two fundamental characters of level 2. $\square$

Remarks: 1) In case a), we can take $\phi_1 = \chi^i \lambda(c^{-1}u)$ when f satisfies the congruences 1.2 or 1.6 and we can take $\phi_1 = \chi^{a+k+1-i}\epsilon\lambda(cu^{-1})$ when f satisfies the congruences 1.3 or 1.7. Since the hypotheses rule out $i \equiv a+k+1-i \pmod{p-1}$, if both 1.6 and 1.7 hold then $* = 0$. According to a conjecture of Serre (proven by Gross [G]), the converse is true as well: if $* = 0$ then there exist forms satisfying both sets of congruences. In this case, the two forms appearing on the right hand sides of 1.6 and 1.7 are called "companions."
2) The theorem shows that the Galois representations attached to certain forms are twists of ordinary representations if and only if the forms have integral slope. It would be interesting to know whether such a statement holds in general. It is true in all examples I know, but the recipe in Corollary 1.10 for the two characters on the diagonal of $\rho_f|_{D_p}$ does not hold in general. For example, there is a unique cusp form f of weight 7, character $\left(\frac{\cdot}{7}\right)$, and slope 3 on $\Gamma_0(7)$. It turns out that the powers of χ appearing on the diagonal in the restriction of the mod 7 representation ρ_f are χ^2 and χ^5, rather than χ^3 and χ^4 as would be predicted by a naive generalization of the theorem.

2. Hecke algebras In this section we will relate the action of Hecke operators on modular forms to their action on the crystalline cohomology of a certain Chow motive. As the arguments are for the most part standard, we will be quite terse.

Let L be a number field with a fixed embedding $L \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and let $\mathcal{O}_L$ be its ring of integers; we will abusively write $\wp$ for the prime of $\mathcal{O}_L$ induced by the prime $\wp$ of $\mathcal{O}_{\overline{\mathbf{Q}}}$ fixed in Section 1. Let $\mathbf{T}$ be the polynomial ring over $\mathcal{O}_L$ generated by symbols T_ℓ for each prime number ℓ and $\langle d\rangle$ for $d \in \mathbf{Z}$. If H is a module for $\mathbf{T}$, we write $\mathbf{T}(H)$ for the algebra of endomorphisms of H generated by $\mathbf{T}$, (i.e., for the image of $\mathbf{T} \to \mathrm{End}(H)$).

Let $M_{k+2}(\Gamma_1(N))$ and $S_{k+2}(\Gamma_1(N))$ be the complex vector spaces of modular forms and cusp forms on $\Gamma_1(N)$. For any prime p dividing N, let $S_{k+2}(\Gamma_1(N))^{p-\mathrm{old}}$ be the space of cusp forms which are old at p, i.e., come from level N/p via the two standard degeneracy maps. Also, define $S_{k+2}(\Gamma_1(N))^{p-\mathrm{new}}$ as the orthogonal complement of $S_{k+2}(\Gamma_1(N))^{p-\mathrm{old}}$ under the Petersson inner product. We let the Hecke algebra $\mathbf{T}$ act on all these spaces in the standard way (via the "upper star" operators T_ℓ^*, U_ℓ^*, and $\langle d\rangle_N^*$, rather than by the "lower star" operators $T_{\ell *}$, $U_{\ell *}$, and $\langle d\rangle_{N*}$, cf. [M-W], 2.5). If ϕ is a character of $(\mathbf{Z}/N\mathbf{Z})^\times$, let $S_{k+2}(\Gamma_1(N))(\phi)$ be the subspace of $S_{k+2}(\Gamma_1(N))$ where the $\langle d \rangle$ act via ϕ. Finally, for sets P_1 and P_2 of prime numbers dividing N and a set Ξ of characters of $(\mathbf{Z}/N\mathbf{Z})^\times$, define

$$S_{k+2}(\Gamma_1(N))^{P_1-\mathrm{old}, P_2-\mathrm{new}}(\Xi) = \bigcap_{p \in P_1} S^{p-\mathrm{old}} \bigcap_{p \in P_2} S^{p-\mathrm{new}} \bigcap \left(\sum_{\phi \in \Xi} S(\phi) \right)$$

where we have written S for $S_{k+2}(\Gamma_1(N))$. All these constructions have an obvious analog for $M_{k+2}(\Gamma_1(N))$.

We use [D-R] and [K-M] as general references for moduli of elliptic curves. Fix an integer $N \geq 5$ and consider the moduli problem $[\Gamma_1(N)]$ on $(Ell/\mathbf{Q})$. Let $X_1(N)$ be the corresponding complete modular curve over $\mathbf{Q}$ and $\overline{X}_1(N) = X_1(N) \times \mathrm{Spec}\ \overline{\mathbf{Q}}$. Under the hypothesis on N, $X_1(N)$ is a fine moduli space and there is a universal curve $\mathcal{E} \xrightarrow{\pi} X_1(N)$. We define $\omega = \pi_* \Omega^1_{reg, \mathcal{E}/X}$ where $\Omega^1_{reg, \mathcal{E}/I}$ is the sheaf of "regular" differentials, i.e., the relative dualizing sheaf (cf. [S], Ch. 4, §3 as well as [D-R], I.2 for a summary of the relevant duality theory). Fix an arbitrary prime number q, let $\overline{\mathbf{Q}}_q$ be the algebraic closure of the q-adic numbers, and choose an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_q$. Then we have a sheaf $\mathcal{F}_k = \mathrm{Sym}^k\, R^1\pi_* \overline{\mathbf{Q}}_q$ for the étale topology on $X_1(N)$ and a cohomology group $H^1_{\text{ét}}(\overline{X}_1(N), \mathcal{F}_k)$. (For $N < 5$ we can define cohomology groups by introducing extra level structure and taking invariants.) These groups are modules for the Hecke algebra (where again we use the "upper star" operators). For sets P_1 and P_2 of prime numbers dividing N and a set Ξ of characters of $(\mathbf{Z}/N\mathbf{Z})^\times$, we define

$$H^1_{\text{ét}}(\overline{X}_1(N), \mathcal{F}_k)^{P_1-\mathrm{old}, P_2-\mathrm{new}}(\Xi)$$

in obvious analogy with the definition for modular forms. (For the new subspaces, we take the orthogonal complement with respect to the cup product.)

All the key ingredients in the proof of the following result appear in [D]. For more details on the problems arising from Hecke operators for primes dividing the level, see [U1, §7].

Proposition 2.1. *Let $N \geq 1$ and $k \geq 0$ be integers and $L \subseteq \overline{\mathbf{Q}}$ a number field. Fix sets P_1 and P_2 of prime numbers dividing N and a set Ξ of characters of $(\mathbf{Z}/N\mathbf{Z})^\times$ with values in L. Then there exists a unique isomorphism of $\mathbf{T}$-algebras*

$$\mathbf{T}(S_{k+2}(\Gamma_1(N))^{P_1-\mathrm{old}, P_2-\mathrm{new}}(\Xi)) \cong \mathbf{T}(H^1_{\text{ét}}(\overline{X}_1(N), \mathcal{F}_k)^{P_1-\mathrm{old}, P_2-\mathrm{new}}(\Xi)). \qquad \square$$

We want to compare these Hecke algebras with ones defined via the cohomology of Igusa curves. Fix an integer $N \geq 5$, a prime $p \not| N$, and integers $k, n \geq 0$ and recall that the $[Ig(p^n)]$ moduli problem on $(Ell/\mathbf{F}_p)$ assigns to E/S the set of $P \in E^{(p^n)}(S)$ which are generators of the kernel of the iterated Verschiebung $V^n : E^{(p^n)} \to E$ ([K-M], 12.3.1). Let $I = Ig_1(p^n N)$ be the complete modular curve over the field of p elements $\mathbf{F}_p$ parameterizing generalized elliptic curves together $[Ig(p^n)]$- and $\Gamma_1(N)$-structures; also let $\overline{I} = I \times \mathrm{Spec}\ \overline{\mathbf{F}_p}$. (If $n = 0$, then I is the reduction of $X_1(N)$ modulo p.) We have a universal generalized elliptic curve $\mathcal{E} \xrightarrow{\pi} I$ and an invertible sheaf $\omega = \pi_* \Omega^1_{\mathcal{E}/I, reg} \cong (R^1\pi_*\mathcal{O}_{\mathcal{E}})^{-1}$ on I. Letting q be a prime distinct from p, we have the étale sheaf $\mathcal{F}_k = \mathrm{Sym}^k\, R^1\pi_*\overline{\mathbf{Q}}_q$ on I and a cohomology group $H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)$. (Again we can define cohomology groups for small N by introducing auxiliary level structure and passing to invariants.)

The group $H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)$ is a module for the Hecke algebra; however, the relation with the Hecke action in characteristic zero is slightly subtle, so we want to be precise about which Hecke operators we are using. For a prime $\ell \not| p^n N$, let I_ℓ be curve parameterizing generalized elliptic curves E with $[Ig(p^n)]$-, $\Gamma_1(N)$-, and $\Gamma_0(\ell)$-structures. There are two maps $\pi_1, \pi_2 : I_\ell \to I$ (forget the subgroup of order ℓ and divide by it, respectively) and a universal isogeny $\Phi : \pi_1^*\mathcal{E} \to \pi_2^*\mathcal{E}$. We then define $T_\ell^* = \pi_{1*}\Phi^*\pi_2^*$ on the cohomology group. For primes $\ell | N$ we have a similar construction with a suitable I_ℓ: write $N = \ell^e N'$ with $\ell \not| N'$ and let I_ℓ be the curve parameterizing generalized elliptic curves with Igusa p^n-structures, $[\Gamma_1(N')]$-structures, and $[\Gamma_0(\ell^{e+1}); e, 0]$-structures (cf. [K-M, 7.9.4] for this moduli problem). The $\langle d \rangle^*_{p^n}$ and $\langle d \rangle^*_N$ operators are defined in the expected way: if x is a point of I representing (E, P, Q) where $P \in E$ has order N and $Q \in E^{(p^n)}$ has order p^n, then $\langle d \rangle_{p^n} x$ represents (E, P, dQ) and $\langle d \rangle_N x$ represents (E, dP, Q). There is no obvious definition of a U_p^* operator on $H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)$, but we do have F^*, induced by the geometric Frobenius on the universal curve, and we define V^* as $p^{k+1}F^{*-1}$. We make $H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)$ into a $\mathbf{T}$-module by letting T_ℓ act by T_ℓ^* if $\ell \not| pN$, by U_ℓ^* if $\ell | N$, and by $\langle p \rangle_N^* V^*$ if $\ell = p$, and by letting $\langle d \rangle$ act by $\langle d \rangle^*_{p^n N} = \langle d \rangle^*_{p^n} \langle d \rangle^*_N$ if $(d, pN) = 1$ and by 0 if $(d, pN) \neq 1$.

Proposition 2.2. *Fix integers $N \geq 1$, $n \geq 0$ and $k \geq 0$, a prime $p \not| N$, and a number field $L \subseteq \overline{\mathbf{Q}}$.*

a) Let Ξ be a set of characters of $(\mathbf{Z}/p^nN\mathbf{Z})^\times$ of conductor divisible by p^n with values in L. Then there is a unique isomorphism of $\mathcal{O}_L$-algebras

$$\phi : \mathbf{T}(H^1_{\text{ét}}(\overline{X}_1(p^nN), \mathcal{F}_k)(\Xi)) \to \mathbf{T}(H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)(\Xi))$$

such that

$$\begin{aligned} \phi(T_\ell^*) &= \langle \ell^{-1}\rangle^*_{p^n} T_\ell^* &\quad& (\ell \not| \, p^nN) \\ \phi(U_\ell^*) &= \langle \ell^{-1}\rangle^*_{p^n} U_\ell^* && (\ell | N) \\ \phi(U_p^*) &= \langle p\rangle^*_N V^* && (\text{if } n \geq 1) \\ \phi(T_p^*) &= T_p^* && (\text{if } n = 0) \\ \phi(\langle d\rangle^*_{p^nN}) &= \langle d^{-1}\rangle^*_{p^n} \langle d\rangle^*_N && (d \in (\mathbf{Z}/p^nN\mathbf{Z})^\times). \end{aligned}$$

b) Let Ξ be a set of characters of $(\mathbf{Z}/N\mathbf{Z})^\times$ with values in L. Then for any $n \geq 1$, there is a unique surjection of $\mathbf{T}$-algebras

$$\phi : \mathbf{T}(H^1_{\text{ét}}(\overline{X}_1(pN), \mathcal{F}_k)^{p-\text{old}}(\Xi)) \to \mathbf{T}(H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)(\Xi)).$$

The kernel of ϕ is nilpotent.

Remarks: 1) In part b), we view elements of Ξ as characters of $(\mathbf{Z}/p^nN\mathbf{Z})^\times$ via the obvious projection to $(\mathbf{Z}/N\mathbf{Z})^\times$. Also, we really do mean $\overline{X}_1(pN)$, not $\overline{X}_1(p^nN)$. The dependence on n is via $\overline{I}$.

2) One could impose old and new conditions at primes dividing N.

Proof: Assume first that we are in case a) and that $n \geq 1$. In this case, we established in [U1] an isomorphism

$$H^1_{\text{ét}}(\overline{X}_1(p^nN), \mathcal{F}_k)(\Xi) \cong H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)(\Xi) \oplus H^1_{\text{ét}}(\overline{Ex}, \mathcal{F}_k)(\Xi)$$

where $\overline{Ex}$ is a certain "exotic" variant of the Igusa curve ([K-M], 12.10). We also checked that the action of U_p^* on the left corresponds to that of $\langle p\rangle^*_N V^* \oplus F^*$ on the right. Similar arguments show that T_ℓ^*, U_ℓ^*, and $\langle d\rangle^*_{p^nN}$ on the left correspond to $\langle \ell^{-1}\rangle^*_{p^n} T_\ell^* \oplus T_\ell^*$, $\langle \ell^{-1}\rangle^*_{p^n} U_\ell^* \oplus U_\ell^*$, and $\langle d^{-1}\rangle^*_{p^n}\langle d\rangle^*_N \oplus \langle d\rangle^*_N$ respectively on the right. On the other hand, there is an "exotic" isomorphism between $\overline{I}$ and $\overline{Ex}$ which shows that restriction induces an isomorphism

$$\mathbf{T}\left((H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)\oplus H^1_{\text{ét}}(\overline{Ex},\mathcal{F}_k))(\Xi)\right)\cong\mathbf{T}\left(H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)(\Xi)\right).$$

This completes the proof of a) in the case $n\geq 1$. The case $n=0$ is similar, but rather simpler: there is a model of $X_1(N)$ with good reduction at p, the reduction is isomorphic to I, and because of the way we have defined the action of the Hecke operators, the proof is essentially trivial.

Now consider case b). Here we have an isomorphism

$$H^1_{\text{ét}}(\overline{X}_1(pN),\mathcal{F}_k)^{p-\text{old}}(\Xi)\cong\left(H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)\oplus H^1_{\text{ét}}(\overline{Ex},\mathcal{F}_k)\right)(\Xi).$$

Arguing as in part a), we see that the actions of T^*_ℓ, U^*_ℓ, and $\langle d\rangle^*_N$ on $H^1_{\text{ét}}(\overline{X}_1(pN),\mathcal{F}_k)^{p-\text{old}}(\Xi)$ intertwine the actions of $T^*_\ell\oplus T^*_\ell$, $U^*_\ell\oplus U^*_\ell$, and $\langle d\rangle^*_N\oplus\langle d\rangle^*_N$ on $(H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)\oplus H^1_{\text{ét}}(\overline{Ex},\mathcal{F}_k))(\Xi)$. On the other hand, using [U1], 8.4, one finds that U^*_p acts as

$$\begin{pmatrix}\langle p\rangle^*_N V^* & 0\\ p^k(p-1)\langle p\rangle^*_N & F^*\end{pmatrix}$$

on $(H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)\oplus H^1_{\text{ét}}(\overline{Ex},\mathcal{F}_k))(\Xi)$. In particular, projection onto the factor $H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)(\Xi)$ induces a surjection

$$\mathbf{T}\left(H^1_{\text{ét}}(\overline{X}_1(pN),\mathcal{F}_k)^{p-\text{old}}(\Xi)\right)\to\mathbf{T}\left(H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)(\Xi)\right).$$

The kernel is the principal ideal generated by $P(U^*_p)$ where P is the minimal polynomial of $\langle p\rangle^*_N V^*$ acting on $H^1_{\text{ét}}(\overline{I},\mathcal{F}_k)(\Xi)$. But by [U1], the eigenvalues of $\langle p\rangle^*_N V^*$ are the same as those of U^*_p on $H^1_{\text{ét}}(\overline{X}_1(pN),\mathcal{F}_k)^{p-\text{old}}(\Xi)$, so $P(U^*_p)$ is nilpotent, as desired. This completes the proof of part b). $\square$

In §2 of [U2], we defined a smooth projective variety $\tilde{X}$ as a certain desingularization of the k-fold fiber product of $\mathcal{E}\to I$ (obtained by blowing up products of double points over the cusps), and we defined an idempotent Π in the group ring $\mathbf{Q}[\mathrm{Aut}_{\mathbf{F}_p}\tilde{X}]$. This construction is a slight variation of that of Scholl in [Sc], but it has the important feature that although we are dealing with modular curves with p in the level, all coefficients appearing in Π are integral at p if $k < p$. We think of the pair $M = (\tilde{X}, \Pi)$ as a Chow motive and we write $H_{\text{ét}}(M)$ for $\Pi H_{\text{ét}}^{k+1}(\tilde{X} \times \operatorname{Spec} \overline{\mathbf{F}_p}, \overline{\mathbf{Q}}_q)$ and $H_{\text{cris}}(M)$ for $\Pi H_{\text{cris}}^{k+1}(\tilde{X} \times \operatorname{Spec} \overline{\mathbf{F}_p}/W(\overline{\mathbf{F}_p})) \otimes_{W(\overline{\mathbf{F}_p})} \mathbf{C}_p$. (Here $\mathbf{C}_p$ is the completion of $\overline{\mathbf{Q}_p}$ with respect to the p-adic absolute value.) As usual, for small N we can define groups $H_{\text{ét}}(M)$ and $H_{\text{cris}}(M)$ by introducing auxiliary level structure and passing to invariants.

We define in the usual fashion operators T_ℓ^* ($\ell \not| p^n N$), U_ℓ^* ($\ell | N$), and $\langle d \rangle_{p^n N}^*$ ($d \in (\mathbf{Z}/p^n N\mathbf{Z})^\times$) acting on $H_{\text{ét}}(M)$ and $H_{\text{cris}}(M)$. There is no obvious definition of a U_p^* operator if $n > 0$, but we do have the Frobenius endomorphism F. More precisely, let F be the absolute Frobenius endomorphism of $\tilde{X}$ and σ the absolute Frobenius of $\overline{\mathbf{F}_p}$. We also write F and σ for the endomorphisms $F \times id$ and $id \times \sigma$ of $\tilde{X} \times \operatorname{Spec} \overline{\mathbf{F}_p}$; clearly $\Phi = F \times \sigma$ is the absolute Frobenius of $\tilde{X} \times \operatorname{Spec} \overline{\mathbf{F}_p}$. Then F induces (linear) automorphisms F^* of $H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)$, $H_{\text{ét}}(M)$, and $H_{\text{cris}}(M)$. (For crystalline cohomology, one usually considers the semi-linear endomorphism induced by Φ.) We define V^* as $p^{k+1} F^{*-1}$ and make $H_{\text{ét}}(M)$ and $H_{\text{cris}}(M)$ into $\mathbf{T}$-modules as follows: let T_ℓ act as T_ℓ^* if $\ell \not| p^n N$, as U_ℓ^* if $\ell | N$, and as $\langle p \rangle_N^* V^*$ if $\ell = p$ and $n > 0$, and let $\langle d \rangle$ act as $\langle d \rangle_{p^n N}^*$ if $(d, p^n N) = 1$ and as 0 if $(d, p^n N) \neq 1$.

For a commutative ring R, let R^{red} be the quotient of R by its nilradical, i.e., by its ideal of nilpotent elements.

Proposition 2.3. *Fix integers $N \geq 1$, $n \geq 0$ and $k \geq 0$, a prime $p \not| N$, and a number field $L \subseteq \overline{\mathbf{Q}}$. Let Ξ be any set of characters of $(\mathbf{Z}/p^n N\mathbf{Z})^\times$ whose values lie in L.*
a) There is a unique isomorphism of $\mathbf{T}$-algebras $\mathbf{T}(H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k)(\Xi)) \cong \mathbf{T}(H_{\text{ét}}(M)(\Xi))$.
b) There is a unique isomorphism of $\mathbf{T}$-algebras $\mathbf{T}(H_{\text{ét}}(M)(\Xi))^{\text{red}} \cong \mathbf{T}(H_{\text{cris}}(M)(\Xi))^{\text{red}}$.

Remark: Again the proposition also holds with old or new restrictions at primes dividing N.

Proof: a) We proved in [U2], 2.1a that there is a canonical isomorphism

$$H^1_{\text{ét}}(\overline{I}, \mathcal{F}_k) \cong H_{\text{ét}}(M).$$

Reviewing the proof, it is immediate that all the maps appearing there commute with the actions of the Hecke operators, so we obtain part a).

b) This follows from a theorem of Katz and Messing. Indeed, the kernel of the structure map $\mathbf{T} \to \mathbf{T}(H_{\text{ét}}(M)(\Xi))^{\text{red}}$ (resp. $\mathbf{T} \to \mathbf{T}(H_{\text{cris}}(M)(\Xi))^{\text{red}}$) is the set of elements of $\mathbf{T}$ which induce a nilpotent endomorphism of $H_{\text{ét}}(M)(\Xi)$ (resp. $H_{\text{cris}}(M)(\Xi)$). But by [K-Me], Thm. 2 (as completed by [Gi-Me]), these two sets are equal. (To be completely precise, the theorem of Katz and Messing was proved only for $\mathbf{Z}$ coefficients, but the same proof works with coefficients in the integers of a number field.) $\square$

We recall that $\mathbf{T}(S_{k+2}(\Gamma_1(N)))$ and its variants are finite and flat over $\mathbf{Z}$ (this follows from [Sh], 3.48) and thus have Krull dimension 1. If f is an eigenform, then the kernel of the homomorphism $\mathbf{T}(S_{k+2}(\Gamma_1(N))) \to \overline{\mathbf{Q}}$ which sends a Hecke operator to its eigenvalue is a minimal prime and each minimal prime corresponds to a $\mathrm{Gal}(\overline{L}/L)$-orbit of normalized eigenforms (where the Galois group acts on q-expansion coefficients). In particular, if the number field L fixed in the definition of $\mathbf{T}$ is sufficiently large (e.g., if it contains the eigenvalues of all Hecke operators on $S_{k+2}(\Gamma_1(N))$) then for every minimal prime $\mathcal{P}$ of $\mathbf{T}(S_{k+2}(\Gamma_1(N)))$, one has $\mathbf{T}(S_{k+2}(\Gamma_1(N)))/\mathcal{P} \cong \mathcal{O}_L$ and there exists a unique normalized eigenform f in $S_{k+2}(\Gamma_1(N))$ whose eigenvalue for each Hecke operator is equal to the image of that operator in $\mathcal{O}_L$. Maximal primes $\mathbf{m}$ containing $\wp$ give rise to systems of eigenvalues mod p: for any minimal prime $\mathcal{P}$ containing $\mathbf{m}$ with associated form f, the eigenvalues of f are congruent mod $\wp$ to the images of the corresponding Hecke operators in $\mathbf{T}(S_{k+2}(\Gamma_1(N)))/\mathbf{m} \cong \mathcal{O}_L/\wp$.

We say that a minimal prime of $\mathbf{T}(H_{\text{cris}}(M))$ has *slope* λ if the valuation of the image of $\langle p\rangle_N^* V^*$ (if $n > 0$) or of T_p^* (if $n = 0$) in $\mathcal{O}_L$ is $k+1-\lambda$. (We use this funny convention to agree with crystalline terminology: if $n > 0$, an element of $H_{\text{cris}}(M)$ which is in the kernel of an ideal of slope λ will have eigenvalue for Frobenius of valuation λ.)

Now take $n = 1$. Henceforth we assume that the number field L is sufficiently large in the sense that it contains the eigenvalues of all Hecke operators acting on $S_{k+2}(\Gamma_1(pN))$ for all k less than some fixed integer (which in the applications will be p). For $0 \le a < p-1$, let Ξ_a be the set of characters of $(\mathbf{Z}/pN\mathbf{Z})^\times \cong (\mathbf{Z}/p\mathbf{Z})^\times \times (\mathbf{Z}/N\mathbf{Z})^\times$ whose restriction to $(\mathbf{Z}/p\mathbf{Z})^\times$ is χ^a where χ is the Teichmüller character.

Corollary 2.4. *Fix a integers $N \ge 1$ and $k \ge 0$, a prime $p \nmid N$, and a number field $L \subseteq \overline{\mathbf{Q}}$*

which is sufficiently large in the sense above. For $0 \le a < p-1$ there is a bijection between normalized eigenforms $f = \sum a_n q^n$ in $S_{k+2}(\Gamma_1(pN))(\Xi_a)$ of slope λ which are old at p if $a = 0$ and minimal primes $\mathcal{P}$ of $\mathbf{T}(H_{\mathrm{cris}}(M)(\Xi_{-a}))$ of slope $k+1-\lambda$. If f has character $\chi^a \epsilon$ then f corresponds to $\mathcal{P}$ if and only if

$$\begin{aligned}
T_\ell^* &\equiv \chi(\ell)^{-a} a_\ell \pmod{\mathcal{P}} \quad (\ell \not| \, pN) \\
U_\ell^* &\equiv \chi(\ell)^{-a} a_\ell \pmod{\mathcal{P}} \quad (\ell | N) \\
V^* &\equiv \epsilon^{-1}(p) a_p \pmod{\mathcal{P}} \\
\langle d \rangle_{pN}^* &\equiv \chi(d)^a \epsilon(d) \pmod{\mathcal{P}} \quad (d \in (\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

□

Remark: There is a variation of the $\Gamma_1(N)$ moduli problem (called $\Gamma_\mu(N)$ in a recent preprint of Diamond and Im) which in many ways is more natural here. Briefly, while a $\Gamma_1(N)$-structure is an embedding of group schemes $\mathbf{Z}/N\mathbf{Z} \hookrightarrow E$, a $\Gamma_\mu(N)$-structure is an embedding $\mu_N \hookrightarrow E$. The $\Gamma_\mu(N)$ problem is well-suited to comparisons in mixed characteristic. In particular, if we replaced $\Gamma_1(N)$ with $\Gamma_\mu(N)$ throughout this section, then the isomorphism ϕ in Proposition 2.2 would be a $\mathbf{T}$-algebra homomorphism (i.e., $T_\ell^* \mapsto T_\ell^*$, $U_\ell^* \mapsto U_\ell^*$, etc.). Moreover, modular forms of slope λ would correspond to ideals of slope λ in Corollary 2.4. We have not taken this route since the vast majority of the literature (in particular [U1-3]) uses the $\Gamma_1(N)$ moduli problem.

3. Hecke operators on I Fix an odd prime p, positive integers n and N with $p \not| \, N$ and $N \ge 5$. In this section we will investigate the action of Hecke operators on sections of various sheaves on I, the Igusa curve of level $p^n N$. These results will be needed to determine, in Section 5, the equivariance for the Hecke action of certain isomorphisms of cohomology groups related to the crystalline cohomology of M.

First of all, if s is a rational function or 1-form on I we define $T_\ell^*(s)$ ($\ell \not| \, p^n N$) and $U_\ell^*(s)$ ($\ell | N$) as $\pi_{1*}\pi_2^*(s)$ where π_1 and π_2 are the maps $I_\ell \to I$ mentioned in Section 2. For $d \in (\mathbf{Z}/p^n N\mathbf{Z})^\times$, we define $\langle d \rangle_{p^n N}^*(s)$ as the usual pull-back of functions or forms. If s is a section of ω^k, we define T_ℓ^* and U_ℓ^* as before, namely as $\pi_{1*}\Phi^*\pi_2^*(s)$. (This definition requires some comment if $k < 0$. Since ω^{-k} is by definition $\underline{\mathrm{Hom}}(\omega^k, \mathcal{O})$, we define $\Phi^* : \pi_2^*\omega^{-k} \to \pi_1^*\omega^{-k}$ as the linear transpose of $\Phi^{*-1} : \pi_1^*\omega^k \to \pi_2^*\omega^k$.) If $\mathcal{F}$ is any invertible sheaf on I and $\mathbf{F}$ is any perfect $\mathbf{F}_p$-algebra, we have a p-linear automorphism σ (defined as in Section 2) of the cohomology groups $H^j(I \times \mathrm{Spec}\ \mathbf{F}, \mathcal{F})$.

We have a canonical regular section ω_c of ω on I, which can be defined as follows: let $K = \mathbf{F}_p(I)$ be the function field of I. The generic fiber of the universal curve $\mathcal{E} \xrightarrow{\pi} I$ is an elliptic curve E over K, equipped with a canonical Igusa structure of level p^n; since E is ordinary, this amounts to an isomorphism of finite group schemes

$$\mathbf{Z}/p^n\mathbf{Z} \xrightarrow{\sim} \mathrm{Ker}\,(V^n : E^{(p^n)} \to E)$$

whose dual is an isomorphism

$$\mathrm{Ker}\,(F : E \to E^{(p^n)}) \xrightarrow{\sim} \mu_{p^n}.$$

There is a unique invariant 1-form on E whose restriction to Ker F is the pull-back of dt/t, where t is the standard coordinate on $\mathbf{G}_m$. This 1-form induces a global rational section of $\pi_*\Omega^1_{reg,\mathcal{E}/I}$ and this section is by definition ω_c. From the definition, one sees that ω_c^{p-1} is the Hasse invariant, viewed as a global section of ω^{p-1}. One can check that ω_c is a generating section of ω away from the supersingular points and vanishes to order p^{n-1} at each supersingular point.

Proposition 3.1. *Let s be a rational function, a rational differential, or a rational section of ω^k, $k \in \mathbf{Z}$, on $I \times \mathrm{Spec}\ \mathbf{F}$ where $\mathbf{F}$ is a perfect field of characteristic p. Then we have*

$$\begin{aligned} T_\ell^*(s\omega_c) &= \ell T_\ell^*(s)\omega_c && (\ell \nmid p^nN) \\ U_\ell^*(s\omega_c) &= \ell U_\ell^*(s)\omega_c && (\ell | N) \\ \sigma(s\omega_c) &= \sigma(s)\omega_c \\ \langle d\rangle_{p^nN}^*(s\omega_c) &= d\langle d\rangle_{p^nN}^*(s)\omega_c && (d \in (\mathbf{Z}/p^nN\mathbf{Z})^\times). \end{aligned}$$

Proof: Considering the definition of ω_c, one finds that $\Phi^*\pi_2^*(\omega_c) = \ell\pi_1^*(\omega_c)$ The first two formulas follow from this. The third formula is equivalent to $\langle d\rangle_{p^nN}^*(\omega_c) = d\omega_c$, which also follows from the definition of ω_c. The last formula follows easily from the fact that ω_c is defined on I itself, i.e., over $\mathbf{F}_p$. $\square$

There is a canonical differential 1-form on I defined as follows: Again consider the generic fiber E/K of the universal curve over I. We claim that E_p, the kernel of p on E, determines canonically an extension of $\mathbf{Z}/p\mathbf{Z}$ by μ_p in the category of finite flat group schemes over K. Indeed, we have

$$0 \to E_F \to E_p \xrightarrow{F} E_V^{(p)} \to 0$$

(where E_F and $E_V^{(p)}$ are the kernels of Frobenius $F : E \to E^{(p)}$ and Verschiebung $V : E^{(p)} \to E$ respectively). The Igusa structure on E identifies $E_V^{(p)}$ with $\mathbf{Z}/p\mathbf{Z}$ and Cartier duality then identifies E_F with μ_p. Let $q \in K^\times/K^{\times p} \cong \mathrm{Ext}^1_K(\mathbf{Z}/p\mathbf{Z}, \mu_p)$ be the class of this extension and form dq/q. This is a rational differential on I which one can check is regular and non-vanishing away from the cusps and supersingular points; its divisor is $p^{2n-1}S - C$ where S is the divisor of supersingular points and C is the divisor of cusps. Define operators θ on rational functions and Θ on rational differentials by the formulas

$$\theta f = \frac{df}{dq/q} \qquad \Theta(\tau) = d\left(\frac{\tau}{dq/q}\right).$$

Proposition 3.2. *For rational functions f and rational differentials τ on $I \times \mathrm{Spec}\,\mathbf{F}$, where $\mathbf{F}$ is a perfect field of characteristic p, we have*

$$\begin{aligned} T_\ell^*\theta f &= \ell^{-1}\theta T_\ell^* f & \qquad T_\ell^*\Theta\tau &= \ell^{-1}\Theta T_\ell^*\tau \\ U_\ell^*\theta f &= \ell^{-1}\theta U_\ell^* f & \qquad U_\ell^*\Theta\tau &= \ell^{-1}\Theta U_\ell^*\tau \\ \sigma\theta f &= \theta\sigma f & \qquad \sigma\Theta\tau &= \Theta\sigma\tau \\ \langle d\rangle_{p^nN}^*\theta f &= d^{-2}\theta\langle d\rangle_{p^nN}^* f & \qquad \langle d\rangle_{p^nN}^*\Theta\tau &= d^{-2}\Theta\langle d\rangle_{p^nN}^*\tau. \end{aligned}$$

Proof: To prove the first two lines, note that the group schemes $\pi_1^*E_p$ and $\pi_2^*E_p$ over $\mathbf{F}_p(X_\ell)$ sit in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mu_p & \to & \pi_1^*E_p & \to & \mathbf{Z}/p\mathbf{Z} & \to & 0 \\ & & \ell\downarrow & & \Phi\downarrow & & 1\downarrow & & \\ 0 & \to & \mu_p & \to & \pi_2^*E_p & \to & \mathbf{Z}/p\mathbf{Z} & \to & 0. \end{array}$$

This shows that the extension class of $\pi_2^*E_p$ is ℓ times that of $\pi_1^*E_p$ (cf. [S], Ch. VII, §1). It follows that $\pi_2^*(dq/q) = \ell\pi_1^*(dq/q)$ and the result follows easily from this.

For the next line, we argue similarly: there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mu_p & \rightarrow & E_p & \rightarrow & \mathbf{Z}/p\mathbf{Z} & \rightarrow & 0 \\ & & d\downarrow & & \langle d\rangle_{p^nN}\downarrow & & d^{-1}\downarrow & & \\ 0 & \rightarrow & \mu_p & \rightarrow & \langle d\rangle^*_{p^nN}E_p & \rightarrow & \mathbf{Z}/p\mathbf{Z} & \rightarrow & 0 \end{array}$$

which shows that $\langle d\rangle^*_{p^nN}(dq/q) = d^2(dq/q)$ and the claimed formulas follow easily.

The last line follows from the fact that dq/q is defined over $\mathbf{F}_p$. □

The proof of the proposition also yields the following, which we record for later use.

Corollary 3.3. *The differential dq/q is an eigenvector for all the Hecke operators. We have $\langle \ell^{-1}\rangle^*_{p^n}T^*_\ell(dq/q) = (1+\ell^{-1})dq/q$, $\langle \ell^{-1}\rangle^*_{p^n}U^*_\ell(dq/q) = dq/q$, and $\langle d\rangle^*_{p^nN}(dq/q) = d^2\,dq/q$.* □

We recall that the projector Π of Section 2 can be applied to certain sheaves on I, since the automorphisms involved cover the identity of I. In particular, we showed in [U3], 4.1 that $\Pi f_*\Omega^{k+1}_{\tilde{X}} \cong \Omega^1_I \otimes \omega^k$ and $\Pi R^k f_*\mathcal{O}_{\tilde{X}} \cong \omega^{-k}$. We need to know the Hecke equivariance of these isomorphisms.

Proposition 3.4. *The isomorphism $\Pi f_*\Omega^{k+1}_{\tilde{X}} \cong \Omega^1_I \otimes \omega^k$ is equivariant for the Hecke operators T^*_ℓ, U^*_ℓ, $\langle d\rangle^*_{p^nN}$, and for σ. Under the isomorphism $\Pi R^k f_*\mathcal{O}_{\tilde{X}} \cong \omega^{-k}$, the operators T^*_ℓ (resp. U^*_ℓ, $\langle d\rangle^*_{p^nN}$, σ) on $\Pi R^k f_*\mathcal{O}_{\tilde{X}}$ correspond to $\ell^k T^*_\ell$ (resp. $\ell^k U^*_\ell$, $\langle d\rangle^*_{p^nN}$, σ) on ω^{-k}.*

Proof: Since the sheaves in question are locally free, it suffices to check the claims away from the cusps. There, the isomorphism $\Pi f_*\Omega^{k+1}_{\tilde{X}} \cong \Omega^1_I \otimes \omega^k$ is the composition of three isomorphisms: $\Pi f_*\Omega^{k+1}_{\tilde{X}} \cong \Omega^1_I \otimes \Pi f_*\Omega^k_{\tilde{X}/I}$ coming from the relative differential sequence; the Künneth isomorphism $\Omega^1_I \otimes \Pi f_*\Omega^k_{\tilde{X}/I} \cong \Omega^1_I \otimes (f_*\Omega^1_{\mathcal{E}/I})^{\otimes^k}$; and the definition of ω (away from the cusps) as $f_*\Omega^1_{\mathcal{E}/I}$. Each of these isomorphisms is clearly equivariant for the Hecke action and since they are defined over $\mathbf{F}_p$, they commute with σ as well.

The second isomorphism is not equivariant because it involves Serre duality. Namely, away from the cusps, it is the composition of the Künneth isomorphism $\Pi R^k f_*\mathcal{O}_{\tilde{X}} \cong (R^1 f_*\mathcal{O}_{\mathcal{E}})^{\otimes^k}$ (which is equivariant) with the isomorphism $(R^1 f_*\mathcal{O}_{\mathcal{E}})^{\otimes^k} \cong \omega^{-k}$ induced by Serre duality. Suppose for the moment that $k = 1$. In the definition $T^*_\ell = \pi_{1*}\Phi^*\pi^*_2$, the isomorphism $\Phi^* : \pi_2^*\omega^{-1} \to \pi_1^*\omega^{-1}$ is the linear transpose of the inverse of the isomorphism $\Phi^* : \pi_2^*\omega = \pi_2^* f_*\Omega^1_{\mathcal{E}/I} \to \pi_1^* f_*\Omega^1_{\mathcal{E}/I} = \pi_1^*\omega$. Thus for sections η of $R^1 f_*\mathcal{O}_E$ and s of $f_*\Omega^1_{\mathcal{E}/I}$, we have

$$\langle \Phi^*\eta, s\rangle = \langle \eta, \check{\Phi}^* s\rangle = \langle \eta, \ell(\Phi^*)^{-1} s\rangle$$

where $\langle\cdot,\cdot\rangle$ is the Serre duality pairing. On the other hand, the transpose of π_2^* (resp. π_{1*}) for both linear and Serre duality is π_{2*} (resp. π_1^*). Thus we find a commutative diagram

$$\begin{array}{cccc} & R^1 f_*\mathcal{O}_{\mathcal{E}} & \to & \omega^{-1} = \underline{\mathrm{Hom}}(\omega, \mathcal{O}_I) & \\ T_\ell^* & \downarrow & & \downarrow & \ell T_\ell^* \\ & R^1 f_*\mathcal{O}_{\mathcal{E}} & \to & \omega^{-1} = \underline{\mathrm{Hom}}(\omega, \mathcal{O}_I) & \end{array}$$

which is the claim when $k = 1$; taking tensor powers yields the claim for a general k. A similar proof works for U_ℓ^*. On the other hand, the transpose of $\langle d\rangle^*_{p^n N}$ for both linear and Serre duality is $\langle d^{-1}\rangle^*_{p^n N}$, so the isomorphism $R^1 f_*\mathcal{O}_{\mathcal{E}} \cong \omega^{-1}$ is equivariant for the actions of this operator. Finally, since this isomorphism is defined over $\mathbf{F}_p$, it too commutes with σ. $\square$

4. Some mod p Hecke algebras As before, we fix an odd prime p and integers $N \geq 5$, k, and a with $p \not| N$, $0 \leq k < p$, and $0 \leq a < p-1$. (From now on, n will be 1.) We also fix a number field $L \subseteq \overline{\mathbf{Q}}$ large enough to contain the eigenvalues of all Hecke operators on $S_w(\Gamma_1(pN))$ for $2 \leq w \leq p+1$. If H is a $\mathbf{T}$-module, we write $H(\chi^a)$ for the submodule where the operators $\langle d\rangle$ act via $\chi^a\epsilon$ where ϵ is any character modulo N. In the notation of Section 2, $H(\chi^a) = H(\Xi_a)$. Recall the motive $M = (\tilde{X}, \Pi)$ of Section 2 attached to the data $(p, n = 1, N, k, a)$. To ease notation, we write $\mathbf{T}_{\mathrm{cris}}$ (or $\mathbf{T}_{\mathrm{cris}}(k, a)$ when we want to emphasize the values of k and a) for $\mathbf{T}(H_{\mathrm{cris}}(M)(\chi^a))$. In this section we will relate certain parts of the spectrum of $\mathbf{T}_{\mathrm{cris}}/\wp\mathbf{T}_{\mathrm{cris}}$ to Hecke algebras arising from the cohomology of logarithmic or exact differentials on M.

For an integer i with $0 \leq i \leq k+1$ we define a Hecke algebra $\mathbf{T}_{\log} = \mathbf{T}_{\log}(k, a, i)$ as follows: On the étale site of $\tilde{X}$, we have the sheaves $\Omega^i_{\log}$ of logarithmic differential i-forms. These sheaves are locally generated additively by sections $\frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_i}{f_i}$ where $f_j \in \mathcal{O}^\times_{\tilde{X}}$. (When $i = 0$, $\Omega^i_{\log}$ is by convention the constant étale sheaf $\underline{\mathbf{Z}/p\mathbf{Z}}$.) Consider the cohomology group

$$H_{\log} = \Pi H^{k+1-i}_{\text{ét}}(\tilde{X} \times \mathrm{Spec}\, \overline{\mathbf{F}_p}, \Omega^i_{\log})(\chi^a).$$

This is an $\mathbf{F}_p$-vector space (not necessarily finite-dimensional) which carries an action of the Hecke operators T_ℓ^*, U_ℓ^*, and $\langle d\rangle_{pN}^*$, acting as in Section 2. It also carries an $\mathbf{F}_p$-linear automorphism σ defined in analogy with the σ acting as in Section 2. Define an action of $\mathbf{T}$ on $H_{\log} \otimes \overline{\mathbf{F}_p}$ by letting T_ℓ act as T_ℓ^* if $\ell \not| \, pN$, as U_ℓ^* if $\ell | N$, and as $\langle p\rangle_N^* \sigma$ if $\ell = p$; and by letting $\langle d\rangle$ act as $\langle d\rangle_{pN}^*$ if $(d, pN) = 1$ or as 0 if $(d, pN) \neq 1$. The base ring $\mathcal{O}_L$ acts via its quotient $\mathcal{O}_L/\wp \subseteq \overline{\mathbf{F}_p}$. We define $\mathbf{T}_{\log}$ as $\mathbf{T}(H_{\log} \otimes \overline{\mathbf{F}_p})$, i.e., as the image of the homomorphism $\mathbf{T} \to \mathrm{End}(H_{\log} \otimes \overline{\mathbf{F}_p})$. The following result says that under suitable hypotheses, $\mathbf{T}_{\log}$ captures the slope i part of $\mathbf{T}_{\mathrm{cris}}$, modulo $\wp$.

Theorem 4.1. *Let p be an odd prime number, N, k, a, and i integers with $p \not| \, N$, $N \geq 5$, $0 \leq k < p$, $0 \leq a < p-1$, and $0 \leq i \leq k+1$. Suppose either $i = 0$; or $i = k+1$; or $a \neq 0$ and i satisfies $i \leq a$ and $k+1-i \leq p-1-a$. Then there is a unique homomorphism of $\mathcal{O}_L$-algebras $\phi : \mathbf{T}_{\mathrm{cris}} \to \mathbf{T}_{\log}$ such that*

$$\begin{aligned}
\phi(T_\ell^*) &= T_\ell^* && (\ell \not| \, pN) \\
\phi(U_\ell^*) &= U_\ell^* && (\ell | N) \\
\phi(\langle p\rangle_N^* V^*) &= p^{k+1-i} \langle p\rangle_N^* \sigma && \\
\phi(\langle d\rangle_{pN}^*) &= \langle d\rangle_{pN}^* && (d \in (\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

For every minimal prime $\mathcal{P} \subseteq \mathbf{T}_{\mathrm{cris}}$ of slope i, there exists a unique maximal prime $\mathbf{m} \subseteq \mathbf{T}_{\log}$ such that

$$\begin{gathered}
\phi^{-1}(\mathbf{m}) = \mathcal{P} + \wp \mathbf{T}_{\mathrm{cris}} \\
\langle p\rangle_N^* V^* \equiv p^{k+1-i} u \pmod{\mathcal{P}} \Rightarrow \langle p\rangle_N^* \sigma \equiv u \pmod{\mathbf{m}}.
\end{gathered} \tag{4.2}$$

Conversely, given a maximal ideal $\mathbf{m}$ of $\mathbf{T}_{\log}$, there exists a minimal prime $\mathcal{P}$ of $\mathbf{T}_{\mathrm{cris}}$ of slope i such that 4.2 holds.

Proof: Consider the integral crystalline cohomology group

$$H = \left(\Pi H_{\mathrm{cris}}^{k+1}(\tilde{X} \times \overline{\mathbf{F}_p}/W(\overline{\mathbf{F}_p})) \otimes_{W(\overline{\mathbf{F}_p})} \mathcal{O}_{\mathbf{C}_p} \right) (\chi^a).$$

It is torsion-free ([U2], 5.6) and so is a lattice in $H_{\mathrm{cris}}(\chi^a) = H \otimes \mathbf{Q}$ on which $\mathbf{T}_{\mathrm{cris}}$ acts. Let $H_{[i]}$ be its slope i subspace $H \cap (H \otimes \mathbf{Q})_{[i]}$.

We showed in [U3], 2.4 that, under the hypotheses, $H_{\log}$ is finite and so by general results of Illusie and Raynaud (cf. [U3] §2) the slope i subspace is a direct factor of H, viewed as F-crystal. The Hecke algebra $\mathbf{T}_{\mathrm{cris}}$ preserves this factor, since the Hecke operators commute with F^*. Moreover, on this factor we have a canonical $\mathbf{Z}_p$-structure

$$H^{\Phi^*=p^i} \subseteq H_{[i]} \qquad H^{\Phi^*=p^i} \otimes_{\mathbf{Z}_p} \mathcal{O}_{\mathbf{C}_p} = H_{[i]}$$

and a canonical isomorphism

$$H^{\Phi^*=p^i}/pH^{\Phi^*=p^i} \cong H_{\log}.$$

(Here as in Section 2, Φ is the semi-linear endomorphism induced by the absolute Frobenius of $\tilde{X} \times \mathrm{Spec}\, \overline{\mathbf{F}_p}$.) This implies that $H_{[i]}/\wp H_{[i]} \cong H_{\log} \otimes \overline{\mathbf{F}_p}$ and this isomorphism is compatible with all the Hecke operators; note that on $H^{\Phi^*=p^i}$, T_p acts by $\langle p \rangle_N^* V^* = \langle p \rangle_N^* \sigma p^{k+1-i}$. This establishes the existence of a homomorphism $\mathbf{T}_{\mathrm{cris}} \to \mathbf{T}_{\log}$ as in the statement of the theorem.

The existence of such a homomorphism gives a relation among ideals in $\mathbf{T}_{\mathrm{cris}}$ and $\mathbf{T}_{\log}$, but we have to do more to obtain the implication in 4.2. (In $\mathbf{T}_{\log}$, a ring of characteristic p, p^{k+1-i} is usually zero.) We will use the following well-known lemma whose proof we recall for the convenience of the reader.

Lemma 4.3. *Let $\mathbf{F}$ be a field, A an $\mathbf{F}$-algebra, and V an A-module which is finite dimensional as an $\mathbf{F}$-vector space. If $\mathcal{P} \subseteq A$ is a prime in the support of V, then*

$$V[\mathcal{P}] = \{v \in V | av = 0 \text{ for all } a \in \mathcal{P}\} \neq 0.$$

Proof: Since $\mathcal{P}$ is in the support of V, we have $\mathrm{Ann}(V) \subseteq \mathcal{P}$ and we can replace A with $A/\mathrm{Ann}(V)$. Thus $A \subseteq \mathrm{End}_{\mathbf{F}}(V)$ and A is Artinian. If $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are the primes of A, with $\mathcal{P} = \mathcal{P}_1$, then V is the direct sum of its localizations $V_{\mathcal{P}_i}$. Moreover, by Nakayama's lemma $\mathcal{P}V_{\mathcal{P}}$ is properly contained in $V_{\mathcal{P}}$; since V is finite dimensional, $\mathcal{P}^m V_{\mathcal{P}} = 0$ for large enough m. If m_0 is the minimal such m, then

$$0 \neq \mathcal{P}^{m_0-1} V_{\mathcal{P}} \subseteq V_{\mathcal{P}} \subseteq V$$

and $P^{m_0-1} V_{\mathcal{P}}$ is killed by $\mathcal{P}$. $\square$

Now let $\mathcal{P}$ be a minimal prime of $\mathbf{T}_{\text{cris}}$ of slope i. Since $\mathbf{T}_{\text{cris}}/\mathcal{P} \cong \mathcal{O}_L$, $\mathcal{P}$ stays prime in $\mathbf{T}_{\text{cris}} \otimes_{\mathcal{O}_L} \mathbf{C}_p$ and we can apply the lemma with $\mathbf{F} = \mathbf{C}_p$, $A = \mathbf{T}_{\text{cris}} \otimes_{\mathcal{O}_L} \mathbf{C}_p$, and $V = H_{\text{cris}}$. An element of $V[\mathcal{P}]$ is just an eigenvector for all the Hecke operators, and scaling suitably, we can take it to be a primitive vector $v \in H$; since $\mathcal{P}$ has slope i, $v \in H_{[i]}$. Thus v projects in $H_{\log} \otimes \overline{\mathbf{F}_p}$ to a non-zero eigenvector for all the Hecke operators. If $\mathbf{m}$ denotes the kernel of the homomorphism from $\mathbf{T}_{\log}$ to $\overline{\mathbf{F}_p}$ which sends an operator to its eigenvalue on v, then $\mathbf{m}$ is the desired maximal ideal.

Conversely, let $\mathbf{m} \subseteq \mathbf{T}_{\log}$ be a maximal ideal. Arguing as in a lemma of Deligne and Serre, we will produce an eigenvector $v \in H_{[i]}$ whose eigenvalue for each $t \in \mathbf{T}$ is congruent to the residue of t modulo $\mathbf{m}$. First, let $\mathbf{T}_{[i]}$ be the subalgebra of $\operatorname{End}(H_{[i]})$ generated by the completion $\mathcal{O}_{L,\wp}$, $\mathbf{T}_{\text{cris}}$, and the operator σ. As $\mathbf{T}_{[i]}$ is finite and torsion free over $\mathcal{O}_{L,\wp}$, it is free over $\mathcal{O}_{L,\wp}$. Also, we have a surjection $\mathbf{T}_{[i]} \xrightarrow{\psi} \mathbf{T}_{\log}$. The ideal $\psi^{-1}(\mathbf{m})$ is maximal and using "going down", we can find a prime $\mathcal{P}'$ of $T_{[i]}$ contained in $\psi^{-1}(\mathbf{m})$ with $\mathcal{P}' \cap \mathcal{O}_{L,\wp} = 0$; since the eigenvalues of all $t \in \mathbf{T}_{\text{cris}}$ lie in $\mathcal{O}_L$, we even have $\mathbf{T}_{[i]}/\mathcal{P}' \cong \mathcal{O}_{L,\wp}$. Now using the lemma as above gives an eigenvector $v \in H_{[i]}$ as desired. The kernel of the homomorphism $\mathbf{T}_{\text{cris}} \to \mathcal{O}_L$ which sends an operator to its eigenvalue on v is then a minimal prime of $\mathbf{T}_{\text{cris}}$ with the desired properties. $\square$

Our next task is to relate the action of Hecke operators on Eisenstein series to mod p cohomology. While it is probably possible to do this by arguing as in Theorem 4.1 (after proving analogues of the results of Illusie and Raynaud in the context of logarithmic schemes ("log-log cohomology"?)), that would take us too far afield so we will use an *ad hoc* method.

Consider the $\mathbf{F}_p$-vector space

$$V = \frac{H^0(I, \Omega^1(C_I))}{H^0(I, \Omega^1)}$$

where C_I denotes the reduced divisor of cusps of I. The Hecke operators T_ℓ^*, U_ℓ^*, and $\langle d \rangle_{pN}^*$ act on this vector space. Moreover, the differential dq/q introduced in Section 3 defines a non-zero element of V and, by Corollary 3.3, dq/q is an eigenvector for all the Hecke operators. On $V \otimes \overline{\mathbf{F}_p}$ we have a p^{-1}-linear action of the Cartier operator $\mathcal{C}$ and a p-linear operator σ. Since dq/q is logarithmic and defined over $\mathbf{F}_p$, it is also fixed by $\mathcal{C}$ and σ. We define

$$D = D(a) = \left(\frac{V}{\mathbf{F}_p dq/q} \right) (\chi^a)$$

and

$$D_{\log} = D_{\log}(a) = \left(D(a) \otimes \overline{\mathbf{F}_p} \right)^{\mathcal{C}=1}.$$

Note that D and $D_{\log}$ are two (possibly distinct) $\mathbf{F}_p$-structures on $D \otimes \overline{\mathbf{F}_p}$. We let $\mathbf{T}$ act on $D(a) \otimes \overline{\mathbf{F}_p}$ by letting T_ℓ act as T_ℓ^* if $\ell \not| pN$, as U_ℓ^* if $\ell | N$, and as $\sigma\mathcal{C}$ if $\ell = p$; $\langle d \rangle$ acts as $\langle d \rangle_{pN}^*$ if $(d, pN) = 1$ and as 0 if $(d, pN) \neq 1$. We then define $\mathbf{T}_{\text{cusps}} = \mathbf{T}_{\text{cusps}}(a)$ as $\mathbf{T}(D(a) \otimes \overline{\mathbf{F}_p})$.

Now consider the space $\mathcal{E}_2(\Gamma_1(pN))$ of Eisenstein series of weight 2 on $\Gamma_1(pN)$, together with its action of the T_ℓ^*, U_ℓ^*, U_p^* and $\langle d \rangle_{pN}^*$. The subspace $\mathcal{E}_2(\Gamma_1(pN))_{[0]}$ spanned by eigenforms for U_p^* with eigenvalues which are units at $\wp$ contains the Eisenstein series $E_{2,\chi^{-2}}^{(N)}$ of Section 1. Let

$$W = W(a) = \begin{cases} \left(\mathcal{E}_2(\Gamma_1(pN))_{[0]} / \mathbf{C}E_{2,\chi^{-2}}^{(N)} \right) (\chi^a) & \text{if } a \neq 0 \\ \mathcal{E}_2(\Gamma_1(pN))_{[0]}^{p-\text{old}} & \text{if } a = 0. \end{cases}$$

We define a Hecke algebra $\mathbf{T}_{\text{Eis}} = \mathbf{T}_{\text{Eis}}(a)$ as $\mathbf{T}(W)$ where $\mathbf{T}$ acts in the obvious way.

Proposition 4.4. *There is a unique surjection of $\mathcal{O}_L$-algebras $\phi : \mathbf{T}_{\text{Eis}} \to \mathbf{T}_{\text{cusps}}$ such that*

$$\begin{aligned} \phi(T_\ell^*) &= \langle \ell^{-1} \rangle_p^* T_\ell^* && (\ell \not| p^n N) \\ \phi(U_\ell^*) &= \langle \ell^{-1} \rangle_p^* U_\ell^* && (\ell | N) \\ \phi(U_p^*) &= \sigma\mathcal{C} && (\ell = p) \\ \phi(\langle d \rangle_{pN}^*) &= \langle d^{-1} \rangle_p^* \langle d \rangle_N^* && (d \in (\mathbf{Z}/p^n N\mathbf{Z})^\times). \end{aligned}$$

The map ϕ induces an isomorphism $(\mathbf{T}_{\text{Eis}}/\wp)^{\text{red}} \cong (\mathbf{T}_{\text{cusps}})^{\text{red}}$.

Proof: This is just a Hecke algebra-theoretic version of the well-known fact that Eisenstein series are determined uniquely by their values at the cusps. The key points in the proof are: i) Eisenstein series of weight 2 correspond via the Kodaira-Spencer isomorphism to differentials on $X_1(pN)$ with poles only at the cusps; ii) the differentials corresponding to Eisenstein series of slope 0 have poles only at cusps whose ramification index is prime to p; and iii) these cusps reduce to the Igusa curve component of a model of $X_1(pN)$ over $\mathbf{Z}_p[\zeta_p]$. In the interest of brevity, we leave the details to the reader. □

Corollary 4.5. *For every prime* $\mathbf{m} \subseteq \mathbf{T}_{\mathrm{cusps}}(a)$*, there exists a Dirichlet character* ϵ *modulo* N *and a normalized Eisenstein series* $E = \sum a_n q^n$ *in* $\mathcal{E}_2(\Gamma_0(pN), \chi^{-a}\epsilon)$ *such that*

$$\begin{aligned} T_\ell^* &\equiv \ell^{-a} a_\ell \pmod{\mathbf{m}} \qquad (\ell \nmid pN) \\ U_\ell^* &\equiv \ell^{-a} a_\ell \pmod{\mathbf{m}} \qquad (\ell | N) \\ \sigma\mathcal{C} &\equiv a_p \pmod{\mathbf{m}} \\ \langle d \rangle_{pN}^* &\equiv \chi^{-a}\epsilon(d) \pmod{\mathbf{m}} \quad (d \in (\mathbf{Z}/pN\mathbf{Z})^\times). \end{aligned}$$

Moreover, we can assume $E \neq E^{(N)}_{2,\chi^{-2}}$ *and if* $a = 0$*, we can assume* E *is old at* p*. Conversely, given such an Eisenstein series, there exists a prime* $\mathbf{m} \subseteq \mathbf{T}_{\mathrm{cusps}}(a)$ *such that these congruences hold.* □

To finish the section, we will relate certain primes of $\mathbf{T}_{\mathrm{cris}}$ of non-integral slope to mod p cohomology. Recall the higher exact differentials on a variety X of characteristic p: let B^i_X and Z^i_X be the sheaves of exact and closed differential i-forms on X. Then $B^i_{1,X}$ is just B^i_X and the $B^i_{n,X} \subseteq Z^i_X$ are defined inductively by the exact sequences

$$0 \to B^i_X \to B^i_{n,X} \xrightarrow{\mathcal{C}} B^i_{n-1,X} \to 0$$

where $\mathcal{C}$ is the Cartier operator. We have inclusions $B^i_{n-1,X} \subseteq B^i_{n,X}$ and each $B^i_{n,X}$ is a locally free sheaf of $\mathcal{O}_X^{p^n}$-modules.

As usual, the Hecke operators T_ℓ^*, U_ℓ^*, and $\langle d \rangle^*_{pN}$ act on the cohomology groups $H^j(M, B^i_{n,\tilde{X}}) = \Pi H^j(\tilde{X}, B^i_{n,\tilde{X}})$. Now consider the cohomology group

$$H_{\mathrm{exact}} = \varprojlim_n H^{k+1-i}(M, B^{i+1}_{n,\tilde{X}})(\chi^a)$$

where the inverse limit is taken with respect to the maps $\mathcal{C}$ and (χ^a) is with respect to the action of the $\langle d\rangle_p^*$. We define an endomorphism V^* by setting $V^*(c_n)_{n\geq 0} = (d_n)_{n\geq 0}$ where d_n is the image of c_{n-1} under the natural map

$$H^{k+1-i}(M, B^{i+1}_{n-1,\tilde{X}}) \to H^{k+1-i}(M, B^{i+1}_{n,\tilde{X}}).$$

We define a $\mathbf{T}$ action on $H_{\text{exact}} \otimes \overline{\mathbf{F}_p}$ by letting T_ℓ act as T_ℓ^* if $\ell \not| pN$, as U_ℓ^* if $\ell | N$, and as $\langle p\rangle_N^* V^*$ if $\ell = p$; $\langle d\rangle_{pN}$ acts as $\langle d\rangle_{pN}^*$ if $(d, pN) = 1$ or as 0 if $(d, pN) \neq 1$; the base ring $\mathcal{O}_L$ acts via its quotient $\mathcal{O}_L/\wp \subseteq \overline{\mathbf{F}_p}$. Define $\mathbf{T}_{\text{exact}} = \mathbf{T}_{\text{exact}}(k, a, i)$ as $\mathbf{T}(H_{\text{exact}} \otimes \overline{\mathbf{F}_p})$. This Hecke algebra captures information modulo p on the part of crystalline cohomology with slopes in the interval $(i, i+1)$.

Proposition 4.6. *Let p be an odd prime number, N, k, a, and i integers with $p \not| N$, $N \geq 5$, $0 \leq k < p$, $0 \leq a < p-1$, and $0 \leq i \leq k$. Suppose either that $i+1 \leq a$ and $k+1-i \leq p-1-a$; or that $i = k$ and $a \geq k$; or that $i = 0$, $a > 0$ and $k \leq p-1-a$. Then there is a unique homomorphism of $\mathcal{O}_L$-algebras $\phi : \mathbf{T}_{\text{cris}} \to \mathbf{T}_{\text{exact}}$ such that*

$$\begin{aligned}
\phi(T_\ell^*) &= T_\ell^* && (\ell \not| pN)\\
\phi(U_\ell^*) &= U_\ell^* && (\ell | N)\\
\phi(\langle p\rangle_N^* V^*) &= p^{k-i}\langle p\rangle_N^* V^* &&\\
\phi(\langle d\rangle_{pN}^*) &= \langle d\rangle_{pN}^* && (d \in (\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

This map identifies Spec $\mathbf{T}_{\text{exact}}$ *with the set of closed points of* Spec $\mathbf{T}_{\text{cris}}$ *lying over $\wp$ and containing a minimal prime whose slope lies in the interval $(i, i+1)$.*

Proof: Let H be the integral crystalline cohomology group appearing in the proof of Theorem 4.1. We proved in [U3], 2.7 that under the hypotheses, the subspace $H_{(i,i+1)} = (H \otimes \mathbf{Q})_{(i,i+1)} \cap H$ with slopes in $(i, i+1)$ is a direct factor of H as F-crystal. This direct factor is preserved by the Hecke operators and we have an isomorphism

$$H_{(i,i+1)} \cong H^{k+1-i}(M, BW\Omega^{i+1}_{\tilde{X}})(\chi^a) \otimes_{W(\mathbf{F}_p)} \mathcal{O}_{\mathbf{C}_p}.$$

We also have

$$H^{k+1-i}(M, BW\Omega^{i+1}_{\tilde{X}})(\chi^a)/F \cong H^{k-i}(M, W\Omega^{i+1}_{\tilde{X}})(\chi^a)/F \cong H_{\text{exact}}$$

and all these isomorphisms are compatible with the Hecke operators. (The F in the last displayed equation is not our F^*, but rather a semi-linear operator defined in the deRham-Witt theory. We have $p^i F = \Phi^*$ on $H_{(i,i+1)}$.) This establishes the existence of a homomorphism $\mathbf{T}_{\text{cris}} \to \mathbf{T}_{\text{exact}}$ and the image of the corresponding map of spectra certainly lies in the set of closed points lying over $\wp$ with slope in $(i, i+1)$. To see that all such arise, we need to use the fact that F is topologically nilpotent on $H^{k+1-i}(M, BW\Omega^{i+1}_{\tilde{X}})(\chi^a)$, so that any operator nilpotent modulo F is also nilpotent modulo p. $\square$

5. Relations between weight $k+2$ and weight 2 In this section we will relate the mod p Hecke algebras studied in the last section for different weights and slopes. This is the key ingredient giving congruences between modular forms of various weights and slopes.

Theorem 5.1. *Suppose that $0 < k < p-1$, $0 < a < p-1$, and i satisfies $1 \le i \le k$, $i \le a$, and $k+1-i \le p-1-a$. Let*

$$c = \binom{a+k+1-i}{i} \Big/ \binom{a}{i}$$

and put $b = a + k - 2i$. Then there exists a unique homomorphism of $(\mathcal{O}_L/\wp)$-algebras $\phi : \mathbf{T}_{\log}(k,a,i) \to \mathbf{T}_{\log}(0, b+2, 1) \oplus \mathbf{T}_{\text{cusps}}(b+2) \oplus \mathbf{T}_{\log}(0,b,0)$ such that

$$\begin{aligned}
\phi(T_\ell^*) &= (\ell^{i-1}T_\ell^*, \ell^{i-1}T_\ell^*, \ell^i T_\ell^*) && (\ell \not| \, pN)\\
\phi(U_\ell^*) &= (\ell^{i-1}U_\ell^*, \ell^{i-1}U_\ell^*, \ell^i U_\ell^*) && (\ell | N)\\
\phi(\sigma) &= (c\sigma, c\sigma\mathcal{C}, c\sigma)\\
\phi(\langle d\rangle_{pN}^*) &= (d^{a-b-2}\langle d\rangle_{pN}^*, d^{a-b-2}\langle d\rangle_{pN}^*, d^{a-b}\langle d\rangle_{pN}^*) && (d \in (\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

The kernel of ϕ is contained in the nilradical of $\mathbf{T}_{\log}(k,a,i)$ and the composition of ϕ with projection to any one of the three factors is surjective.

Corollary 5.2. *The map of topological spaces underlying*

$$\text{Spec } (\mathbf{T}_{\log}(0, b+2, 1) \oplus \mathbf{T}_{\text{cusps}}(b+2) \oplus \mathbf{T}_{\log}(0,b,0)) \to \text{Spec } \mathbf{T}_{\log}(k,a,i)$$

is surjective and its restriction to each of the closed subschemes Spec $\mathbf{T}_{\log}(0, b+2, 1)$, Spec $\mathbf{T}_{\text{cusps}}(b+2)$, *and* Spec $\mathbf{T}_{\log}(0,b,0)$ *is injective.* $\square$

Proof of 5.1: The proof will use the detailed information on $H_{\log}$ worked out in [U3] and the results on Hecke operators on I in Section 3.

Let

$$c_1 = \binom{a+k+1-i}{i} \qquad c_2 = \binom{a}{i}$$

so that $c = c_1/c_2$. According to Theorem 2.4a) of [U3], there is a three step filtration of $H_{\log}$ whose graded pieces are isomorphic to $H^0_{\text{ét}}(I \otimes \overline{\mathbf{F}_p}, \Omega^1_{\log})(\chi^{b+2})$, $D_{\log}(b+2)$, and $H^1_{\text{ét}}(I \otimes \overline{\mathbf{F}_p}, \Omega^0_{\log})(\chi^b)$. The filtration on $H_{\log}$ is preserved by the Hecke operators, but the action on the graded pieces is not the usual one. In order to make this precise, we will have to recall some of the proof of 2.4a) from [U3]. Let $f : \tilde{X} \to I$ be the projection. Using the Leray spectral sequence, one finds that

$$H_{\log} = H^0(I \otimes \overline{\mathbf{F}_p}, \Pi R^{k+1-i} f_* \Omega^i_{log})(\chi^a).$$

Let $\pi : I \to X_1(N)_{/\mathbf{F}_p}$ be the natural map; this is a Galois cover with group $(\mathbf{Z}/p\mathbf{Z})^\times$ and we can define $(\pi_* \Pi R^{k+1-i} f_* \Omega^i_{log})(\chi^a)$. In [U3], §8, we defined a sheaf $\mathcal{F}(\chi^b)$ (which is a subsheaf of π_* of the constant sheaf of rational functions on I) and an isomorphism

$$(\pi_* \Pi R^{k+1-i} f_* \Omega^i_{log})(\chi^a) \cong \mathcal{F}(\chi^b).$$

This isomorphism introduces a twist in the Hecke action, which we will now make explicit.

First of all, the sheaf sequence

$$0 \to \Omega^i_{log,\tilde{X}} \to Z^i_{\tilde{X}} \xrightarrow{1-\mathcal{C}} \Omega^i_{\tilde{X}} \to 0$$

induces an inclusion $\Pi R^{k+1-i} f_* \Omega^i_{log,\tilde{X}} \subseteq \Pi R^{k+1-i} f_* Z^i_{\tilde{X}}$ of sheaves on I. In [U3], §7, we defined subsheaves $D_i \subseteq \Pi R^k f_* \mathcal{O}_{\tilde{X}}$ and $C_{i+1} \subseteq f_* \Omega^{k+1}_{\tilde{X}}$ and homomorphisms $D_i \to \Pi R^{k+1-i} f_* Z^i_{\tilde{X}}$ and $C_{i+1} \hookrightarrow \Pi R^{k+1-i} f_* Z^i_{\tilde{X}}$. These homomorphisms are defined by chasing through cohomology exact sequences and are Hecke equivariant. Using the isomorphisms $\Pi R^k f_* \mathcal{O}_{\tilde{X}} \cong \omega^{-k}$ and $f_* \Omega^{k+1}_{\tilde{X}} \cong \Omega^1_I \otimes \omega^k$, for suitable functions f and g on I the section $(f\omega_c^{-k}, dg\omega_c^k)$ of $\omega^{-k} \oplus \Omega^1_I \otimes \omega^k$ gives rise to a section s of $\Pi R^{k+1-i} f_* Z^i_{\tilde{X}}$; we gave necessary and sufficient conditions in [U3], §8 for a pair of functions (f, g) to give rise to a section of $\Pi R^{k+1-i} f_* \Omega^i_{log,\tilde{X}}$. In view of Proposition 3.4, we have that the action of $(\ell^k T_\ell^*, T_\ell^*)$

on $(f\omega_c^{-k}, dg\omega_c^k)$ corresponds to the action of T_ℓ^* on s. Similarly, the action of $(\ell^k U_\ell^*, U_\ell^*)$ (resp. $(\langle d\rangle_{pN}^*, \langle d\rangle_{pN}^*)$, (σ,σ)) on $(f\omega_c^{-k}, dg\omega_c^k)$ corresponds to the action of U_ℓ^* (resp. $\langle d\rangle_{pN}^*$, σ) on s.

The isomorphism $\Pi R^{k+1-i} f_* \Omega^i_{log,\tilde{X}}(\chi^a) \to \mathcal{F}(\chi^b)$ alluded to above sends a section s corresponding to $(f\omega_c^{-k}, dg\omega_c^k)$ to the rational function h defined as

$$h = \frac{1}{i!}\theta^i(f) + \frac{(-1)^k k!}{i!}\theta^{p-1-(k-i)}.$$

Using Propositions 3.1 and 3.2, we see that the action of T_ℓ^* (resp. U_ℓ^*, $\langle d\rangle_p^*$, $\langle d\rangle_N^*$, σ) on $\pi_* \Pi R^{k+1-i} f_* \Omega^i_{log,\tilde{X}}(\chi^a)$ corresponds to the action of $\ell^i T_\ell^*$ (resp. $\ell^i U_\ell^*$, $d^{a-b}\langle d\rangle_p^*$, $\langle d\rangle_N^*$, σ) on $\mathcal{F}(\chi^b)$.

In the last part of the proof of 2.4a) of [U3], we have a short exact sequence

$$0 \to \left(\frac{H^0(I\otimes \overline{\mathbf{F}_p}, \Omega^1(C_I))}{\mathbf{F}_p x^{-1} dq/q}\right)(\chi^{b+2})^{c_1\mathcal{C}-c_2=0} \to H^0(I\otimes\overline{\mathbf{F}_p}, \mathcal{F})(\chi^b) \to$$
$$H^1(I\otimes\overline{\mathbf{F}_p}, \mathcal{O})(\chi^b)^{c_1-c_2F^*=0} \to 0$$

where x satisfies $x^{p-1} = c_2/c_1$. The maps are as follows: given a section h of $\mathcal{F}(\chi^b)$, there exist rational functions h_x in the local ring at each supersingular point x which are well-defined up to regular functions, and are such that $h - c_1 h_x + c_2 h_x^p$ is regular at x. The map to $H^1(I,\mathcal{O})$ sends h to the répartition of $\mathcal{O}_I$ with h_x at the supersingular point x and zeroes elsewhere. This map is evidently equivariant for the Hecke operators. The kernel turns out to be the set of global sections h of $\mathcal{F}(\chi^b)$ which can be written globally in the form $h = c_1 H - c_2 H^p$ and such a section is mapped to the 1-form $H\frac{dq}{q}$ which lies in $H^0(I,\Omega^1(C_I))(\chi^{b+2})^{c_1\mathcal{C}-c_2=0}$ and is well-defined up to the addition of an $\mathbf{F}_p$ multiple of $x^{-1}dq/q$. Using Proposition 3.2, we have that the action of T_ℓ^* (resp. U_ℓ^*, $\langle d\rangle_p^*$, $\langle d\rangle_N^*$, σ) on $H^0(I,\mathcal{F})(\chi^b)$ corresponds to the action of $\ell^{-1}T_\ell^*$ (resp. $\ell^{-1}U_\ell^*$, $d^{-2}\langle d\rangle_p^*$, $\langle d\rangle_N^*$, σ) on $H^0(I,\Omega^1(C_I))(\chi^{b+2})$.

Finally, multiplication by x defines isomorphisms

$$\left(\frac{H^0(I\otimes \overline{\mathbf{F}_p}, \Omega^1(C_I))}{\mathbf{F}_p x^{-1} dq/q}\right)(\chi^{b+2})^{c_1\mathcal{C}-c_2=0} \cong \left(\frac{H^0(I\otimes \overline{\mathbf{F}_p}, \Omega^1(C_I))}{\mathbf{F}_p dq/q}\right)(\chi^{b+2})^{\mathcal{C}=1}$$
$$H^1(I\otimes\overline{\mathbf{F}_p}, \mathcal{O})(\chi^b)^{c_1-c_2F^*=0} \cong H^1(I\otimes\overline{\mathbf{F}_p}, \mathcal{O})(\chi^b)^{F^*=1}$$
$$\cong H^1(I\otimes\overline{\mathbf{F}_p}, \Omega^0_{\log})(\chi^b)$$

and $\left(\frac{H^0(I\otimes\overline{\mathbf{F}_p},\Omega^1(C_I))}{\mathbf{F}_p dq/q}\right)(\chi^{b+2})^{\mathcal{C}=1}$ is in an obvious way an extension of $D_{\log}(b+2)$ by $H^0(I\otimes\overline{\mathbf{F}_p},\Omega^1_{\log})(\chi^{b+2})$. These isomorphisms are equivariant for all the Hecke operators except σ on the left hand side corresponds to $c\sigma$ on the "log" groups.

Combining all of the above, we have an action of $\mathbf{T}_{\log}(k,a,i)$ on the group

$$H^0_{\text{ét}}(I,\Omega^1_{\log})(\chi^{b+2})\oplus D_{\log}(b+2)\oplus H^1_{\text{ét}}(I,\Omega^0_{\log})(\chi^b)$$

such that

$$\begin{aligned}
T^*_\ell &\leftrightarrow (\ell^{i-1}T^*_\ell,\ell^{i-1}T^*_\ell,\ell^iT^*_\ell) && (\ell\not| pN)\\
U^*_\ell &\leftrightarrow (\ell^{i-1}U^*_\ell,\ell^{i-1}U^*_\ell,\ell^iU^*_\ell) && (\ell|N)\\
\sigma &\leftrightarrow (c\sigma,c\sigma\mathcal{C},c\sigma)\\
\langle d\rangle^*_{pN} &\leftrightarrow (d^{a-b-2}\langle d\rangle^*_{pN},d^{a-b-2}\langle d\rangle^*_{pN}\,d^{a-b}\langle d\rangle^*_{pN}) && (d\in(\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

This proves the existence of the homomorphism in the theorem and its uniqueness is obvious. It is also clear that this homomorphisms has a nilpotent kernel and that its composition with the projection to each of the factors is surjective. Thus the proof of the theorem is complete. $\square$

The next result gives a relation between the algebras $\mathbf{T}_{\text{exact}}$ attached to different weights and characters.

Theorem 5.3. *Let k, a, and i be integers with $0<k<p$, $0<a<p-1$, and $0\le i\le k$, and let $b=a+k-2i$. Suppose either that $i+1\le a$ and $k+1-i\le p-1-a$; or that $i=k$ and $a\ge k$; or that $i=0$ and $p-1-a\ge k$. Then there exists a unique isomorphism of $(\mathcal{O}_L/\wp)$-algebras $\phi:\mathbf{T}_{\text{exact}}(k,a,i)\to\mathbf{T}_{\text{exact}}(0,b,0)$ such that*

$$\begin{aligned}
\phi(T^*_\ell) &= \ell^iT^*_\ell && (\ell\not| pN)\\
\phi(U^*_\ell) &= \ell^iU^*_\ell && (\ell|N)\\
\phi(\langle p\rangle^*_NV^*) &= \langle p\rangle^*_NV^*\\
\phi(\langle d\rangle^*_{pN}) &= d^{a-b}\langle d\rangle^*_{pN} && (d\in(\mathbf{Z}/pN\mathbf{Z})^\times).
\end{aligned}$$

Proof: Using [U3], 9.2, we have, for all n, isomorphisms

$$H^{k+1-i}(M, B^{i+1}_{n,\tilde{X}})(\chi^a) \cong H^1(I, B^1_{n,I})(\chi^b)$$

compatible with the Cartier operators. These isomorphisms are also compatible with the maps induced by the inclusions $B^{i+1}_{n-1} \hookrightarrow B^{i+1}_n$, and thus with the V^* operators . So to prove the theorem, we need only examine their equivariance with respect to the operators T^*_ℓ, U^*_ℓ, and $\langle d\rangle^*_{pN}$.

To that end, let us recall how these isomorphisms were defined. In fact, we defined isomorphisms

$$\left(\pi_*\Pi R^{k-i} f_* B^{i+1}_{n,\tilde{X}}\right)(\chi^a) \cong \left(\pi_* B^1_{n,I}\right)(\chi^b)$$

where $\pi : I \to X_1(N)_{/\mathbf{F}_p}$ is the projection. Given a section $s \in \left(\pi_*\Pi R^{k-i} f_* B^{i+1}_{n,\tilde{X}}\right)(\chi^a)$, we have n sections $s_1, \ldots, s_n$ of $\left(\pi_*\Pi R^{k-i} f_* Z^{i+1}_{\tilde{X}}\right)(\chi^a)$, namely the images of s under the maps

$$\Pi R^{k-i} f_* B^{i+1}_{n,\tilde{X}} \xrightarrow{\mathcal{C}^{n-j}} \Pi R^{k-i} f_* B^{i+1}_{j,\tilde{X}} \to \Pi R^{k-i} f_* Z^{i+1}_{\tilde{X}}.$$

As in the proof of 5.1, attached to each s_j we have a pair of sections $(f_j\omega_c^{-k}, dg_j\omega_c^k)$ of ω^{-k} and $\Omega^1_I \otimes \omega^k$ whose images in $\Pi R^{k-i} f_* Z^{i+1}_{\tilde{X}}$ under the d- and $\mathcal{C}$-construction maps sum to s_j. Using 3.4, we have that the action of T^*_ℓ (resp. U^*_ℓ, $\langle d\rangle^*_{pN}$) on s intertwines the action of $(\ell^k T^*_\ell, T^*_\ell)$ (resp. $(\ell^k U^*_\ell, U^*_\ell)$, $(\langle d\rangle^*_{pN}, \langle d\rangle^*_{pN})$) on $(f_j\omega_c^{-k}, dg_j\omega_c^k)$. Now set

$$h_j = \theta\phi(f_j, dg_j) = \frac{1}{i!}\theta^{i+1}(f_j) + \frac{(-1)^k k!}{i!}\theta^{p-k+i}(g_j)$$

and

$$c' = \binom{k-i+a}{i+1} \Big/ \binom{a}{i+1}.$$

Then the image of s in $B^1_{n,I}$ is the 1-form

$$\tau = \left(h_1^{p^{n-1}} + c' h_2^{p^{n-2}} + \cdots + c'^{n-1} h_n\right)\frac{dq}{q}.$$

Using Propositions 3.1 and 3.2, we see that the action of T^*_ℓ (resp. U^*_ℓ, $\langle d\rangle^*_{pN}$) on s intertwines the action of $\ell^i T^*_\ell$ (resp. $\ell^i U^*_\ell$, $d^{a-b}\langle d\rangle^*_{pN}$) on τ. This is exactly the needed equivariance. $\square$

6. Proof of Theorem 1.1 Suppose for the moment that $N \geq 5$. We prove part a) first: let k, a, and i be integers satisfying the hypotheses of the theorem and let $f = \sum a_n q^n$ be a normalized eigenform in $S_{k+2}(\Gamma_1(pN))$ of slope i with character $\chi^a\epsilon$. Set $b = a + k - 2i$, $a' = p - 1 - a$, $i' = k + 1 - i$, and $b' = a' + k - 2i'$. By Corollary 2.4 and Theorem 4.1, we have a maximal ideal $\mathbf{m} \subseteq \mathbf{T}_{\log} = \mathbf{T}_{\log}(k, a', i')$ such that $\mathbf{T}_{\log}/\mathbf{m} \cong \mathcal{O}_L/\wp$ and

$$\begin{aligned} T_\ell^* &\equiv \ell^{a'} a_\ell \pmod{\mathbf{m}} \qquad (\ell \nmid pN) \\ U_\ell^* &\equiv \ell^{a'} a_\ell \pmod{\mathbf{m}} \qquad (\ell | N) \\ \langle p \rangle_N^* \sigma &\equiv p^{-i} a_p \pmod{\mathbf{m}} \\ \langle d \rangle_{pN}^* &\equiv \chi^{a'} \epsilon(d) \pmod{\mathbf{m}} \quad (d \in (\mathbf{Z}/pN\mathbf{Z})^\times). \end{aligned}$$

Now using Theorem 5.1, we get a maximal ideal $\mathbf{m}'$ in one of the three rings $\mathbf{T}_{\log}(0, b'+2, 1)$, $\mathbf{T}_{\text{cusps}}(b' + 2)$, or $\mathbf{T}_{\log}(0, b', 0)$ such that certain congruences are satisfied. Assume for definiteness that $\mathbf{m}'$ is in the first ring; the other two are entirely analogous. Then we have

$$\begin{aligned} T_\ell^* &\equiv \ell^{a'-i'} a_\ell \pmod{\mathbf{m}'} \qquad (\ell \nmid pN) \\ U_\ell^* &\equiv \ell^{a'-i'} a_\ell \pmod{\mathbf{m}'} \qquad (\ell | N) \\ \langle p \rangle_N^* \sigma &\equiv c^{-1} p^{-i} a_p \pmod{\mathbf{m}'} \\ \langle d \rangle_{pN}^* &\equiv \chi^{b'+2} \epsilon(d) \pmod{\mathbf{m}'} \quad (d \in (\mathbf{Z}/pN\mathbf{Z})^\times). \end{aligned}$$

Using Theorem 4.1 and Corollary 2.4 again ("in reverse") we find a cusp form $g = \sum c_n q^n$ of weight 2, level pN, and character $\chi^b\delta$ for some $\delta \equiv \epsilon \pmod{\wp}$. It has slope 0, it is old at p if $b = 0$, and the displayed congruences imply that

$$\begin{aligned} a_\ell &\cong \ell^i c_\ell \pmod{\wp} \quad \text{for all } \ell \neq p \\ p^{-i} a_p &\equiv c c_p \pmod{\wp} \\ \epsilon(d) &\equiv \delta(d) \pmod{\wp} \quad \text{for all } d \in \mathbf{Z}. \end{aligned}$$

If the ideal $\mathbf{m}'$ lies in one of the other two rings $\mathbf{T}_{\text{Eis}}(b' + 2)$ or $\mathbf{T}_{\log}(0, b', 0)$ we proceed analogously, using 4.5 in place of 4.1 and 2.5 in the Eisenstein case.

This proves that given f of weight $k + 2$ and slope i we get the desired g of weight 2. Conversely, given a normalized eigenform g of weight 2 as in part a), we reverse the above argument and find a form f of weight $k+2$ satisfying the desired congruences. This proves case a) of the theorem. Case b) is entirely similar, using 2.5, 4.6, and 5.3 in place of 2.5, 4.1, and 5.1.

Finally, it remains to treat the cases $N \leq 4$. When $p = 3$ we checked the truth of the theorem in Section 1 by exhibiting congruent forms. If $p > 3$ we use a standard invariants argument: choose an auxiliary integer N' such that $N' > 2$, $(N, N') = 1$, and p does not divide the order of $H = \mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z})$. We have a variety $\tilde{X}$ based on the modular curve for simultaneous $\Gamma_1(pN)$ and $\Gamma(N')$ level structures (by our assumptions on N', these are fine moduli spaces) and a projector $\Pi \in \mathbf{Q}[G']$ where $G' = H \times G$ and $G = (\mu_2^k \rtimes (\mathbf{Z}/N\mathbf{Z} \times (\mathbf{Z}/N'\mathbf{Z})^2) \rtimes S_k$. The crystalline cohomology of $(\tilde{X}, \Pi)$ is related to modular forms in $S_{k+2}(\Gamma_1(pN))$ by an analogue of 2.5 and the arguments involving results of [U3] go through *mutatis mutandis*. In the end, we find congruences between forms $f = \sum a_n q^n$ of level pN and weight $k+2$ and forms g of level pN and weight 2.

Note, however, that because we use full level N' structure, we do not have Hecke operators T_ℓ for primes $\ell | N'$ and so *a priori* we do not have congruences between the Fourier coefficients indexed by primes $\ell | N'$. But the coefficients for these ℓ are determined by the other coefficients, via the associated Galois representation modulo p. In the case where 1.2 holds for $(n, N') = 1$ with $g = \sum b_n q^n$ we have

$$b_\ell \equiv \mathrm{Tr}\,(\rho_g(\mathrm{Fr}_\ell)) = \mathrm{Tr}\,(\rho_f \otimes \chi^{1-i}(\mathrm{Fr}_\ell)) \equiv a_\ell \ell^{1-i} \pmod{\wp}$$

for each prime $\ell | N'$ and in the case where 1.3 holds for $(n, N') = 1$ with $h = \sum c_n q^n$ we have

$$c_\ell \equiv \mathrm{Tr}\,(\rho_h(\mathrm{Fr}_\ell)) = \mathrm{Tr}\,(\rho_f \otimes \chi^{-i}(\mathrm{Fr}_\ell)) \equiv a_\ell \ell^{-i} \pmod{\wp}$$

for each prime $\ell | N'$. Thus we obtain the desired congruences for all coeffieicents.

This completes the proof of Theorem 1.1 in all cases. $\square$

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