On the Fourier coefficients of modular forms II

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In this paper we continue our study of the p-adic valuations of eigenvalues of the Hecke operator U_p. In [U2], we proved that the Newton polygon of the characteristic polynomial of U_p on certain spaces of cusp forms of level divisible by p is bounded below by an explicitly given (Hodge) polygon. Here, we investigate the extent to which this result is sharp. In particular, we want to find the highest polygon with integer slopes which lies below the Newton polygon of U_p (its "contact polygon"). Knowledge of this polygon yields non-trivial upper bounds on dimensions of spaces of forms defined by slope conditions. In some cases, we can go much further, giving *formulae* for the dimensions of spaces of forms of certain slopes in terms of forms of weight 2. This can be viewed as a generalization to higher slope of well-known results of Hida [H] on the number of ordinary eigenforms, i.e., eigenforms of slope 0. What underlies all of our results is very fine information on a certain crystalline cohomology group associated to modular forms. In a future paper we will exploit this information further and prove congruences between modular forms of various weights and slopes. This allows us to get good control on the Galois representations modulo p attached to certain forms of weight > 2.

The first section of the paper gives our results on modular forms and then in Section 2 we give the cohomological results underlying these theorems. The main results on modular forms are 1.4-1.8 and the most important technical result is Theorem 2.4. There is a summary of the rest of the paper at the end of Section 2.

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1. Slope decompositions Fix a positive integer M and a non-negative integer k and let $S_{k+2}(\Gamma_1(M))$ be the complex vector space of cusp forms of weight $k+2$ for the congruence subgroup $\Gamma_1(M)$ of $\mathrm{SL}_2(\mathbf{Z})$. Let $S_{k+2}(\Gamma_1(M);\mathbf{Q})$ be the $\mathbf{Q}$-vector space of forms all of whose Fourier coefficients at the standard cusp ∞ are rational numbers; according to a theorem of Shimura ([Sh], 3.52) this is a $\mathbf{Q}$-structure on $S_{k+2}(\Gamma_1(M))$. For any commutative $\mathbf{Q}$-algebra R define

$$S_{k+2}(\Gamma_1(M);R) = S_{k+2}(\Gamma_1(M);\mathbf{Q}) \otimes_{\mathbf{Q}} R.$$

This space carries an action of Hecke operators T_ℓ for all primes $\ell \nmid M$, U_ℓ for $\ell | M$, $\langle d\rangle_M$ for $d \in (\mathbf{Z}/M\mathbf{Z})^\times$; if R contains an M-th root of unity ζ, we also have an action of the operator w_ζ ([Sh], Ch. 3 or [Mi], 4.5 and 4.6). If $M = M_1M_2$ with $(M_1, M_2) = 1$, then we have operators $\langle d\rangle_{M_i}$ for $d \in (\mathbf{Z}/M_i\mathbf{Z})^\times$ and $\langle d\rangle_M = \langle d\rangle_{M_1}\langle d\rangle_{M_2}$. If R contains the $\phi(M_1)$-th roots of unity $\mu_{\phi(M_1)}$ (where ϕ is Euler's function), then we have a direct sum decomposition

$$S_{k+2}(\Gamma_1(M);R) \cong \bigoplus_{\psi:(\mathbf{Z}/M_1\mathbf{Z})^\times \to R} S_{k+2}(\Gamma_1(M);R)(\psi)$$

where $f \in S_{k+2}(\Gamma_1(M);R)$ lies in $S_{k+2}(\Gamma_1(M);R)(\psi)$ if and only if $\langle d\rangle_{M_1} f = \psi(d) f$ for all $d \in (\mathbf{Z}/M_1\mathbf{Z})^\times$.

Now fix a prime number p, a non-negative integer k, and an integer N prime to p. Let $R = \mathbf{Q}_p$ be the p-adic numbers. We can apply the above constructions with $M_1 = p$, $M_2 = N$, and we get a direct sum decomposition:

$$S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p) \cong \bigoplus_{a=0}^{p-2} S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)$$

where $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p$ is the Teichmüller character (characterized by $\chi(d) \equiv d \pmod{p}$). (Note that we are not decomposing for characters modulo N.)

We want to consider the action of U_p on $S = S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)$ when $0 < a < p-1$. As is well-known, this action is semi-simple (since S is then a space of forms new at p) and its eigenvalues α are algebraic integers all of whose complex embeddings satisfy $\alpha\overline{\alpha} = p^{(k+1)}$ ([Mi], 4.6.17). Let v be the valuation of $\overline{\mathbf{Q}_p}$ normalized so that $v(p) = 1$. Then if α is an eigenvalue of U_p on S, we have $0 \leq v(\alpha) \leq k+1$. We define $v(\alpha)$ to be the *slope* of the eigenvalue, and if $f \in S \otimes \overline{\mathbf{Q}_p}$ is an eigenvector for U_p with eigenvalue α, we will also call $v(\alpha)$ the slope of f. If f is an eigenform with slope i then for all $\zeta \in \mu_{pN}$, $w_\zeta f$ is an eigenform with slope $k+1-i$.

We will also have occasion to consider the action of U_p on $S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)^{p-\mathrm{old}}$, the space of forms which are old at p; this is isomorphic to the sum of two copies of $S_{k+2}(\Gamma_1(N); \mathbf{Q}_p)$ embedded via the standard degeneracy maps. Although this action may not be semi-simple, the eigenvalues of U_p are again algebraic integers α which have absolute value $p^{(k+1)/2}$ in every complex embedding. Thus the slopes of forms in $S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)^{p-\mathrm{old}}$ are also in the interval $[0, k+1]$. We also note that

$$\det\left(1 - U_p T | S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)^{p-\mathrm{old}}\right) = \det\left(1 - T_p T + \langle p\rangle_N p^{k+1} T^2 | S_{k+2}(\Gamma_1(N); \mathbf{Q}_p)\right).$$

To ease notations, we define

$$S = S(k,b) = \begin{cases} S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^b) & \text{if } b \not\equiv 0 \pmod{p-1} \\ S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)^{p-\mathrm{old}} & \text{if } b \equiv 0 \pmod{p-1}. \end{cases} \tag{1.1}$$

Elementary linear algebra gives a unique decomposition

$$S \cong \bigoplus_{\lambda} S_\lambda$$

(compatible in an evident sense with the Hecke operators) such that all of the eigenvalues of U_p on S_λ have slope λ. More generally, for I an interval of real numbers, let

$$S_I = \bigoplus_{\lambda \in I} S_\lambda.$$

We want to study the dimensions of the S_I for certain intervals I, especially those of the form $[i]$ or $(i, i+1)$ where i is an integer in the range $0 \le i \le k+1$. In particular, we want to relate these dimensions for different values of k and a.

A convenient way to package information about dimensions of subspaces defined by slopes is via Newton polygons. Recall that if

$$P(T) = 1 + a_1 T + \cdots + a_d T^d = \prod_{i=1}^{d} (1 - \alpha_i T)$$

is a polynomial with $\overline{\mathbf{Q}_p}$ coefficients and with the factors α_i ordered so that $v(\alpha_i) \le v(\alpha_{i+1})$, then the *Newton polygon* of P with respect to v is defined to be the graph of the continuous, piecewise linear, convex function f on $[0, d]$ with $f(0) = 0$ and $f'(x) = v(\alpha_i)$ for all $x \in (i-1, i)$. This polygon can be seen to be part of the boundary of the convex hull in the plane of the points $(0,0)$ and $(i, v(a_i))$ for $i = 1, \ldots, d$. In particular, if the $v(a_i) \in \mathbf{Z}$, then the breakpoints of the Newton polygon (i.e., the points (x, y) on the polygon where f changes slope) have integer coordinates.

Given non-negative real numbers $l_0, \dots, l_n$, define the associated *Hodge polygon* to be the graph of the continuous, piecewise linear, convex function f on $[0, l_0 + \cdots + l_n]$ with $f(0) = 0$ and $f'(x) = i$ for all $x \in (l_0 + \cdots + l_{i-1}, l_0 + \cdots + l_i)$. In [U2], we proved that the Newton polygon of the polynomial

$$P(k,a) = \det\left(1 - U_p T | S(k,a)\right)$$

is bounded below by an explicit Hodge polygon. Let g be the genus of $X_1(N)$, c the number of cusps on this curve, and set $w = g - 1 + c/2$. Then main theorem of [U2] says that when $N \geq 5$ and $0 < a < p-1$, the Newton polygon of $P(k,a)$ lies on or above the Hodge polygon attached to the integers

$$\begin{aligned} l_0 &= (k+a+1)w - c/2 \\ l_1 = \cdots = l_k &= (p-1)w \\ l_{k+1} &= (k+p-a)w - c/2 \end{aligned} \tag{1.2}$$

and the two polygons have the same endpoints. (We also treated the cases where $N \leq 4$ if $p > 3$ in [U2], Theorem 7.1, but the formulae are much more complicated.) Since the middle Hodge numbers are not zero, this result shows that the eigenvalues of U_p are more divisible by p than one might *a priori* expect.

It is natural to ask how sharp this result is, so let us consider the highest Hodge polygon (i.e., polygon with integral slopes) lying on or below the Newton polygon of $P(k,a)$. Generally, given any Newton polygon, define its associated *contact polygon* as the highest Hodge polygon lying on or below it and having the same endpoints. By definition, the Newton and contact polygons meet at some point on every edge of the latter, and any edge of the Newton polygon with integer slope is contained in the corresponding edge of the contact polygon. It is not hard to check the following formula for the lengths of the sides of the contact polygon: if $\lambda_1, \cdots, \lambda_d$ are the slopes of the Newton polygon (with multiplicities) and if

$$m_i = \sum_{v(\alpha_j)\in(i-1,i)} (v(\alpha_j) - (i-1)) + \sum_{v(\alpha_j)=i} 1 + \sum_{v(\alpha_j)\in(i,i+1)} (i+1-v(\alpha_j)),$$

then the contact polygon is the Hodge polygon attached to the numbers $m_0, m_1, \ldots$. We note that $m_i = 0$ if and only if none of the $v(\alpha_j)$ lie in the interval $(i-1, i+1)$. The polygon attached to the m_i was called the Hodge-Newton polygon by Crew and Ekedahl and the slope polygon by Illusie, but both of these terminologies seem to cause confusion. I propose to call it the contact polygon (and the m_i contact numbers) in view of its contact properties with respect to the Newton polygon. For more discussion on this polygon and its applications to modular forms, see [U3].

Evidently the difference between the Hodge polygon of [U2] and the contact polygon of $P(k,a)$ is a measure of the sharpness of the result of [U2]. To measure the difference, let $t^i = t^i(k,a)$ $(i = 1, \ldots, k)$ be the number of units the slope i edge of the Hodge polygon should be raised so that it meets the slope i edge of the contact polygon (or equivalently, so that it meets the Newton polygon). Since the Hodge polygon of [U2] and the Newton polygon of $P(k,a)$ have the same endpoints, it follows that $t^0 = t^{k+1} = 0$. With the convention that $t^i = 0$ for $i < 0$ or $i > k+1$, the t^i are determined by the relations

$$m_i = l_i - t^{i-1} + 2t^i - t^{i+1} \tag{1.3}$$

for $i = 0, \ldots, k+1$. Our first result determines when the t^i vanish. To avoid constant repetition, throughout the paper we assume the following.

Standing Hypotheses. *Let p be an odd prime number, N an integer prime to p, k an integer with $0 \leq k < p$, and a an integer with $0 < a < p-1$. If $N \leq 2$, assume that $k \equiv a \pmod 2$. After Section 1, we assume also that $N \geq 5$.*

Theorem 1.4. *Let i be an integer with $1 \leq i \leq k$.*

a) If $i \leq a$ and $k+1-i \leq p-1-a$ then $t^i(k,a) = 0$.

b) If $i = a+1$ then $t^i(k,a) = 0$ if and only if $S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{k+1-i})_{(0,1)} = 0$. If $k+1-i = p-a$ then $t^i(k,a) = 0$ if and only if $S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{-i})_{(0,1)} = 0$.

c) If $N \geq 5$ and if $i > a+1$ or $k+1-i > p-a$ then $t^i(k,a) > 0$. For any N, if $i = a+2$ we have $t^i(k,a) \geq l_{i-1}$ and if $i = k+a-p$ we have $t^i(k,a) \geq l_{i+1}$. (Here the l_i are defined by 1.2 if $N \geq 5$ and by [U2], 7.1 if $N \leq 4$.)

The theorem says that the Newton polygon attached to modular forms touches the Hodge polygon attached to the l_i at some point on every edge of middle slope, where the definition of "middle" depends on k, a, and i. Figure 1 below may be helpful in organizing the hypotheses. In it, k is fixed and there is one box for each pair (i,a). For the shaded boxes $t > 0$, for the clear boxes $t = 0$, while for the boxes with stars, the behaviour of t is "arithmetical": it depends on eigenvalues of modular forms of weight 2. Note that there are no $*$'s if $k \leq 1$ and no shaded region if $k \leq 2$; the shaded region becomes larger as k increases until there are no clear squares when $k = p-1$.

The theorem together with obvious properties of the contact polygon give upper bounds on the dimensions of spaces of modular forms with certain slopes. For example, if (i,a) is in the clear region, then dim $S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)_{[i]} \leq (p-1)w$, since this is the maximum possible length of the slope i edge of the contact polygon.

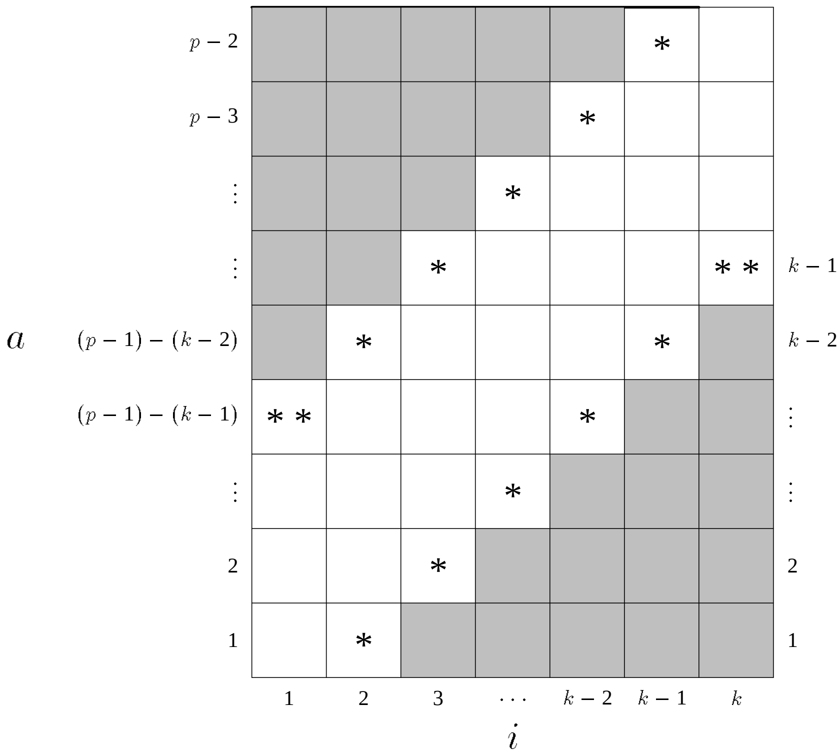

Figure 1

We can obtain much more precise information in the clear region, relating dimensions of spaces of forms of weight $k+2$ and weight 2. Recall (from 1.1) the notation $S(k,b)$ for spaces of modular forms of level pN. To put our result in context, we first recall that by results of Hida [H], for all $k > 0$ we have

$$\begin{aligned}\dim\, S_{k+2}(\Gamma_1(pN);\mathbf{Q}_p)(\chi^a)_{[0]} &= \dim\, S_2(\Gamma_1(pN);\mathbf{Q}_p)(\chi^{a+k})_{[0]} \\ &= \dim\, S(0,a+k)_{[0]} + \begin{cases} 0 & \text{if } a+k \not\equiv 0 \pmod{p-1} \\ S-1 & \text{if } a+k \equiv 0 \pmod{p-1}\end{cases}\end{aligned}$$

(where S is the number of supersingular points on $X_1(N)$ in characteristic p) and

$$\begin{aligned}\dim\, S_{k+2}(\Gamma_1(pN);\mathbf{Q}_p)(\chi^a)_{[k+1]} &= \dim\, S_2(\Gamma_1(pN);\mathbf{Q}_p)(\chi^{a-k})_{[1]} \\ &= \dim\, S(0,a-k)_{[1]}.\end{aligned}$$

Thus, the dimensions of spaces of forms of weight $k+2$ and minimal or maximal slope (i.e., slope 0 or $k+1$) are determined by slopes in weight 2. We can generalize this to forms of slope i and character χ^a when (a,i) lies in the clear part of Figure 1.

Theorem 1.5. *Let i be an integer $1 \leq i \leq k$ such that $i \leq a$ and $k+1-i \leq p-1-a$ and set $b = a+k-2i$. Let c be the number of cusps on the modular curve $X_1(N)$. Then we have*

$$\begin{aligned}\dim\, S_{k+2}(\Gamma_1(pN);\mathbf{Q}_p)(\chi^a)_{[i]} = &\dim\, S(0,b+2)_{[1]} \\ &+ \begin{cases} c & \text{if } b \not\equiv 0,-2 \pmod{p-1} \\ c-1 & \text{if } p>3 \text{ and } b \equiv 0 \text{ or } -2 \pmod{p-1} \\ c-2 & \text{if } p=3 \text{ and } b \equiv 0 \equiv -2 \pmod{p-1} \end{cases} \\ &+ \dim\, S(0,b)_{[0]}.\end{aligned}$$

Remarks: 1) Implicit in this result are upper and lower bounds on the dimensions of the spaces $S_{k+2}(\Gamma_1(pN);\mathbf{Q}_p)(\chi^a)_{[i]}$. If $N \geq 5$, the upper bound is equal to $(p-1)w$, which is the number of supersingular points on $X_1(N)$ in characteristic p. The lower bound (c, $c-1$, or $c-2$) seems remarkable. Is there a distinguished subspace of $S_{k+2}(\Gamma_1(pN);\mathbf{Q}_p)(\chi^a)_{[i]}$ of this dimension?

2) Using a theorem of Hida [H], the right hand side of this equation can be rewritten in terms of forms of higher weight but level N. Precisely, if $0 \leq b \leq p-2$, one has

$$\dim\, S(0,b)_{[0]} = \dim\, S_{b+2}(\Gamma_1(N);\mathbf{Q}_p)_{[0]}$$

and if $-1 \leq b \leq p-3$, one has

$$\dim\, S(0,b+2)_{[1]} = \dim\, S_{p-1-b}(\Gamma_1(N);\mathbf{Q}_p)_{[0]}.$$

(Here the slope decomposition on the right hand side is for the operator T_p instead of U_p.)

We also have a result for slopes in an open interval $(i,i+1)$ when both (a,i) and $(a,i+1)$ lie in the clear region of Figure 1.

Theorem 1.6. *Let i be an integer $0 \leq i \leq k$ and assume that either: $i+1 \leq a$ and $k+1-i \leq p-1-a$; or $i=0$ and $a \leq p-1-k$; or $i=k$ and $a \geq k$. Set $b = a+k-2i$. Then*

$$\dim S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)_{(i,i+1)} = \dim S(0,b)_{(0,1)}.$$

Remark: It is natural to ask whether a similar statement holds where the interval $(i, i+1)$ is replaced by a single slope λ and $(0,1)$ is replaced by $\lambda - i$.

Here is an easy consequence of Theorem 1.4 in the $*$ region:

Corollary 1.7. *Let i be an integer with $2 \leq i \leq k-1$. Then*

$$S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{k+1-i})_{(0,1)} = 0 \Rightarrow S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{i-1})_{(i-1,i+1)} = 0$$

and

$$S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{-i})_{(0,1)} = 0 \Rightarrow S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{i-k})_{(i-1,i+1)} = 0.$$

Proof: Under the hypotheses, the combination of formula 1.3 for the contact numbers, formula 1.2 for the Hodge numbers, and Theorem 1.4 implies that $m_i \leq 0$; but by definition $m_i \geq 0$, so $m_i = 0$. As noted above, this implies there are no slopes in $(i-1, i+1)$. $\square$

The $**$ boxes behave slightly differently:

Theorem 1.8. *In addition to the standing hypotheses, suppose that $k > 1$ and $N \geq 5$.*

a) (The case $a = (p-1)-(k-1)$ and $i = 1$.)

$$S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{-1})_{(0,1)} = 0 \Rightarrow \begin{cases} S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{1-k})_{[1,2)} = 0 \\ \text{and} \\ \dim\, S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{1-k})_{(0,1)} = 2(p-1)w. \end{cases}$$

b) (The case $a = k-1$ and $i = k$.)

$$S_2(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{1})_{(0,1)} = 0 \Rightarrow \begin{cases} S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{k-1})_{(k-1,k]} = 0 \\ \text{and} \\ \dim\, S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^{k-1})_{(k,k+1)} = 2(p-1)w. \end{cases}$$

Remarks: 1) It would be interesting to construct explicitly some forms with slope in $(0,1)$ or $(k,k+1)$ which would "explain" this result. We remark that if $N \geq 5$, $(p-1)w$ is the number of supersingular points on $X_1(N)$ in characteristic p.

2) A version of this theorem also holds for $3 \leq N \leq 4$ as long as $p > 3$. (If $N = 1$ or 2 and a is as in the statement of the theorem, then $k \not\equiv a \pmod 2$ and so $S(k,a) = 0$.) Writing $l(k,a,i)$ for the integers defined in [U2], Theorem 7.1, the theorem holds with $2(p-1)w$ replaced with $l(k,p-k,1)+l(k,p-k,0)-l(0,1,0)$ in case a) and with $l(k,k-1,k)+l(k,k-1,k+1)-l(0,p-2,k+1)$ in case b).

To finish this introduction, let us give two examples. First consider the case where $p=5$ and $N=1$. We have $S_2(\Gamma_1(5);\mathbf{Q}_5)=0$ and so the Theorems 1.5-8 show that there are no forms of weights 3 or 4, and that in weight 5, there are no forms with character χ^1 and slopes in $[0,2)$ or with character χ^{-1} and slopes in $(2,4]$. Also, Theorem 1.7 shows that in weight 5, there are no forms with character χ^1 or χ^{-1} and slopes in $(1,3)$. Theorem 7.1 of [U2] shows that there are no forms of weight 5, character χ^1 and slope in $(3,4]$ and none of character χ^{-1} and slope in $[0,1)$ (because the slope 0 or slope 4 edge of the Hodge polygon has length 0). Thus the only allowable slope in weight 5 and character χ^1 is 3 and the only allowable slope in weight 5 and character χ^{-1} is 1 and there should be exactly one eigenform with each of these two slopes. In weight 6, the only relevant character is χ^2 and by Theorem 1.7, no slopes in $(1,4)$ occur. Again, [U2] rules out slopes in $[0,1)$ and $(4,5]$ so the only allowable slopes are 1 and 4. Since $S_6(\Gamma_1(5);\overline{\mathbf{Q}}_5)(\chi^2)$ is w-invariant, these two slopes must occur with the same multiplicity, and since the dimension of the space is 2, both should occur exactly once. Warren Staley, a graduate student at the University of Arizona, has kindly supplied me with data confirming that these predictions are accurate.

Thus, in this very simple case, the slope structure of spaces of forms of small enough weight is determined *a priori* by that in weight 2.

For a second example, let $p = 3$, $N = 5$, $k = 2$, and $a = 1$. The genus of $X_1(5)$ is 1 and it has 4 cusps; it has 2 supersingular points in characteristic 3. Also, $S(0,b) = 0$ for all b. Using well-known results of Hida on ordinary forms, we find that there are no forms with slope 0 and (using the w-operator) there are none of slope 3. Theorem 1.8 implies that there are also no forms with slopes in $[1,2]$, and that the spaces of forms of slopes in $(0,1)$ and $(2,3)$ are both 4-dimensional. Computation with Pari reveals that in fact there are 4 forms of slope 1/2 and 4 forms with slope 5/2.

2. Slopes and cohomology All the results of the previous section follow from a comparison between modular forms and cohomology proved in [U2] and a calculation of certain cohomology groups of logarithmic or exact differentials. In this section we introduce these sheaves of differentials and explain the relation with modular forms. We will state three theorems (2.3, 2.4, and 2.6) and explain how all the results of Section 1 follow from them; the proof of these three theorems will take up the remainder of the paper.

We retain the odd prime p, and the integers N, k, and a with $p \not| N$, $0 \le k < p$, and $0 < a < p-1$; from now on we assume $N \ge 5$. Recall the Igusa curve $I = Ig(pN)$ over $\mathbf{F}_p$ and its universal curve $\mathcal{E} \to I$. As in [U2], we have the k-fold fibre product $\mathcal{E} \times_I \cdots \times_I \mathcal{E}$ and its canonical desingularization X, both of which are acted on by $G = (\mathbf{Z}/N\mathbf{Z} \rtimes \mu_2)^k \rtimes S_k$ and by $(\mathbf{Z}/p\mathbf{Z})^\times$. The map $f : X \to I$ is equivariant for the actions of G and $(\mathbf{Z}/p\mathbf{Z})^\times$, where G acts trivially on I and $(\mathbf{Z}/p\mathbf{Z})^\times$ acts via the $\langle d\rangle_p$. From now on we write $\langle d\rangle$ for $\langle d\rangle_p$. (In [U2], we wrote X for the k-fold fiber product and $\tilde{X}$ for the desingularization.

As we will use only the latter variety here, we change to the simpler notation X for the desingularized k-fold fiber product.)

Let $\epsilon : G \to \{\pm 1\}$ be the character of G which is 1 on the factors $\mathbf{Z}/N\mathbf{Z}$, the identity on the factors μ_2 and the sign character on S_k. Let $\Pi \in \mathbf{Z}_p[G]$ be the projector attached to ϵ and let $\Pi_a \in \mathbf{Z}_p[G \times (\mathbf{Z}/p\mathbf{Z})^\times]$ be the projector attached to the character $\epsilon\chi^a$. Let $\mathbf{F}$ be a perfect field of characteristic p, $W = W(\mathbf{F})$ its ring of Witt vectors, and $\sigma : W \to W$ the automorphism induced by the absolute Frobenius of $\mathbf{F}$. We view X and I as varieties over $\mathbf{F}$ by extension of scalars. As explained in Section 3 of [U2], the groups G and $(\mathbf{Z}/p\mathbf{Z})^\times$ act on the crystalline cohomology groups $H^n_{cris}(X/W)$. We adopt the following notational convention, suggested by the idea that the pairs (X, Π_a) should be thought of as "motives define by p-integral projectors": M and M_a will stand for suitable combinations of X and Π or X and Π_a respectively. For example, $H^j(M, \Omega^i)$ means $\Pi H^j(X, \Omega^i)$ and $H^n_{cris}(M_a/W)$ means $\Pi_a H^n_{cris}(X/W)$.

We proved in [U2] that $H^{k+1}_{cris}(M/W)$ is a free W-module ([U2], Corollary 5.6) and that the characteristic polynomial of the absolute Frobenius on $H^{k+1}_{cris}(M_a/W)$ is equal to

$$P(T) = \det\left(1 - U_p\, T | S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)\right) \tag{2.1}$$

([U2], Corollary 2.2). Thus we can apply the arsenal of crystalline cohomology to analyze these spaces of modular forms. We will prove the assertions of Section 1 by using the deRham-Witt complex and logarithmic differentials to study $H^{k+1}_{cris}(M_a/W)$.

For any smooth, complete, irreducible variety Y over a perfect field $\mathbf{F}$, recall the deRham-Witt sheaves $W_n\Omega^i$ on the étale site of Y, introduced by Illusie in [I1], and further developed in [I-R], [M], and [E]. For each n these fit into a complex

$$\cdots \xrightarrow{d} W_n\Omega^i \xrightarrow{d} W_n\Omega^{i+1} \xrightarrow{d} \cdots$$

and as n varies, there are projections $W_{n+1}\Omega^i \to W_n\Omega^i$ giving pro-sheaves $W\Omega^i$ and a pro-complex $W\Omega^{\cdot}$. We recall also the (pro-) sheaves $ZW\Omega^i$ and $BW\Omega^{i+1}$ which are the kernel and image respectively of the differential $d : W\Omega^i \to W\Omega^{i+1}$, as well as the (pro-) sheaf

$$\mathcal{H}^i W\Omega^{\cdot} = \frac{ZW\Omega^i}{BW\Omega^i}.$$

The $W\Omega^i$ are endowed with (σ and σ^{-1}-linear) endomorphisms F and V such that $FV = p$ and $FdV = d$; there is an endomorphism Φ of the complex $W\Omega^{\cdot}$ which is p^iF on $W\Omega^i$. There are two spectral sequences (the slope and conjugate spectral sequences)

$$H^j(Y, W\Omega^i) \Rightarrow H^{i+j}_{cris}(Y/W) \text{ and } H^i(Y, \mathcal{H}^j W\Omega^{\cdot}) \Rightarrow H^{i+j}_{cris}(Y/W)$$

which give information on slopes of crystalline cohomology. The endomorphism Φ of $W\Omega^{\cdot}$ induces the absolute Frobenius endomorphism on the abutment of the slope spectral sequence. In order to avoid notational confusion, we will write Φ for the absolute Frobenius endomorphism of H_{cris}. We will use [I2] as a highly readable general reference for results about the deRham-Witt sheaves and their cohomology.

Consider the subcomplex of $W_n\Omega^{\cdot}$ whose degree i component is generated additively by sections

$$d\underline{x_1}/\underline{x_1} \wedge \cdots \wedge d\underline{x_i}/\underline{x_i}$$

where $x_j \in \mathcal{O}_Y^{\times}$ and $\underline{x}_j = (x_j, 0, \ldots, 0)$ is the Teichmüller representative of x_j. The degree i component is denoted $W_n\Omega^i_{log}$ in [I-R] and $\nu_n(i)$ in [M], and as n varies, they form a projective system of sheaves with the maps induced from the projections $W_n\Omega^i \to W_{n-1}\Omega^i$. By convention $W_n\Omega^0_{log} = \nu_n(0) = \mathbf{Z}/p^n\mathbf{Z}$. Also, we will write Ω^i_{log} for $W_1\Omega^i_{log}$; it is just the usual sheaf of logarithmic differential i-forms. We have exact sequences

$$0 \to W_m\Omega^i_{log} \to W_{m+n}\Omega^i_{log} \to W_n\Omega^i_{log} \to 0$$

and

$$0 \to W\Omega^i_{log} \xrightarrow{p} W\Omega^i_{log} \to \Omega^i_{log} \to 0.$$

Now consider the presheaf

$$A \mapsto H^j_{ét}(Y \times \text{Spec } A, W_n\Omega^i_{log}) = H^j_{ét}(Y \times \text{Spec } A, \nu_n(i))$$

on the category of perfect $\mathbf{F}$-algebras equiped with the étale topology. One knows that the associated sheaf $\underline{H}^j(Y, W_n\Omega^i_{log})$ is represented by a commutative algebraic perfect group scheme killed by p^n (see [I-R] Ch. 4 or [M] Section 1 for definitions and proofs). If A is an algebraically closed field containing $\mathbf{F}$, then the presheaf and the sheaf have the same value on A, i.e.,

$$H^j(Y \times \text{Spec } A, W_n\Omega^i_{log}) = \underline{H}^j(Y, W_n\Omega^i_{log})(\text{Spec } A).$$

The connected component U^{ij}_n of this representing group is a unipotent algebraic perfect group scheme of finite dimension. A fundamental duality theorem of Milne ([M], 1.4) says that

$$U^{ij}_n \cong \text{Ext}^1(U^{d-i,d+1-j}_n, \mathbf{Q}_p/\mathbf{Z}_p)$$

where d is the dimension of Y; in particular, the groups U^{ij}_n and $U^{d-i,d+1-j}_n$ have the same dimension. Finally, $U^{ij} = \varprojlim_n U^{ij}_n$ is also finite dimensional ([M], 1.8, 3.1 and [I-R], IV.3). We let T^{ij} denote the dimension of $U^{i+1,j}$. (The reason for this funny shift is that the T^{ij} were originally defined in terms of some other cohomological object and were later proved to be equal to the number defined here.) If Y has dimension n, one knows that $T^{ij} = 0$ if $i \geq n-1$ or $j \leq 1$; in particular, all $T^{ij} = 0$ when Y is a curve. As remarked by Milne ([M] 3.5), the T^{ij} are in general rather difficult to calculate.

These integers are closely related to slopes of Frobenius on crystalline cohomology via a result of Ekedahl. Suppose that the Hodge to deRham spectral sequence $H^j(Y, \Omega^i) \Rightarrow H^{i+j}_{dR}(Y)$ degenerates at E_1 and that the crystalline cohomology groups $H^n_{cris}(Y/W)$ are torsion-free. Fix an integer n and let $m_0, \ldots, m_n$ be the contact numbers attached to the characteristic polynomial of Frobenius on $H^n_{cris}(Y/W)$. Let h^{ij} be the Hodge numbers $h^{ij} = \dim_{\mathbf{F}} H^j(Y, \Omega^i)$ and let T^{ij} be the dimensions of the group schemes $U^{i+1,j}$ attached to Y as above. Then Ekedahl's result is

$$m_i = h^{i,n-i} - T^{i,n-i} + 2T^{i-1,n+1-i} - T^{i-2,n+2-i}. \tag{2.2}$$

(This is a combination of [I2], 6.3.11 and 6.3.1.)

We want to apply this result to the M_a. As explained in [U2], the groups G and $(\mathbf{Z}/p\mathbf{Z})^\times$ act on the deRham-Witt cohomology groups $H^j(X, W\Omega^i)$ and $H^i(X, \mathcal{H}^j W\Omega^\cdot)$, and on the spectral sequences converging to $H^n_{cris}(X/W)$. Thus we can apply the projectors Π and Π_a to these objects. Moreover, G acts on X covering the trivial action on I, so it makes sense to apply Π to certain sheaves on I, such as $R^j f_* \Omega^i_X$, $R^j f_* Z^i_X$, $R^j f_* B^i_X$, and $R^j f_* \Omega^i_{X/I}$. We extend our basic notational convention to cover these groups and sheaves: thus $H^j(M_a, W\Omega^i)$ means $\Pi_a H^j(X, W\Omega^i)$ and $R^j f_* \Omega^i_M$ means $\Pi R^j f_* \Omega^i_X$.

More generally, we can define objects $R\Gamma(M_a, W\Omega^\cdot)$ in the derived category $D^b_c(R)$ of [I-R] (here R is the Dieudonné-Raynaud ring $W_\sigma[F,V][d]$); these are direct factors of $R\Gamma(X, W\Omega^\cdot)$ and their cohomology groups are the R-modules

$$\cdots \to H^j(M_a, W\Omega^i) \xrightarrow{d} H^j(M_a, W\Omega^{i+1}) \to \cdots.$$

Indeed, the canonical flasque resolutions $W\Omega^i \to L_0^i \to L_1^i \to \cdots$ of the $W\Omega^i$ fit together to form a flasque resolution $W\Omega^\cdot \to L_0^\cdot \to L_1^\cdot \to \cdots$ of $W\Omega^\cdot$ by R-modules $L_j^\cdot$. Moreover, from the definition, it is clear that $G \times (\mathbf{Z}/p\mathbf{Z})^\times$ acts on the R-modules $L_j^\cdot$. Applying the functor $R\Gamma(X, -)$ followed by Π_a, we get a complex of R-modules whose cohomology groups are the desired R-modules. This complex is clearly coherent (in the sense of [I2], 2.4.6) since it is a direct factor of $R\Gamma(X, W\Omega^\cdot)$, which is itself coherent. Thus we can apply many of the results of [E], which are formulated for an arbitrary object of $D_c^b(R)$, to the $\Pi_a R\Gamma(X, W\Omega^\cdot)$.

Now apply these constructions to M_a: let U^{ij} be the inverse limit of the connected components of the groups representing $\underline{H}^j(M_a, W_n\Omega^i_{log}) = \Pi_a \underline{H}^j(X, W_n\Omega^i_{log})$ and let T^{ij} be the corresponding dimensions. We proved in [U2], 5.6 that the Hodge to deRham spectral sequence of M degenerates and that $H_{cris}(M) = H^{k+1}_{cris}(M)$ is torsion free, and, in 5.8, that the Hodge numbers $h^{i,k+1-i}$ of M_a are the integers l_i of Section 1. Thus the numbers t^i of Section 1 are given by the T^{ij}: comparing 1.2 and 2.2 we have $t^i = T^{i-1,k+2-i}$. In particular, we can determine the contact numbers m_i attached to modular forms cohomologically.

Here is the first main theorem on the T^{ij}. Part a) (which is easy) will be proven in Section 4 and part b) will be proven in Section 6.

Theorem 2.3. *Assume the standing hypotheses and let $a' = p - 1 - a$.*

a) We have

$$H^j(M, W\Omega^i_{log}) = H^j(M, \Omega^i_{log}) = 0$$

unless $i + j = k + 1$ or $k + 2$; moreover, $T^{ij} = 0$ unless $i + j = k + 1$.

b) For $1 \leq i \leq k$,

$$T^{i-1,k+2-i} \geq \begin{cases} 0 & \text{if } k-a' \leq i \\ (p-1)w(k-a'-i) & \text{if } (k-a')/2 \leq i \leq k-a' \\ (p-1)w\, i & \text{if } i \leq (k-a')/2 \end{cases}$$
$$+ \begin{cases} 0 & \text{if } i \leq a+1 \\ (p-1)w(i-a-1) & \text{if } a+1 \leq i \leq (k+a+2)/2 \\ (p-1)w(k+1-i) & \text{if } (k+a+2)/2 \leq i \end{cases}.$$

This lower bound is non-zero if and only if $a+1 < i$ or $i < k-a'$.

For $H = H^{k+1}_{cris}(M_a/W)$ or $H = H^1_{cris}(I/W)(\chi^b)$ and for any interval $J \subseteq [0, k+1]$, define

$$H_J = (H \otimes \mathbf{Q})_J \cap H$$

(where the slope decomposition of crystalline cohomology is with respect to the action of Frobenius and the canonical valuation of $W \otimes \mathbf{Q}$). The following result, besides giving the vanishing of certain T^{ij}, will be crucial in relating forms of higher weight to forms of weight 2. We will establish it in Section 8. In the statement, C is the reduced divisor of cusps on I, dq/q is a certain logarithmic differential on I which is defined in Section 3, and $\mathcal{C}$ and F are the Cartier operator and Frobenius respectively.

Theorem 2.4. *Assume the standing hypotheses and let $a' = p-1-a$.*

a) Fix an integer i with $1 \leq i \leq k$ and suppose $i \leq a$ and $k+1-i \leq p-1-a$. Let $b = a+k-2i$. Then $H^{k+1-i}(M_a, \Omega^i_{log})$ is finite. If the ground field $\mathbf{F}$ is algebraically closed then $H^{k+1-i}(M_a, \Omega^i_{log})$ has a three step filtration whose graded pieces are isomorphic to

$$H^0(I, \Omega^1_{log})(\chi^{b+2}), \qquad \left(\frac{H^0(I, \Omega^1(C))}{\mathbf{F}dq/q + H^0(I, \Omega^1)}(\chi^{b+2})\right)^{\mathcal{C}=1}, \text{ and } \qquad H^1(I, \Omega^0_{log})(\chi^b).$$

Moreover, $T^{i-1,k+2-i} = 0$.

b) Fix an integer i with $1 \leq i \leq k$ and let $b = a+k-2i$. Suppose $\mathbf{F}$ is algebraically closed.

If $i = a+1$ then

$$H^{k+1-i}(M_a, \Omega^i_{log}) \cong H^0(I, \Omega^1)(\chi^{b+2})^{\mathcal{C}=0} \cong H^0(I, B^1_I)(\chi^{b+2})$$

and $T^{i-1,k+2-i} = 0$ if and only if $H^0(I, B^1_I)(\chi^{b+2}) = 0$, which holds if and only if $H^1_{cris}(I/W)(\chi^{b+2})_{(0,1)} = 0$. If $k+1-i = p-a$ then

$$H^{k+1-i}(M_a, \Omega^i_{log}) \cong H^1(I, \mathcal{O})(\chi^b)^{F=0} \cong H^0(I, B^1_I)(\chi^b)$$

and $T^{i-1,k+2-i} = 0$ if and only if $H^0(I, B^1_I)(\chi^b) = 0$ if and only if $H^1_{cris}(I/W)(\chi^b)_{(0,1)} = 0$.

In view of the fact that $t^i = T^{i-1,k+2-i}$, Theorem 1.4 follows immediately from Theorems 2.3 and 2.4.

The vanishing of T's recorded in 2.4 and results of Illusie and Raynaud [I-R] allow one to determine the ranks of certain pieces of $H^{k+1}_{cris}(M_a/W)$ defined by slopes. Here is the cohomological result underlying Theorem 1.5:

Corollary 2.5. *Fix an integer $1 \le i \le k$ and let $b = a+k-2i$.*

a) If $i \le a$ or $k+1-i \le p-1-a$ then we have

$$\begin{aligned}\mathrm{Rk}_W\, H^{k+1}_{cris}(M_a/W)_{[i]} = {} & \mathrm{Rk}_W\, H^1_{cris}(I/W)(\chi^{b+2})_{[1]} \\ & + \begin{cases} c & \text{if } b \not\equiv 0, -2 \pmod{p-1} \\ c-1 & \text{if } p>3 \text{ and } b \equiv 0, -2 \pmod{p-1} \\ c-2 & \text{if } p=3 \text{ and } b \equiv 0 \equiv -2 \pmod{p-1} \end{cases} \\ & + \mathrm{Rk}_W\, H^1_{cris}(I/W)(\chi^b)_{[0]}.\end{aligned}$$

b) If $i = a+1$ then

$$H^1_{cris}(I/W)(\chi^{b+2})_{(0,1)} = 0 \Rightarrow H^{k+1}_{cris}(M_a/W)_{[i]} = 0$$

and if $k+1-i = p-a$ then

$$H^1_{cris}(I/W)(\chi^b)_{(0,1)} = 0 \Rightarrow H^{k+1}_{cris}(M_a/W)_{[i]} = 0.$$

Proof: The argument of [I-R] IV.4.5 shows that if $T^{i-1,k+2-i} = T^{i-1,k+3-i} = 0$, then $H^{k+1}_{cris}(M_a/W)_{[i]}$ is a direct factor of $H^{k+1}_{cris}(M_a/W)$ as F-crystal and the slope and conjugate spectral sequences define an isomorphism

$$H^{k+1}_{cris}(M_a/W)_{[i]} \cong H^{k+1-i}(M_a, ZW\Omega^i).$$

Since neither side of the equality to be proved depends on the ground field $\mathbf{F}$, we can assume it is algebraically closed. Then using [I-R] IV.3, we have a canonical $\mathbf{Z}_p$-structure on $H^{k+1-i}(M_a, ZW\Omega^i)$:

$$H^{k+1-i}(M_a, ZW\Omega^i)^{F=1} \otimes_{\mathbf{Z}_p} W \cong H^{k+1-i}(M_a, ZW\Omega^i)$$

and there is an isomorphism

$$H^{k+1-i}(M_a, ZW\Omega^i)^{F=1} \cong H^{k+1-i}(M_a, W\Omega^i_{log}).$$

This shows that $H^{k+1-i}(M_a, W\Omega^i_{log})$ is torsion free; a similar argument gives the vanishing of $H^{k+2-i}(M_a, W\Omega^i_{log})$ so we have

$$H^{k+1-i}(M_a, W\Omega^i_{log})/p \cong H^{k+1-i}(M_a, W\Omega^i_{log}/p) \cong H^{k+1-i}(M_a, \Omega^i_{log}).$$

Putting all this together, for i such that $T^{i-1,k+2-i} = T^{i-1,k+3-i} = 0$, we have

$$\mathrm{Rk}_W\, H^{k+1}_{cris}(M_a/W)_{[i]} = \dim_{\mathbf{F}_p} H^{k+1-i}(M_a, \Omega^i_{log}).$$

Similar arguments go through in general for a curve and we have

$$\begin{aligned} \mathrm{Rk}_W\, H^1(I/W)(\chi^{b+2})_{[1]} &= \dim_{\mathbf{F}_p} H^0(I, \Omega^1_{log})(\chi^{b+2}) \\ \mathrm{Rk}_W\, H^1(I/W)(\chi^{b})_{[0]} &= \dim_{\mathbf{F}_p} H^1(I, \Omega^0_{log})(\chi^{b}) \end{aligned} .$$

Part a) is now an immediate consequence of Theorem 2.4a and the fact (cf. Section 3) that the differential dq/q lies in the χ^2 eigenspace for the $\langle d \rangle$ action. Similarly, part b) follows from Theorem 2.4b. $\square$

Here is the main cohomological result related to fractional slopes. We will prove it in Section 9.

Theorem 2.6. *Assume the standing hypotheses, fix an integer i with $0 \leq i \leq k$, and set $b = a + k - 2i$.*

a) Suppose either $i + 1 \leq a$ and $k + 1 - i \leq p - 1 - a$; or $i = 0$ and $a \leq p - 1 - k$; or $i = k$ and $a \geq k$. Then

$$H^{k+1-i}(M_a, BW\Omega^{i+1})/F \cong H^1(I, BW\Omega^1)(\chi^b)/F.$$

b) If $i = a$ then

$$H^1(I/W)(\chi^{k+1-i})_{(0,1)} = 0 \Rightarrow H^{k+1-i}(M_a, BW\Omega^{i+1}) = 0$$

and if $k - i = p - 1 - a$ then

$$H^1(I/W)(\chi^{-i})_{(0,1)} = 0 \Rightarrow H^{k+1-i}(M_a, BW\Omega^{i+1}) = 0.$$

Using 2.6, we can establish the cohomological result underlying Theorem 1.6.

Corollary 2.7. *Fix an integer i with $0 \leq i \leq k$, and set $b = a + k - 2i$.*

a) Suppose either $i + 1 \leq a$ and $k + 1 - i \leq p - 1 - a$; or $i = 0$ and $a \leq p - 1 - k$; or $i = k$ and $a \geq k$. Then

$$\mathrm{Rk}_W\, H^{k+1}_{cris}(M_a/W)_{(i,i+1)} = \mathrm{Rk}_W\, H^1_{cris}(I/W)(\chi^b)_{(0,1)}.$$

b) If $i = a$ then

$$H^1(I/W)(\chi^{k+1-i})_{(0,1)} = 0 \Rightarrow H^{k+1}_{cris}(M_a/W)_{(i,i+1)} = 0$$

and if $k-i=p-1-a$ *then*

$$H^1(I/W)(\chi^{-i})_{(0,1)}=0 \Rightarrow H^{k+1}_{cris}(M_a/W)_{(i,i+1)}=0.$$

Proof: The hypotheses and Theorems 2.3 and 2.4 imply that we have

$$T^{i-1,k+2-i}=T^{i-1,k+3-i}=T^{i,k+1-i}=T^{i,k+2-i}=0.$$

Using these facts, the argument of [I-R] IV.4.5 shows that $H^{k+1}_{cris}(M_a/W)_{(i,i+1)}$ is a direct factor of $H^{k+1}_{cris}(M_a/W)$ as F-crystal and we have an isomorphism

$$H^{k+1}_{cris}(M_a/W)_{(i,i+1)} \cong H^{k+1-i}(M_a, BW\Omega^{i+1})$$

with Φ on the left corresponding to p^iF on the right. Similarly,

$$H^1_{cris}(I/W)_{(0,1)} \cong H^1(I, BW\Omega^1).$$

Theorem 2.6 thus shows that

$$H^{k+1}_{cris}(M_a/W)_{(i,i+1)}/p^{-i}\Phi \cong H^1_{cris}(I/W)_{(0,1)}/\Phi.$$

This is already enough to prove part b): we have

$$H^{k+1}_{cris}(M_a/W)_{(i,i+1)}/p^{-i}\Phi=0 \Rightarrow H^{k+1}_{cris}(M_a/W)_{(i,i+1)}=0$$

since $p^{-i}\Phi$ is topologically nilpotent on $H^{k+1}_{cris}(M_a/W)_{(i,i+1)}$.

To finish the proof of a), suppose that $\lambda_1,\ldots,\lambda_d$ are the slopes (taken with multiplicities) occuring in $H^{k+1}_{cris}(M_a/W)_{(i,i+1)}$ and set

$$h^i=\sum_j (i+1)-\lambda_j \qquad h^{i+1}=\sum_j \lambda_j - i$$

(so h^i and h^{i+1} are the contact numbers attached to $H^{k+1}_{cris}(M_a/W)_{(i,i+1)}$). We have

$$\mathrm{Rk}_W\, H^{k+1}_{cris}(M_a/W)_{(i,i+1)} = h^i + h^{i+1}$$

$$\dim_{\mathbf{F}} H^{k+1}_{cris}(M_a/W)_{(i,i+1)}/p^{-i}\Phi = h^{i+1}$$

and

$$\dim_{\mathbf{F}} H^{k+1}_{cris}(M_a/W)_{(i,i+1)}/p^{i+1}\Phi^{-1} = h^i$$

Similarly we have integers h^0 and h^1 attached to $H^1_{cris}(I/W)(\chi^b)$ and the argument above shows that $h^{i+1} = h^1$. On the other hand, the hypotheses of the Corollary are invariant under replacing i with $k-i$ and a with $p-1-a$ so using Poincaré duality we have

$$\begin{aligned} h^i(H^{k+1}_{cris}(M_a/W)_{(i,i+1)}) &= h^{k-i}(H^{k+1}_{cris}(M_{-a}/W)_{(k-i,k+1-i)}) \\ &= h^1(H^1_{cris}(I/W)(\chi^{-b})_{(0,1)}) = h^0(H^1_{cris}(I/W)(\chi^b)_{(0,1)}). \end{aligned}$$

Thus $h^0 + h^1 = h^i + h^{i+1}$ which completes the proof of the corollary. $\square$

Finally, let us establish the cohomological result underlying 1.8.

Corollary 2.8. *In addition to the standing hypotheses (in particular, $N \geq 5$), assume $k > 1$. Then*

$$H^1(I/W)(\chi^{-1})_{(0,1)} = 0 \Rightarrow \begin{cases} H^{k+1}_{cris}(M_{1-k}/W)_{[1,2)} = 0 \\ \text{and} \\ \mathrm{Rk}_W\, H^{k+1}_{cris}(M_{1-k}/W)_{(0,1)} = 2(p-1)w. \end{cases}$$

Similarly,

$$H^1(I/W)(\chi^{1})_{(0,1)} = 0 \Rightarrow \begin{cases} H^{k+1}_{cris}(M_{k-1}/W)_{(k-1,k]} = 0 \\ \text{and} \\ \mathrm{Rk}_W\, H^{k+1}_{cris}(M_{k-1}/W)_{(k,k+1)} = 2(p-1)w. \end{cases}$$

Proof: Let us prove the first assertions; the other can be proven analogously or deduced from Poincaré duality. The vanishing implication already follows from 2.5b and 2.7b (both applied with $i = 1$ and $a = p - k$); it follows that the Newton polygon attached to $H^{k+1}_{cris}(M_{1-k}/W)$ first meets the slope 1 edge of the contact polygon at the point where its slope 1 and slope 2 edges meet. On the other hand, the hypothesis $H^1(I/W)(\chi^{-1})_{(0,1)} = 0$ implies that $\mathrm{Rk}\, H^1(I/W)(\chi^{-1})_{[0]} = 2w - c/2$ and so (for example by well-known results of Hida [H] together with 2.1) $\mathrm{Rk}\, H^{k+1}_{cris}(M_{1-k}/W)_{[0]} = 2w - c/2$. By Theorem 2.4, $T^{0,k+1} = T^{1,k} = 0$ so using the formulas 1.2 and 1.3, the horizontal distance between the point where the Newton polygon leaves the slope 0 edge of the contact polygon and the point where it meets the slope 1 edge (i.e., the rank of $H^{k+1}_{cris}(M_{1-k}/W)_{(0,1)}$) is $2(p-1)w$. $\square$

Now we explain how the results of this section imply the theorems of Section 1. Indeed, Formula 2.1 says that the Newton polygon of U_p on $S = S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)$ is equal to the Newton polygon of Frobenius on $H = H^{k+1}_{cris}(M_a/W)$. Thus for all λ, the rank over W of the slope λ part of H is equal to the dimension over $\mathbf{Q}_p$ of the slope λ subspace of S. This means that 2.5, 2.7, and 2.8 immediately imply 1.5, 1.6, and 1.8 respectively.

As we have seen, all the results of Section 2 (and thus of Section 1 as well) follow from 2.3, 2.4, and 2.6, so we are reduced to proving those theorems. We will carry this out in the rest of the paper, but for simplicity, we assume throughout that $N \geq 5$. We explain here how to obtain the results of Section 1 when $N \leq 4$. The point is that when $N \leq 4$, there is no universal curve, so we have to replace the motive (X, Π) with one based on a higher level modular curve. First of all, if $p = 3$, $N \leq 4$, all of the assertions of Section 1 are true, as can be checked with a finite amount of computation. (We will explain a more refined version of this computation, including congruences as well as slopes, in a future paper.) Let us then assume that $p > 3$. We choose an auxiliary integer $M > 2$ with p not dividing the order of $H = \mathrm{GL}_2(\mathbf{Z}/M\mathbf{Z})$ and replace the $\Gamma_1(N)$ moduli problem with the simultaneous moduli problem $\Gamma_1(N) \cap \Gamma(M)$. We have a modular curve I for Igusa structures of level p and $\Gamma_1(N) \cap \Gamma(M)$ structures and a construction analogous to that at the beginning of this section gives pairs (X, Π) and (X, Π_a) whose cohomology is related to modular forms on $\Gamma_1(N) \cap \Gamma(M)$. The variety X also has an action of H and we define $\Pi' \in \mathbf{Z}_p[G \times H]$ and $\Pi'_a \in \mathbf{Z}_p[G \times (\mathbf{Z}/p\mathbf{Z})^\times \times H]$ as the projectors cutting out the H-invariant part of the cohomology of (X, Π) or (X, Π_a). We can then carry out all the arguments of the paper with the motives (X, Π') and (X, Π'_a). The characteristic polynomial of Frobenius on $\Pi'_a H^{k+1}_{cris}(X)$ is equal to the right hand side of 2.1 and Theorems 2.3-2.8 hold if we replace cohomology groups of I with their H-invariant parts. (The integers on the right hand side of the inequality in 2.3b implicitly involve a cohomology group on I. The correct values when $N \leq 4$ are given in Section 6.) Using these versions of 2.3-2.8, one finds that the theorems of Section 1 hold as stated, i.e., for all N.

To end this section, we give some idea of the contents of the rest of the paper. It will turn out that the two crucial technical results are these: $H^{k+2-i}(M_a, \Omega^i_{log})$ has a positive dimensional $\mathbf{F}$-vector space as a quotient for certain values of k, a, and i (this is essentially 2.3b); and, for certain other values of k, a, and i, the group $H^{k+1-i}(M_a, \Omega^i_{log})$ is finite and closely related to $H^0(I, \Omega^1_{log})(\chi^{b+2})$ and $H^1(I, \Omega^0_{log})(\chi^b)$ (this is part of 2.4a). The cohomology of the Ω^i_{log} is mysterious, but we can relate it to more tractable coherent sheaves: the exact sequence

$$0 \to \Omega^i_{log} \to Z^i_X \xrightarrow{1-\mathcal{C}} \Omega^i_X \to 0$$

leads to a surjection $H^{k+2-i}(M_a, \Omega^i_{log}) \to H^{k+2-i}(M_a, Z^i)$ and an isomorphism

$$R^{k+1-i} f_* \Omega^i_{M,log} \cong \operatorname{Ker}\left(1 - \mathcal{C} : R^{k+1-i} f_* Z^i_M \to R^{k+1-i} f_* \Omega^i_M\right). \tag{2.9}$$

We know all about the cohomology of the Ω^i_M from [U2], so here we concentrate on the cohomology of Z^i_M together with its two maps 1 and $\mathcal{C}$ to the cohomology of Ω^i_M; along the way we will also need to study the cohomology of exact differentials.

In Section 3 we make a preliminary study of the relative deRham cohomology of the universal curve $\mathcal{E} \to I$, with its two filtrations and Gauss-Manin connection. The interaction between these three structures near the supersingular points plays a key role throughout the paper. Using this, in Section 4 we consider the $R^j f_* Z^i_M$ and the maps 1 and $\mathcal{C}$ in the very simple cases where $i = 0$ (where $Z^0_X = \mathcal{O}^p_X$) and $i = k+1$ (where $Z^{k+1}_X = \Omega^{k+1}_X$). In order to study 1 and $\mathcal{C}$ for other values of i we use the deRham cohomology sheaves $\underline{H}^{i+j}_{dR}(M)$ of M on I, with their Hodge and conjugate filtrations $F^{\cdot}$ and $F_{\cdot}$. There is a natural surjection $R^j f_* Z^i_M \to (F^i \cap F_i)\underline{H}^{i+j}_{dR}(M)$ and its compositions with the two maps $(F^i \cap F_i)\underline{H}^{i+j}_{dR}(M) \to \operatorname{Gr}^i \underline{H}^{i+j}_{dR}(M)$ and $(F^i \cap F_i)\underline{H}^{i+j}_{dR}(M) \to \operatorname{Gr}_i \underline{H}^{i+j}_{dR}(M)$ are closely related to 1 and $\mathcal{C}$. Indeed, the conjugate spectral sequence degenerates, $\operatorname{Gr}_i \underline{H}^{i+j}_{dR}(M) \cong R^j f_* \Omega^i_M$, and $\mathcal{C} : R^j f_* Z^i_M \to R^j f_* \Omega^i_M$ is the composition above. On the other hand, the Hodge spectral sequence turns out not to degenerate. We analyze it in Section 5 and then in Section 6 use these results to compute the cokernels of 1 and $\mathcal{C} : R^j f_* Z^i_M \to R^j f_* \Omega^i_M$. Chasing through some exact sequences leads to computations of $R^{k+2-i} f_* Z^i_M$ and $R^{k+2-i} f_* B^i_M$ and to the positive dimensionality of $H^{k+2-i}(M_a, Z^i)$ for certain k, a, and i. Since this group is a quotient of $H^{k+2-i}(M_a, \Omega^i_{log})$, this yields the first main technical result.

Sections 7 and 8 contain a rather indirect computation of the $H^{k+1-i}(M_a, \Omega^i_{log})$ for the remaining k, a, and i. First we observe that the non-degeneration of the Hodge spectral sequence leads to a construction of a map (with well-understood kernel) from certain subsheaves of the invertible sheaf $R^k f_* \mathcal{O}_M$ to the $R^{k+1-i} f_* Z^i_M$. Next we construct analogous (injective) maps from certain subsheaves of the invertible sheaf $R^0 f_* \Omega^{k+1}_M$ to the $R^{k+1-i} f_* Z^i_M$, and we show that, for suitable k, a, and i, the images of the two maps generate $(R^{k+1-i} f_* Z^i_M)(\chi^a)$. Thus we can reduce questions on the hard-to-compute sheaves $R^{k+1-i} f_* Z^i_M$ to questions on well-understood invertible sheaves on I. This idea and the isomorphism 2.9 allow us to establish an isomorphism between $(R^{k+1-i} f_* \Omega^i_{M,log})(\chi^a)$ and a certain sheaf $\mathcal{F}$ of functions. Finally, we show that the group of global sections of $\mathcal{F}$ has a natural filtration whose graded pieces are closely related to $H^0(I, \Omega^1_{log})(\chi^{b+2})$ and $H^1(I, \Omega^0_{log})(\chi^b)$. (This last point suggests that there may be a more natural description of $R^{k+1-i} f_* \Omega^i_{M,log}$ in a suitable derived category. However, the isomorphism between the graded pieces of $H^0(I, R^{k+1-i} f_* \Omega^i_{M,log})(\chi^a)$ and the logarithmic cohomology groups on I is twisted, both with respect to Frobenius and to the relevant ring of Hecke operators, and the twisting depends in a complicated way on the data k, a, and i. At the moment, I know of no simple complex which has all the required properties.)

3. Relative deRham cohomology We keep the notational conventions of the last section; in particular, M stands for the appropriate combination of the variety X and the projector Π. For example, $f_* \Omega^i_{M/I}$ means $\Pi f_* \Omega^i_{X/I}$. Also, if $\mathcal{F}$ is a sheaf we will frequently write $s \in \mathcal{F}$ when we mean "s is a section of $\mathcal{F}$ over some open set" and we also denote the restriction of $\mathcal{F}$ to any open set by $\mathcal{F}$.

Our goal in this section is to collect some facts we need on the relative deRham cohomology of M over I with its two filtrations, pairing, and Gauss-Manin connection. We will also construct some canonical cohomology classes by using the Igusa structure on $\mathcal{E}/I$.

Recall the diagram of Frobenius:

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/I}} & X' & \xrightarrow{G_{X/I}} & X \\ & & \downarrow & & \downarrow \\ & \searrow & I & \xrightarrow{F_I} & I \end{array}$$

where F_I is the absolute Frobenius of I, the square is Cartesian and defines X', and $F_X = G_{X/I} \circ F_{X/I}$ is the absolute Frobenius of X. We will frequently use a prime to denote the pull back under F_I or $G_{X/I}$; for example, if ω_c is a section of $\Omega^1_{X/I}$, we write ω'_c for $G^*_{X/I}\omega_c$. Recall also the relative Cartier isomorphism: let $Z^i_{X/I} = \mathrm{Ker}\,(d : \Omega^i_{X/I} \to \Omega^{i+1}_{X/I})$, $B^i_{X/I} = \mathrm{Im}\,(d : \Omega^{i-1}_{X/I} \to \Omega^i_{X/I})$, and $\mathcal{H}^i_{X/I} = Z^i_{X/I}/B^i_{X/I}$. Then we have an $\mathcal{O}_{X'}$-linear isomorphism

$$\mathcal{C}_{X/I} : F_{X/I*}\mathcal{H}^i_{X/I} \to \Omega^i_{X'/I}. \tag{3.1}$$

We will sometimes also view this as a semi-linear map

$$\mathcal{C}_{X/I} : Z^i_{X/I} \to \Omega^i_{X'/I}$$

where semilinearity here means that $\mathcal{C}_{X/I}(f^p\sigma) = f'\mathcal{C}_{X/I}(\sigma)$. Note that this last map is still $\mathcal{O}_I$-linear.

We are going to study the relative deRham cohomology of X over I. In fact, most of our calculations in this section will actually take place over $I^0 = I \setminus \{cusps\}$, where X is the k-fold fiber product of a smooth elliptic fibration.

Recall the relative deRham complex of X over I:

$$\Omega^{\cdot}_{X/I} = \left(0 \to \mathcal{O}_X \to \Omega^1_{X/I} \to \Omega^2_{X/I} \to \cdots\right).$$

By definition, the relative deRham cohomology sheaves are the hypercohomology sheaves of this complex:

$$\underline{H}^j_{dR}(X/I) = R^j f_* \Omega^{\cdot}_{X/I}.$$

The complex $\Omega^{\cdot}_{X/I}$ has two filtrations by subcomplexes, the Hodge filtration and the conjugate filtration, which we denote by $F^{\cdot}$ and $F_{\cdot}$:

$$F^j(\Omega^{\cdot}_{X/I}) = \left(0 \to \Omega^j_{X/I} \to \Omega^{j+1}_{X/I} \to \cdots\right)$$

and

$$F_j(\Omega^{\cdot}_{X/I}) = \left(0 \to \mathcal{O}_X \to \cdots \to \Omega^{j-1}_{X/I} \to Z^j_{X/I} \to 0\right).$$

These filtrations give rise to two spectral sequences, the Hodge spectral sequence

$$E_1^{ij} = R^j f_* \Omega^i_{X/I} \Longrightarrow \underline{H}^{i+j}_{dR}(X/I)$$

and the conjugate spectral sequence

$$E_2^{ij} = R^i f_* \mathcal{H}^j_{X/I} \Longrightarrow \underline{H}^{i+j}_{dR}(X/I)$$

(which is actually the shift $E_1^{p,q} \to E_2^{q+2p,-p}$ of the usual spectral sequence associated to the conjugate filtration). We denote again by $F^{\cdot}$ and $F_{\cdot}$ the filtrations on the abutment $\underline{H}^{i+j}_{dR}(X/I)$ and by $\mathrm{Gr}^{\cdot}$ and $\mathrm{Gr}_{\cdot}$ the associated graded sheaves.

We will need two other structures on the relative deRham cohomology: the deRham pairing and the Gauss-Manin connection. Over I^0, where the fibers of f are smooth complete varieties of dimension k, we have the deRham pairing

$$\langle , \rangle_{dR} : \underline{H}^j_{dR}(X/I) \otimes \underline{H}^{2k-j}_{dR}(X/I) \to \mathcal{O}_I$$

induced by cup product, the wedge product $\Omega^{\cdot}_{X/I} \otimes \Omega^{\cdot}_{X/I} \to \Omega^{\cdot}_{X/I}$, and the trace map $\underline{H}^{2k}_{dR}(X/I) \cong \mathcal{O}_I$. This is an alternating, $\mathcal{O}_I$-linear, non-degenerate pairing which has the following compatibility with Serre duality: if $s \in f_*\Omega^j_{X/I}$, $\tilde{t} \in R^j f_*\mathcal{O}_X$, and $\tilde{s},\ t \in \underline{H}^j_{dR}(X/I)$ are sections over some open with $\tilde{s}$ and $\tilde{t}$ the images of s and t under the edge maps $f_*\Omega^j_{X/I} \to \underline{H}^j_{dR}(X/I)$ and $\underline{H}^j_{dR}(X/I) \to R^j f_*\mathcal{O}_X$ of the Hodge spectral sequence, then $\langle \tilde{s}, t\rangle_{dR} = \langle s, \tilde{t}\rangle_{Serre}$.

The Gauss-Manin connection is the derivation of $\mathcal{O}_I$-modules

$$\nabla : \underline{H}^j_{dR}(X/I) \to \Omega^1_I \otimes \underline{H}^j_{dR}(X/I)$$

defined as the coboundary in the long exact sequence of hypercohomology deduced from the short exact sequence of complexes

$$0 \to f^*\Omega^1_I \otimes \Omega^{\cdot -1}_{X/I} \to \Omega^{\cdot}_X \to \Omega^{\cdot}_{X/I} \to 0.$$

(See [K-O] for a proof that this is a derivation and that it defines an integrable connection on $\underline{H}^j_{dR}(X/I)$.) It follows immediately from the definition that the conjugate filtration is horizontal with respect to ∇ (i.e., $\nabla(F_j\underline{H}^i_{dR}(X/I)) \subseteq \Omega^1_I \otimes F_j\underline{H}^i_{dR}(X/I)$) and ∇ shifts the Hodge filtration by at most one (Griffiths transversality: $\nabla(F^j\underline{H}^i_{dR}(X/I)) \subseteq \Omega^1_I \otimes F^{j-1}\underline{H}^i_{dR}(X/I)$).

Since G acts on X covering the trivial action on I, we can apply Π to the $\underline{H}^j_{dR}(X/I)$ and define, following our usual notational convention,

$$\underline{H}^j_{dR}(M/I) = \Pi\underline{H}^j_{dR}(X/I).$$

We can also apply Π to each term of the spectral sequences above and get two spectral sequences converging to $\underline{H}^*_{dR}(M/I)$; we will again denote the resulting filtrations by $F^{\cdot}$

and F.. Since the character ϵ giving rise to Π satisfies $\epsilon = \epsilon^{-1}$ and over I^0 the trace map $\underline{H}^{2k}_{dR}(X/I) \cong \mathcal{O}_I$ is equivariant for G (i.e., G acts trivially on $\underline{H}^{2k}_{dR}(X/I)$), the deRham pairing restricts to give an alternating, non-degenerate, $\mathcal{O}_I$-linear pairing

$$\langle , \rangle_{dR} : \underline{H}^j_{dR}(M/I) \otimes \underline{H}^{2k-j}_{dR}(M/I) \to \mathcal{O}_I$$

over I^0. Finally, the Gauss-Manin connection commutes with the action of G, so ∇ defines an integrable connection on the $\underline{H}^j_{dR}(M/I)$.

In [U2], 5.2 we showed that over I^0,

$$R^j f_* \Omega^i_{M/I} = \Pi R^j f_* \Omega^i_{X/I} \cong \begin{cases} \omega^{2i-k} & \text{if } i+j=k \\ 0 & \text{otherwise} \end{cases}$$

and we have $R^i f_* \mathcal{H}^j_{M/I} = \Pi R^i f_* \mathcal{H}^j_{X/I} \cong \Pi R^i f_* \Omega^j_{X'/I}$. It is immediate that over I^0, the spectral sequences of deRham cohomology of M/I degenerate and $\underline{H}^i_{dR}(M/I) = 0$ for $i \neq k$.

Recall that when $k = 1$, X is $\mathcal{E}$, the universal curve over I, and $\underline{H}^1_{dR}(X/I) = \underline{H}^1_{dR}(\mathcal{E}/I)$. We want to define some interesting sections of $\underline{H}^1_{dR}(\mathcal{E}/I)$. The Hodge and conjugate spectral sequences give rise to two exact sequences

$$0 \to f_* \Omega^1_{\mathcal{E}/I} \to \underline{H}^1_{dR}(\mathcal{E}/I) \to R^1 f_* \mathcal{O}_{\mathcal{E}} \to 0$$

and

$$0 \to R^1 f_* \mathcal{H}^0_{\mathcal{E}/I} \to \underline{H}^1_{dR}(\mathcal{E}/I) \to f_* \mathcal{H}^1_{\mathcal{E}/I} \to 0.$$

Since $f : \mathcal{E} \to I$ is a flat family of curves of arithmetic genus 1, the sheaf $R^1 f_* \mathcal{O}_{\mathcal{E}}$ is invertible on I; we define ω to be its inverse. Over I^0, Serre duality defines an isomorphism $\omega \cong f_* \Omega^1_{\mathcal{E}/I}$. Also, the relative Cartier isomorphism 3.1 defines isomorphisms $R^1 f_* \mathcal{H}^0_{\mathcal{E}/I} \cong R^1 f'_* \mathcal{O}_{\mathcal{E}'} \cong \omega^{-p}$ and $f_* \mathcal{H}^1_{\mathcal{E}/I} \cong f'_* \Omega^1_{\mathcal{E}'/I} \cong \omega^p$ over I^0.

Fix a splitting

$$\underline{H}^1_{dR}(\mathcal{E}/I) \cong \omega \oplus \omega^{-1} \tag{3.2}$$

of the Hodge filtration sequence

$$0 \to \omega \to \underline{H}^1_{dR}(\mathcal{E}/I) \to \omega^{-1} \to 0$$

over the open curve I^0. (If $p > 3$, then Katz [K1, A1.2] has defined a distinguished splitting in terms of a Weierstrass model.) Recall also that there is a canonical section ω_c of ω characterized by the fact that $\mathcal{C}_{X/I}(\omega_c) = \omega_c'$. (Cf. [K-M], 12.8. The existence of ω_c is one possible definition of the Igusa curve.) We write ω_c for its image in $\underline{H}^1_{dR}(\mathcal{E}/I)$ as well. As a section of ω, ω_c vanishes to order 1 at each supersingular point and is a generating section elsewhere. We let η_K be the rational section of $\underline{H}^1_{dR}(\mathcal{E}/I)$ mapping to $(0, \omega_c^{-1})$ under the splitting 3.2. It is regular away from the supersingular points. By the compatibility between Serre duality and the deRham pairing mentioned above, $\langle \omega_c, \eta_K \rangle_{dR} = \langle \omega_c, \omega_c^{-1} \rangle_{Serre} = 1$.

We also have the conjugate filtration sequence which over I^0 is isomorphic to

$$0 \to \omega^{-p} \to \underline{H}^1_{dR}(\mathcal{E}/I) \to \omega^p \to 0;$$

let η_c be the image in $\underline{H}^1_{dR}(\mathcal{E}/I)$ of the rational section ω_c^{-p} of ω^{-p}. This is an analogue of ω_c for the conjugate filtration. We will find below a natural rational section of $\underline{H}^1_{dR}(\mathcal{E}/I)$ lifting the section ω_c^p of ω^p.

Next we introduce $\mathcal{O}_I$-linear maps $F : \underline{H}^1_{dR}(\mathcal{E}'/I) \to \underline{H}^1_{dR}(\mathcal{E}/I)$ and $V : \underline{H}^1_{dR}(\mathcal{E}/I) \to \underline{H}^1_{dR}(\mathcal{E}'/I)$ which were first defined by Tadao Oda in his thesis [O]. By definition, F is the composition

$$\underline{H}^1_{dR}(\mathcal{E}'/I) \to R^1 f'_* \mathcal{O}_{\mathcal{E}'} \xrightarrow{\mathcal{C}^{-1}} R^1 f_* \mathcal{H}^0_{\mathcal{E}/I} \to \underline{H}^1_{dR}(\mathcal{E}/I)$$

and V is the composition

$$\underline{H}^1_{dR}(\mathcal{E}/I) \to f_*\mathcal{H}^1_{\mathcal{E}/I} \xrightarrow{\mathcal{C}} f'_*\Omega^1_{\mathcal{E}'/I} \to \underline{H}^1_{dR}(\mathcal{E}'/I).$$

Explicitly, if $(\omega'_i, \lambda'_{ij})$ is a Čech hyper 1-cocycle (for some cover) representing a class of $\underline{H}^1_{dR}(\mathcal{E}'/I)$ over some open, then $(0, \lambda^p_{ij})$ represents its image under F. If (ω_i, λ_{ij}) represents a class of $\underline{H}^1_{dR}(\mathcal{E}/I)$, then $(\mathcal{C}_{\mathcal{E}/I}\omega_i, 0)$ represents its image under V. In particular, the following transposition formula for F and V holds:

$$\langle F(x), y\rangle_{dR,\ \mathcal{E}/I} = \langle x, V(y)\rangle_{dR,\ \mathcal{E}'/I}. \tag{3.3}$$

Note also that Ker $V = F_0\underline{H}^1_{dR}(\mathcal{E}/I)$ and Ker $F = F^1\underline{H}^1_{dR}(\mathcal{E}'/I)$.

Define rational functions A and B on I by the formula

$$F(\eta'_K) = A\eta_K + B\omega_c.$$

Then the sections $A\omega_c^{p-1}$ and $B\omega_c^{p+1}$ extend to regular sections of ω^{p-1} and ω^{p+1} respectively, over all of I. (Cf. [K1] and [K2] for this and other assertions in this paragraph.) As is well-known, $A\omega_c^{p-1}$ is the value of the Hasse invariant on $\mathcal{E}$ (viewed as a section of ω^{p-1}); thus by the definition of ω_c, $A = 1$. Strictly speaking, $B\omega_c^{p+1}$ is not the value of a modular form, since it depends on the splitting 3.2; the possible splittings are a homogeneous space for $H^0(I^0, \omega^2)$ and if we change the splitting by s, the value of $B\omega_c^{p+1}$ changes by $s\omega_c^{p-1}$. In particular, $B\omega_c^{p+1}$ is well-defined where ω_c^{p-1} vanishes, i.e., on the "supersingular scheme" defined by the vanishing of ω_c^{p-1}. If $p > 3$ and we use the splitting of Katz, then the value of $B\omega_c^{p+1}$ is the reduction mod p of $-\frac{1}{12}E_{p+1}$ where E_{p+1} is the Eisenstein series of weight $p+1$ and level 1, viewed as a section of ω^{p+1}. By a theorem of

Igusa, the section $B\omega_c^{p+1}$ of ω^{p+1} does not vanish at any point where the section $A\omega_c^{p-1}$ of ω^{p-1} does (i.e., at the supersingular points). This implies that B has poles of order $p+1$ at each supersingular point; it is regular elsewhere. Calculating as in [U2], (above 5.7), we find that $\langle d\rangle^*\omega_c = d\omega_c$ and $\langle d\rangle^* B = d^{-p-1}B = d^{-2}B$ for all $d \in (\mathbf{Z}/p\mathbf{Z})^\times$.

We have $F(\omega_c') = 0$ and $F(\eta_K') = \eta_K + B\omega_c$ by definition. Using the transposition formula 3.3, we compute

$$V(\omega_c) = A\omega_c' = \omega_c'$$

and

$$V(\eta_K) = -B\omega_c'.$$

Also, it follows from the definitions that $F(\eta_K') = \eta_c$, so we find

$$\eta_c = \eta_K + B\omega_c.$$

Moreover, since $V(\eta_K) = -B\omega_c'$, we have $-B^{-1}\eta_K \in \underline{H}^1_{dR}(\mathcal{E}/I)$ lifts $\omega_c^p \in f_*\mathcal{H}^1_{\mathcal{E}/I} \cong \omega^p$.

To summarize, we have three "bases" (at the generic point) of $\underline{H}^1_{dR}(\mathcal{E}/I)$, namely (ω_c, η_K), (η_c, η_K), and (ω_c, η_c). The first is adapted to the Hodge filtration in the sense that a section $s = x\omega_c + y\eta_K$ $(x, y \in \mathcal{O}_I)$ lies in $F^1\underline{H}^1_{dR}(\mathcal{E}/I)$ if and only if $y = 0$; s is regular at ordinary points where x and y are regular, and at supersingular points where $\mathrm{ord}(x) \geq -1$, $\mathrm{ord}(y) \geq 1$. The basis (η_c, η_K) is adapted to the conjugate filtration in a similar sense; a section $s = x\eta_c + y\eta_K$ is regular at ordinary points where x and y are regular and at supersingular points where $\mathrm{ord}(x) \geq p$, $\mathrm{ord}(y) \geq 1$. Finally, the basis (ω_c, η_c) is convenient for comparing filtrations: $s = x\omega_c + y\eta_c$ lies in $F^1\underline{H}^1_{dR}(\mathcal{E}/I)$ if and only if $y = 0$ and it lies in $F_0\underline{H}^1_{dR}(\mathcal{E}/I)$ if and only if $x = 0$; unfortunately, the regularity conditions in this basis (and its symmetric powers) are not as attractive: s is regular at ordinary points where x and y are regular and at supersingular points where $\mathrm{ord}(x) \geq -1$ and $\mathrm{ord}(x + By) \geq 1$.

(We remark that η_c stands for η_{can} and η_K stands for η_{Katz}. η_K is what Katz calls η_{can} in [K1], but η_c seems more canonical—it does not depend on any choice of splitting.)

To discuss the Gauss-Manin connection, recall the differential dq/q on the Igusa curve I; dq/q was characterized in [U1, 7.7] in terms of the the kernel of p on the universal curve $\mathcal{E}$. Alternatively, dq/q can be defined on the Tate curve as the image of ω_c^2 under the Kodaira-Spencer isomorphism $\omega^2 \cong \Omega^1(log\ cusps)$ and then extended to the Igusa curve. On I, dq/q vanishes to order p at each supersingular point, has simple poles at each cusp, and is regular and non-vanishing at the other ordinary points. Also, we have $\langle c\rangle^* dq/q = c^2\, dq/q$ for all $c \in (\mathbf{Z}/p\mathbf{Z})^\times$. In terms of this differential, Katz [K1, A1.3.16] has shown that

$$\nabla(\omega_c) = (dq/q)\eta_K + B(dq/q)\omega_c = (dq/q)\eta_c$$

and

$$\begin{aligned}\nabla(\eta_K) &= -(B^2 dq/q + dB)\omega_c - B(dq/q)\eta_K \\ &= \frac{dB}{B}\eta_K - (B\frac{dq}{q} + \frac{dB}{B})\eta_c.\end{aligned}$$

(This last formula corrects a sign mistake in [K1]; the problem starts at A1.3.14 and continues through A1.3.16 where the lower left entry of the matrix on the right hand side has the wrong sign.) It follows that $\nabla(\eta_c) = 0$.

Lemma 3.4. *The differential $B(dq/q)$ has a simple pole with residue $+1$ at each supersingular point. The differential $B^2(dq/q) + dB$ vanishes to order at least $p - 4$ at each supersingular point.*

Proof: This follows from the fact that ∇ takes regular sections to regular sections and the formula for $\nabla(\eta_K)$: we work in an open neighborhood of some fixed supersingular point. Let t be a uniformiser there, so that $t\eta_K$ is a regular section of $\underline{H}^1_{dR}(\mathcal{E}/I)$. We have

$$\nabla(t\eta_K) = (dt - tB(dq/q))\eta_K - t(B^2(dq/q) + dB)\omega_c.$$

That this section is regular implies that $\mathrm{ord}(dt - tB(dq/q)) \geq 1$, which in turn implies that $B(dq/q)$ has a simple pole with residue 1. Similarly, we must have $\mathrm{ord}(B^2(dq/q) + dB) \geq -2$. But both dq/q and B are eigenvectors for the action of $(\mathbf{Z}/p\mathbf{Z})^\times$, with corresponding characters χ^2 and χ^{-2}. Since $(\mathbf{Z}/p\mathbf{Z})^\times$ acts on the cotangent space at a supersingular point via the character χ, we must have $\mathrm{ord}(B^2(dq/q) + dB) \equiv -3 \pmod{p-1}$. Thus $\mathrm{ord}(B^2(dq/q) + dB) \geq p - 4$. (Alternatively, if $p > 3$, Katz has shown [K1, A1.4.3] that $B^2(dq/q) + dB = \frac{Q}{144}\frac{dq}{q}$ where $Q\omega_c^4$ is the reduction modulo p of the Eisenstein series E_4 of weight 4 and level 1; it follows that $(B^2(dq/q) + dB)$ vanishes to order at least $p - 4$ at every supersingular point, and to order exactly $p - 4$ at every supersingular point with $j \neq 0$.) $\square$

Finally, we define operators θ and Θ on functions and differentials on I by the formulas

$$\theta f = \frac{df}{dq/q}$$

and

$$\Theta\sigma = d\left(\frac{\sigma}{dq/q}\right).$$

We have (because dq/q is a logarithmic differential, for example) that $\theta^p = \theta$ and $\Theta^p = \Theta$ and Θ defines an automorphism of the space of exact rational differential 1-forms on I.

Lemma 3.5. *Let f be a rational function on I, σ a rational differential on I, j an integer, and x a supersingular point.*

a) $\mathrm{ord}_x(\theta f) \geq \mathrm{ord}_x(f) - (p+1)$ *with equality if and only if* $\mathrm{ord}_x(f) \not\equiv 0 \pmod{p}$.

b) $\mathrm{ord}_x(\Theta\sigma) \geq \mathrm{ord}_x(\sigma) - (p+1)$ *with equality if and only if* $\mathrm{ord}_x(\sigma) \not\equiv 0 \pmod{p}$.

c) $\mathrm{ord}_x(df - jfB\frac{dq}{q}) \geq \mathrm{ord}_x(f) - 1$ *with equality if and only if* $\mathrm{ord}_x(f) \not\equiv j \pmod{p}$.

Proof: Parts a) and b) are immediate from the fact that dq/q vanishes to order p at each supersingular point. Part c) follows from Lemma 3.4. □

Let $\mathcal{C} = \mathcal{C}_I$ be the Cartier operator on rational differentials on I.

Lemma 3.6. *For any rational function f on I, $f - \theta^{p-1}f$ is a p-th power and*

$$\mathcal{C}(f\frac{dq}{q}) = \left(f - \theta^{p-1}f\right)^{1/p}\frac{dq}{q}.$$

Proof: It is clear from the definition that the kernel of θ consists exactly of the p-powers of rational functions on I. Since $\theta f = \theta^p f$, $f - \theta^{p-1}f$ is a p-power. Also, $\theta^{p-1}f\, dq/q$ is an exact differential, so

$$\mathcal{C}(f\frac{dq}{q}) = \mathcal{C}\left(f - \theta^{p-1}f\right)\frac{dq}{q} = \left(f - \theta^{p-1}f\right)^{1/p}\frac{dq}{q}.$$

□

Our calculations on $\mathcal{E}$ are then related to X by the following lemma, whose proof is a standard argument using the Künneth formula, the cohomology of abelian varieties, and linear algebra.

Lemma 3.7. *There is a canonical isomorphism*

$$\underline{H}^k_{dR}(M/I) \cong \mathrm{Sym}^k\, \underline{H}^1_{dR}(\mathcal{E}/I)$$

of sheaves on I^0. Under this isomorphism, the filtrations $F^{\cdot}$, and $F_{\cdot}$, the deRham pairing, and the Gauss-Manin connection on $\underline{H}^k_{dR}(M/I)$ correspond to the k-th symmetric power of the corresponding structures on $\underline{H}^1_{dR}(\mathcal{E}/I)$. $\square$

For convenience, we make explicit some consequences of the lemma. A rational section $s = f_0\eta_K^k + \cdots + f_k\omega_c^k \in \underline{H}^k_{dR}(M/I)$ (written in the (ω_c, η_K) basis) is regular over I^0 if and only if $\mathrm{ord}_x(f_j) \geq k - 2j$ at each supersingular point x and $\mathrm{ord}_y(f_j) \geq 0$ at other points y. The section s lies in $F^i\underline{H}^k_{dR}(M/I)$ if and only if $f_j = 0$ for $j < i$, in which case its image in $\mathrm{Gr}^{\,i}\, \underline{H}^k_{dR}(M/I) \cong R^{k-i}f_*\Omega^i_{X/I} \cong \omega^{2i-k}$ is $f_i\omega_c^{2i-k}$. We have

$$\nabla(s) = \sum \left((j+1)f_{j+1}\frac{dq}{q} + df_j - (k-2j)f_jB\frac{dq}{q} - (k+1-j)f_{j-1}(B^2\frac{dq}{q} + dB) \right) \omega_c^j\eta_K^{k-j}. \tag{3.8}$$

A rational section $s = g_k\eta_K^k + \cdots + g_0\eta_c^k \in \underline{H}^k_{dR}(M/I)$ (written in the (η_c, η_K) basis) is regular over I^0 if and only if $\mathrm{ord}_x(g_j) \geq pk - j(p-1)$ at each supersingular point x and $\mathrm{ord}_y(g_j) \geq 0$ at other points y. The section s lies in $F_i\underline{H}^k_{dR}(M/I)$ if and only if $g_j = 0$ for $j > i$, in which case its image in $\mathrm{Gr}_{\,i}\, \underline{H}^k_{dR}(M/I) \cong R^{k-i}f_*\mathcal{H}^i_{X/I} \cong \omega^{p(2i-k)}$ is $(-B)^i g_i\omega_c^{p(2i-k)}$. We have

$$\begin{aligned}\nabla(s) &= \sum \left(dg_j + jg_j\frac{dB}{B} - (j+1)g_{j+1}(B\frac{dq}{q} + \frac{dB}{B}) \right) \eta_K^j\eta_c^{k-j} \\ &= \sum \left(\frac{d(g_jB^j)}{B^j} - (j+1)g_{j+1}(B\frac{dq}{q} + \frac{dB}{B}) \right) \eta_K^j\eta_c^{k-j}.\end{aligned} \tag{3.9}$$

Finally, if $s = \sum f_j \omega_c^j \eta_c^{k-j}$ is a rational section of $\underline{H}_{dR}^k(M/I)$ written in the (ω_c, η_c) basis, we have

$$\nabla(s) = \sum \left(df_j + (j+1) f_{j+1} \frac{dq}{q} \right) \omega_c^j \eta_c^{k-j}. \tag{3.10}$$

Rewriting s in either the (ω_c, η_K) basis or the (η_c, η_K) basis, we see that s is regular over I^0 if and only if the f_j are regular at ordinary points and one of the following two systems of inequalities are satisfied at each supersingular point x: either

$$\mathrm{ord}_x \left(\sum_{j \le l} \binom{k-j}{l-j} f_j B^{l-j} \right) \ge k - 2l \tag{3.11}$$

for $l = 0, \ldots, k$, or

$$\mathrm{ord}_x \left(\sum_{j \ge l} \binom{j}{l} f_j B^{-j} \right) \ge pk - l(p-1) \tag{3.12}$$

for $l = 0, \ldots, k$. Note that if $f_0, \ldots, f_i$ are functions satisfying the inequalities 3.11 for $l = 0, \ldots, i$, then we can find functions $f_{i+1}, \ldots, f_k$ such that $s = \sum f_j \omega_c^j \eta_c^{k-j}$ is a regular section of $\underline{H}_{dR}^k(M/I)$. Similarly, if $f_i, \ldots, f_k$ are functions satisfying the inequalities 3.12 for $l = i, \ldots, k$, then we can find functions $f_1, \ldots, f_{i-1}$ such that $s = \sum f_j \omega_c^j \eta_c^{k-j}$ is a regular section of $\underline{H}_{dR}^k(M/I)$.

4. Cohomology of some exact differentials For an integer i, let Z_X^i and B_X^{i+1} be the sheaves of closed and exact differentials on X, defined as the kernel and image of the homomorphism

$$\Omega_X^i \xrightarrow{d} \Omega_X^{i+1}.$$

We continue to use our notational convention regarding M: $R^j f_* \Omega_M^i$ means $\Pi R^j f_* \Omega_X^i$ and similarly for Z_M^i and B_M^i. In order to calculate the cohomology of logarithmic differentials we are going to eventually need fairly precise information on the sheaves $R^j f_* B_M^i$ and

$R^j f_* Z^i_M$. In this section we begin these calculations by considering the higher direct images $R^j f_* B^1_M$ and $R^j f_* B^{k+1}_M$. We will then prove Theorem 2.3a.

We need to review our calculation of the sheaves $R^j f_* \Omega^i_M$ from [U2]. First recall the sheaves $\underline{C}^i_k$: for $1 \le i \le k$, $\underline{C}^i_k$ is by definition the quotient of $\Omega^1_I \otimes \omega^{2i-2-k}$ by its subsheaf of sections vanishing to order at least $p-2$ at each supersingular point. Next, note that if $\mathcal{F}$ is a $(\mathbf{Z}/p\mathbf{Z})^\times$-equivariant sheaf on I (i.e., we are given maps $\langle d\rangle^* \mathcal{F} \to \mathcal{F}$ for every $d \in (\mathbf{Z}/p\mathbf{Z})^\times$ satisfying an obvious compatibility), then $(\mathbf{Z}/p\mathbf{Z})^\times$ acts on the stalks $\mathcal{F}_x$ at supersingular points x, since these points are fixed by the $\langle d\rangle$. If moreover $\mathcal{F}$ is a sheaf of $\mathbf{F}_p$ vector spaces then we can decompose $\mathcal{F}_x$ into eigenspaces for the action of $(\mathbf{Z}/p\mathbf{Z})^\times$. These remarks apply in particular to $\underline{C}^i_k$, $R^j f_* \Omega^i_M$, $R^j f_* Z^i_M$, and $R^j f_* B^i_M$.

Proposition 4.1. *a) We have isomorphisms of sheaves on I:*

$$R^j f_* \Omega^i_M \cong \begin{cases} \Omega^1_I \otimes \omega^k & \text{if } (i,j) = (k+1, 0) \\ \underline{C}^i_k & \text{if } i+j = k+1 \text{ and } 1 \le i \le k \\ \omega^{-k} & \text{if } (i,j) = (0,k) \\ 0 & \text{for all other } (i,j) \end{cases}$$

b) For $1 \le i \le k$,

$$\dim_{\mathbf{F}_p} (R^{k+1-i} f_* \Omega^i_M)_x(\chi^b) = \begin{cases} 1 & \text{if } b \not\equiv 0 \pmod{p-1} \\ 0 & \text{if } b \equiv 0 \pmod{p-1}. \end{cases}$$

Proof: This is essentially the main result of [U2], although it was not stated in this form there. To prove a), note that the arguments of Sections 4 and 5 of [U2] can be sheafified on I. Precisely, replacing X by $f^{-1}(U)$, where U is a Zariski open subset of I, the arguments of Section 4 relate the direct images of the Ω^i to the direct images of differentials on a certain log scheme $f^{-1}(U)^\times$ and the arguments of Section 5 compute these direct images in terms of the sheaves Ω^1_U and ω on U. As U varies, we obtain the desired isomorphisms of sheaves.

Part b) follows from a) and the fact that $(\mathbf{Z}/p\mathbf{Z})^\times$ acts on the cotangent space at a supersingular point via the character χ and acts trivially on a suitable generating section of ω at each supersingular point x (cf. [U2], 5.5 and 5.7). $\square$

We now want to compute Πf_* of the absolute Cartier operator $\mathcal{C}_X : F_{X*}Z_X^{k+1} \to \Omega_X^{k+1}$. Note that since X has dimension $k+1$, $Z_X^{k+1} = \Omega_X^{k+1}$. By Proposition 4.1, we have an isomorphism $f_*\Omega_M^{k+1} \cong \Omega_I^1 \otimes \omega^k$, and thus a map $f_*\mathcal{C}_X : F_{I*}(\Omega_I^1 \otimes \omega^k) \to \Omega_I^1 \otimes \omega^k$, or equivalently, a p^{-1}-linear map $f_*\mathcal{C}_X : \Omega_I^1 \otimes \omega^k \to \Omega_I^1 \otimes \omega^k$

Proposition 4.2. *Suppose that s is a section of $\Omega_I^1 \otimes \omega^k$ over some open set of I^0 of the form $s = \sigma\omega_c^k$ where σ is a regular section of Ω_I^1. Then $(f_*\mathcal{C})s = \mathcal{C}_I(\sigma)\omega_c^k$ where $\mathcal{C}_I : \Omega_I^1 = Z_I^1 \to \Omega_I^1$ is the (p^{-1}-linear) Cartier operator on I.*

Remark: Since $\Omega_I^1 \otimes \omega^k$ is a locally free sheaf of $\mathcal{O}_I^p$ modules and $f_*\mathcal{C}_X$ is $\mathcal{O}_I^p$-linear, the proposition completely determines $f_*\mathcal{C}_X$ on all of I (with the same formula).

Proof: First note that we can compute $f_*\mathcal{C}$ locally on X. If s is a section of Ω_X^{k+1} which can be written as $f^{-1}(\sigma) \wedge \tau$ where $\tau \in \Omega_{X/I}^k = Z_{X/I}^k$ has a lift to $\tilde{\tau} \in Z_X^k$, then

$$\mathcal{C}_X(s) = \mathcal{C}_X(f^{-1}(\sigma)) \wedge \mathcal{C}_X(\tilde{\tau}) = f^{-1}(\mathcal{C}_I(\sigma)) \wedge \mathcal{C}_X(\tilde{\tau}) = f^{-1}(\mathcal{C}_I(\sigma)) \wedge \mathcal{D}(\tilde{\tau})$$

where $\mathcal{D}$ is the composition of $\mathcal{C}_X : F_{I*}Z_X^k \to \Omega_X^k$ with the natural projection $\Omega_X^k \to \Omega_{X/I}^k$.

The following lemma gives a criterion in terms of Gauss-Manin and the conjugate filtration for such liftings $\tilde{\tau}$ to exist.

Lemma 4.3. *A section τ of $f_*\Omega_{X/I}^k = f_*Z_{X/I}^k$ over some open of I has a lift, locally on X, to a section of Z_X^k if and only if the image of τ under the composition*

$$f_*\Omega_{X/I}^k \to \underline{H}_{dR}^k(X/I) \xrightarrow{\nabla} \Omega_I^1 \otimes \underline{H}_{dR}^k(X/I)$$

lies in $\Omega^1_I \otimes F_{k-1}\underline{H}^k_{dR}(X/I)$.

Proof: This is an immediate consequence of the definition of ∇ as the coboundary in the long exact hypercohomology sequence of the short exact sequence of complexes

$$0 \to f^*\Omega^1_I \otimes \Omega^{\cdot -1}_{X/I} \to \Omega^{\cdot}_X \to \Omega^{\cdot}_{X/I} \to 0.$$

Indeed, working with Čech hypercochains for some cover, the class of τ in $\underline{H}^k_{dR}(X/I)$ is represented by $(\tau_i, 0, \ldots, 0)$ where τ_i is the restriction of τ to the i-th open of the cover. The image of this class under ∇ lands in the piece $\Omega^1_I \otimes F_{k-1}\underline{H}^k_{dR}(X/I)$ of the conjugate filtration if and only if (after refining the cover) the τ_i can be lifted to forms $\tilde{\tau}_i$ in Ω^k_X with $d\tilde{\tau}_i = 0$. □

Now the section ω^k_c of $f_*\Omega^k_{X/I}$ satisfies

$$\nabla(\omega^k_c) = k\omega^{k-1}_c \eta_c(dq/q) \in \Omega^1_I \otimes F_{k-1}\underline{H}^k_{dR}(X/I),$$

so it has locally on X a lift to a closed k-form.

Recall the diagram of Frobenius:

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/I}} & X' & \xrightarrow{G_{X/I}} & X \\ & \searrow & \downarrow & & \downarrow \\ & & I & \xrightarrow{F_I} & I \end{array}$$

where F_I is the absolute Frobenius of I and $F_X = G_{X/I} \circ F_{X/I}$ is the absolute Frobenius of X. From this we deduce the following diagram of sheaves on X:

$$\begin{array}{ccccc} F_{X*}Z^k_X & & \xrightarrow{\mathcal{C}_X} & & \Omega^k_X \\ \downarrow & & & & \downarrow \\ F_{X*}Z^k_{X/I} & = G_{X/I*}F_{X/I*}Z^k_{X/I} & \xrightarrow{G_{X/I*}\mathcal{C}_{X/I}} & G_{X/I*}\Omega^k_{X'/I} & \xleftarrow{G^*_{X/I}} \quad \Omega^k_{X/I}. \end{array}$$

It follows from the definitions of the Cartier operators that the diagram commutes. Now the map $G^*_{X/I}$ is injective, so to determine the image in $\Omega^k_{X/I}$ of a closed lift of ω_c^k to $F_{X*}Z^k_X$ which the lemma guarantees us, it is enough to determine the image of ω_c^k under $G_{X/I*}\mathcal{C}_{X/I}$, i.e., under the relative Cartier operator. But ω_c is the differential which is fixed by the relative Cartier operator, so the image of ω_c^k in $G_{X/I*}\Omega^k_{X'/I}$ is $\omega_c'^k = G^*_{X/I}(\omega_c^k)$. Thus $\mathcal{C}_X(f^{-1}(\sigma) \wedge \omega_c^k) = f^{-1}(\mathcal{C}_I(\sigma)) \wedge \omega_c^k$ and this completes the proof of Proposition 4.2. ☐

Applying Proposition 4.1 again, we have an isomorphism $R^k f_* \mathcal{O}_M \cong \omega^{-k}$ of sheaves on I. Thus we have a p^{-1}-linear map $R^k f_* F_X : \omega^{-k} \to \omega^{-k}$. Using Serre duality and the transpose property of $\mathcal{C}_X$ and F_X with respect to it (analogous to 3.3 above), we find the action of F_X on $R^k f_* \mathcal{O}_M$:

Corollary 4.4. *The map $R^k f_* F_X$ sends a section $f\omega_c^{-k}$ to $f^p\omega_c^{-k}$.*

We can now compute the direct images of B^{k+1}_X and B^1_X.

Theorem 4.5. *We have*

$$R^j f_* B^{k+1}_M = 0$$

for all $j \geq 2$. Let S be the reduced divisor of supersingular points on I. Then there is an isomorphism of sheaves on I:

$$R^1 f_* B^{k+1}_M \cong \frac{\Omega^1_I(kS)}{\Omega^1_I(S)}$$

and the right hand side is a skyscraper consisting of a $(k-1)$-dimensional vector space at each supersingular point. Also,

$$f_* B^{k+1}_M \cong B^1_I(kS),$$

the sheaf of exact differential 1-forms with poles of order at worst k at each supersingular point.

We also have isomorphisms:

$$R^j f_* B^1_M = 0$$

for all $j \neq k$ and

$$R^k f_* B^1_M \cong \frac{\mathcal{O}_I(-kS)}{\mathcal{O}_I(-kS)^p}.$$

Moreover, the sheaf $\mathcal{O}_I(-kS)/\mathcal{O}_I(-kS)^p$ is isomorphic to an extension of $B^1_I((1-k)S)$ by a skyscraper consisting of a vector space of dimension $k-1$ at each supersingular point.

(In this last sheaf, the $\mathcal{O}_I^p$-torsion sections near a supersingular point are represented by the classes of $t^p, t^{2p}, \ldots, t^{(k-1)p}$, where t is a uniformiser at the point.)

Proof: This follows immediately by taking the direct images of the exact sequences

$$0 \to B^{k+1}_X \to \Omega^{k+1}_X \xrightarrow{\mathcal{C}_X} \Omega^{k+1}_X \to 0$$

and

$$0 \to \mathcal{O}_X \xrightarrow{F_X} \mathcal{O}_X \xrightarrow{d} B^1_X \to 0,$$

and applying Propositions 4.1 and 4.2 and Corollary 4.4. $\square$

We now collect some easy consequences of Theorems 4.1 and 4.5. Note that Theorem 2.3a) follows immediately from parts a) and b) of the following result.

Proposition 4.6. *Under the standing hypotheses, we have:*

a) For all i and j with $i+j \neq k+1, k+2$, $H^j(M, Z^i) = H^j(M, B^i) = H^j(M, \Omega^i_{log}) = 0$. Similarly, $R^j f_* Z^i_M = R^j f_* B^i_M = R^j f_* \Omega^i_{M,log} = 0$ for $i+j \neq k+1, k+2$. The *sheaves* $R^{k+2-i}f_*Z^i_M$, $R^{k+2-i}f_*B^i_M$, *and* $R^{k+2-i}f_*\Omega^i_{log,M}$ *are supported at the supersingular points.*

b) $H^j(M, W\Omega^i_{log}) = 0$ *if* $i+j \neq k+1, k+2$ *and* $T^{ij} = 0$ *if* $i+j \neq k+1$.

c) For all $n \geq 1$, $\dim U_n^{i,k+1-i} = \dim U_n^{i,k+2-i}$.

d) $T^{i-1,k+2-i} = 0$ *if and only if* $\dim U_1^{i,k+1-i} = 0$.

e) $T^{i-1,k+2-i} \geq \dim U_1^{i,k+2-i} \geq \dim_{\mathbf{F}} H^{k+2-i}(M_a, Z^i)$

Proof: Consider the exact sequences

$$0 \to Z^i_X \to \Omega^i_X \xrightarrow{d} B^{i+1}_X \to 0, \tag{4.7}$$

$$0 \to B^i_X \to Z^i_X \xrightarrow{\mathcal{C}} \Omega^i_X \to 0, \tag{4.8}$$

and

$$0 \to \Omega^i_{log,X} \to Z^i_X \xrightarrow{1-\mathcal{C}} \Omega^i_X \to 0. \tag{4.9}$$

(Here 1 is the abusive but standard notation for the natural inclusion $Z^i_X \subseteq \Omega^i_X$.) It follows from Theorem 4.1 (and was proven already in [U2], 5.5) that $H^j(M, \Omega^i) = 0$ unless $i+j = k+1$. Taking cohomology of 4.7-9 on M immediately gives the first part of a). The rest follows similarly by taking the higher direct images of 4.7-9 and using 4.1 and 4.5.

Taking cohomology of the exact sequences

$$0 \to W_m\Omega^i_{X,log} \to W_{m+n}\Omega^i_{X,log} \to W_n\Omega^i_{X,log} \to 0,$$

and using induction, we have that if $i+j \neq k+1, k+2$, then $H^j(M, W\Omega^i_{log}) = 0$ and the groups U_n^{ij} and their inverse limit U^{ij} are zero. Thus $T^{ij} = 0$ unless $i+j = k$ or $k+1$. Also, for all $m, n \geq 1$ we have 6 term exact sequences

$$0 \to H^{k+1-i}(M, W_m\Omega^i_{log}) \to H^{k+1-i}(M, W_{m+n}\Omega^i_{log}) \to H^{k+1-i}(M, W_n\Omega^i_{log}) \to$$
$$H^{k+2-i}(M, W_m\Omega^i_{log}) \to H^{k+2-i}(M, W_{m+n}\Omega^i_{log}) \to H^{k+2-i}(M, W_n\Omega^i_{log}) \to 0. \tag{4.10}$$

But the inverse limit of the connected components $U_n^{i,k+2-i}$ of the groups representing $H^{k+2-i}(M, W_n\Omega^i_{log})$ is finite dimensional, and so the coboundary map in 4.10 must have a zero-dimensional kernel and cokernel for all sufficiently large m and n; this implies that $\varprojlim_n H^{k+1-i}(M, W_n\Omega^i_{log})$ is pro-finite and so $T^{i,k-i} = 0$. This completes the proof of b).

Before proving c), we note that the cohomology groups $H^j(M \times \text{Spec } A, \Omega^i)$, considered as functors on perfect $\mathbf{F}$ algebras A, are (trivially) represented by vector groups. If G represents $H^j(M \times \text{Spec } A, \Omega^i)$, then $\dim G = \dim_{\mathbf{F}} H^j(M, \Omega^i)$. Similar remarks apply with Ω^i replaced by any coherent sheaf, for example Z^i or B^i. Moreover, exact sequences of cohomology give rise to exact sequences of perfect group schemes.

To prove c), we note that taking cohomology of 4.7 and 4.8 on M, one finds that

$$\dim_{\mathbf{F}} H^{k+1-i}(M, B^i) = \dim_{\mathbf{F}} H^{k+2-i}(M, B^i)$$

and

$$\dim_{\mathbf{F}} H^{k+1-i}(M, Z^i) - \dim_{\mathbf{F}} H^{k+1-i}(M, \Omega^i) - \dim_{\mathbf{F}} H^{k+2-i}(M, Z^i) = 0$$

for all i. Taking cohomology of 4.9, and using the last displayed equation and the remarks above, we find that $\dim U_1^{i,k+1-i} = \dim U_1^{k+2-i}$. This proves c) for $n = 1$ and the general case follows from 4.10.

Now d) follows immediately from 4.10 and c). Taking cohomology of 4.9 on M and using 4.10 yields e) and this completes the proof of the proposition. $\square$

5. The d construction In this section we will study the deRham cohomology sheaves of M on I and the associated Hodge and conjugate spectral sequences. The former turns out not to degenerate, and we will need to determine the graded pieces of the filtration on the abutment. Moreover, the non-zero differentials will be used in Section 7 to construct interesting cohomology class with coefficients in closed or exact differentials.

To that end, we introduce the deRham cohomology sheaves of M on I: Let

$$\Omega^{\cdot}_X = \left(0 \to \mathcal{O}_X \to \Omega^1_X \to \cdots\right)$$

be the deRham complex of X and set

$$\underline{H}^j_{dR}(M) = R^j f_* \Omega^{\cdot}_M = \Pi R^j f_*(\Omega^{\cdot}_X);$$

these are sheaves of $\mathcal{O}^p_I$-modules. The Hodge filtration

$$F^i \Omega^{\cdot}_X = \left(0 \to \Omega^i_X \to \Omega^{i+1}_X \to \cdots\right)$$

and the conjugate filtration

$$F_i \Omega^{\cdot}_X = \left(\cdots \to \Omega^{i-1}_X \to Z^i_X \to 0\right)$$

give rise to two spectral sequences converging to $\underline{H}^*_{dR}(M)$, namely the Hodge spectral sequence

$$E_1^{ij} = R^j f_* \Omega^i_M = \Pi R^j f_* \Omega^i_X \Rightarrow \underline{H}^{i+j}_{dR}(M) \tag{5.1}$$

and (a shift of) the conjugate spectral sequence

$$E_2^{ij} = R^i f_* \mathcal{H}^j_M = \Pi R^i f_* \mathcal{H}^j_X \Rightarrow \underline{H}^{i+j}_{dR}(M) \tag{5.2}$$

where $\mathcal{H}^j_X = Z^j_X / B^j_X$. We denote the resulting filtrations on $\underline{H}^*_{dR}(M)$ by $F^{\cdot}$ and $F_{\cdot}$ and their gradeds by $\mathrm{Gr}^{\cdot}$ and $\mathrm{Gr}_{\cdot}$. Although these are the same notations used for the filtrations and gradeds on relative deRham cohomology, the context should make the difference clear.

The Cartier operator of X over $\mathbf{F}$ induces isomorphisms

$$F_{I*}R^i f_* \mathcal{H}^j_M \to R^i f_* \Omega^j_M$$

which we may also view as p^{-1}-linear maps $R^i f_* Z^j_M \to R^i f_* \Omega^j_M$. It follows immediately from these isomorphisms and Proposition 4.1 that the conjugate spectral sequence 5.2 degenerates at E_2 and we have $\underline{H}^i_{dR}(M) = 0$ if $i \neq k$, $k+1$. Also

$$\underline{H}^k_{dR}(M) \cong R^k f_* \mathcal{H}^0_M \xrightarrow[\sim]{\mathcal{C}} R^k f_* \mathcal{O}_M \cong \omega^{-k}.$$

On the other hand, as we will see presently, the Hodge spectral sequence 5.1 definitely does not degenerate at E_1. From the form of the initial term, it is clear that the only possibly non-zero maps in 5.1 are

$$d^{0k}_r : E^{0k}_r \to E^{r,k+1-r}_r = R^{k+1-r} f_* \Omega^r_M$$

and $E^{0k}_r = \mathrm{Ker}\ d^{0k}_{r-1}$ is a subsheaf of $R^k f_* \mathcal{O}_M$. To simplify, we write d_i for d^{0k}_i, $1 \leq i \leq k+1$.

Recall the Gauss-Manin connection on relative deRham cohomology discussed in Section 3. It sits in a 4-term exact sequence

$$0 \to H^k_{dR}(M) \to H^k_{dR}(M/I) \xrightarrow{\nabla} \Omega^1_I \otimes H^k_{dR}(M/I) \to H^{k+1}_{dR}(M) \to 0. \tag{5.3}$$

The following proposition relates the d_i to ∇.

Proposition 5.4. *Let ϕ be the projection*

$$\underline{H}^k_{dR}(M/I) \to \mathrm{Gr}^0\, \underline{H}^k_{dR}(M/I) \cong R^k f_* \mathcal{O}_M = E^{0k}_1.$$

By convention set Ker $d_0 = R^k f_* \mathcal{O}_M$.

a) For $0 \le i \le k$,

$$\phi\left(\nabla^{-1}(\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I))\right) \subseteq \text{Ker } d_i.$$

b) For $0 \le i \le k$, *the diagram*

$$\begin{tikzcd}
\nabla^{-1}(\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)) \arrow[r, "\nabla"] \arrow[ddd, "\phi"] & \Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I) \arrow[d] \\
 & \Omega^1_I \otimes \mathrm{Gr}^i \, \underline{H}^k_{dR}(M/I) \arrow[d, "\wr"] \\
 & \Omega^1 \otimes R^{k-i} f_* \Omega^i_{M/I} \arrow[d] \\
\text{Ker } d_i \arrow[r, "d_{i+1}"] & R^{k-i} f_* \Omega^{i+1}_M
\end{tikzcd}$$

commutes.

c) For $0 \le i \le k$, *the map*

$$\phi : \nabla^{-1}(\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)) \to \text{Ker } d_i$$

is surjective.

Proof: In fact, all these assertions actually hold at the level of the presheaves $U \mapsto H^k(f^{-1}U, \Omega^{\cdot}_{M/I})$, etc. Thus we fix an open $U \subseteq I$ and work with Čech hypercohomology over U. We will abuse notation slightly and write $\underline{H}^k_{dR}(M/I)$ for $\underline{H}^k_{dR}(M/I)(U)$, etc., and write D for the hypercoboundary map on both $\Omega^{\cdot}_X$ and $\Omega^{\cdot}_{X/I}$.

a) A class in $R^k f_* \mathcal{O}_M$ lies in Ker d_i if and only if it can be represented by a k-cocycle λ of $\mathcal{O}_X$ which can be extended to a k-hypercochain $\tilde{\lambda}$ of $\Omega^{\cdot}_X$ with $D\tilde{\lambda}$ a hypercocycle of $F^{i+1}\Omega^{\cdot}_X$. On the other hand, a class of $\underline{H}^k_{dR}(M/I)$ lies in $\nabla^{-1}(\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I))$ if and only if is represented by a k-hypercocycle μ of $\Omega^{\cdot}_{X/I}$ which has a lift $\tilde{\mu}$ to a hypercochain of $\Omega^{\cdot}_X$ with $D\tilde{\mu}$ a hypercocycle of $f^*\Omega^1_I \otimes F^i \Omega^{\cdot}_{X/I} \hookrightarrow F^{i+1}\Omega^{\cdot}_X$. Thus the image of the class of

μ in $R^k f_* \mathcal{O}_M$ clearly lies in Ker d_i. (We cannot yet conclude that we have equality in a) because the projection to $\Omega^{\cdot}_{M/I}$ of the hypercochain $\tilde{\lambda}$ might not define a hyperco*cycle*.)

b) Suppose μ represents a section of $\nabla^{-1}(\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I))$, i.e., μ is a k-hypercocycle of $\Omega^{\cdot}_{X/I}$ with a lift $\tilde{\mu}$ to a k-hypercochain of $\Omega^{\cdot}_X$ such that $D\tilde{\mu}$ is a hypercocycle of $f^*\Omega^1_I \otimes F^i\Omega^{\cdot}_{X/I}$. If $\tilde{\mu} = (\tilde{\mu}_0, \ldots, \tilde{\mu}_k)$ where $\tilde{\mu}_j$ is a Čech $(k-j)$-cochain of Ω^j_X, then the image of the class of $\nabla\tilde{\mu}$ in $R^{k-i} f_* \Omega^{i+1}_M$ is represented by $d\tilde{\mu}_i$ (up to a sign depending on conventions), where d is the deRham differential. But this is also the image under $d_{i+1} \circ \phi$ (up to the same conventions) of the class of μ_0 in $R^k f_* \mathcal{O}$.

c) We work by induction on i, the case $i = 0$ being easy: $\nabla^{-1}(\Omega^1_I \otimes F^0 \underline{H}^k_{dR}(M/I)) = \underline{H}^k_{dR}(M/I)$ and $\phi : \underline{H}^k_{dR}(M/I) \to R^k f_* \mathcal{O}_M$ is onto, by the degeneration of the relative Hodge to deRham spectral sequence. Now suppose $i \geq 1$. By b) and c) for $i-1$, a section of Ker d_i can be lifted to a section s of $\underline{H}^k_{dR}(M/I)$ whose image in $\Omega^1 \otimes \mathrm{Gr}^{\,i-1}\, \underline{H}^k_{dR}(M/I) = \Omega^1 \otimes R^{k+1-i} f_* \Omega^{i-1}_{M/I}$ lies in

$$\mathrm{Ker}\left(\Omega^1 \otimes R^{k+1-i} f_* \Omega^{i-1}_{M/I} \to R^{k+1-i} f_* \Omega^i_M\right).$$

But this kernel is equal to the image of the (Kodaira-Spencer) map

$$\mathrm{Gr}^{\,i}\, \underline{H}^k_{dR}(M/I) = R^{k-i} f_* \Omega^i_{M/I} \to \Omega^1_I \otimes R^{k+1-i} f_* \Omega^{i-1}_{M/I}$$

so modifying our class s in $\underline{H}^k_{dR}(M/I)$ by a section of $F^i \underline{H}^k_{dR}(M/I)$ (which lies in Ker $\phi = F^1 \underline{H}^k_{dR}(M/I)$ as $i \geq 1$), we can find a lift s whose image under ∇ lies in $\Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)$, as needed. $\square$

We can use this result to determine the image of d_i and thus the graded pieces of the Hodge filtration on $\underline{H}^{k+1}_{dR}(M)$. To state the result we introduce a convenient piece of notation which will reoccur several times. Let t be an "eigenuniformiser" at a supersingular point x: $\operatorname{ord}_x(t) = 1$ and $\langle c\rangle^* t = ct$ for all $c \in (\mathbf{Z}/p\mathbf{Z})^\times$. We let $W_x \subseteq \mathcal{O}_{I,x}$ be the subvector space of functions whose t-expansions involve only $t^{b(p-1)}$ with $b \geq 0$ (i.e., the $(\mathbf{Z}/p\mathbf{Z})^\times$-invariant subspace) and let V_x be $\mathcal{O}_{I,x}^p W_x$. We have $B \in t^{-p-1} W_x$ and $dq/q \in t^p W_x \, dt$. Also, both V_x and W_x are stable under multiplication: $V_x V_x \subseteq V_x$ and $W_x W_x \subseteq W_x$.

Proposition 5.5. (The d construction) *For $1 \leq i \leq k$ we have:*

a) A section $s = f\omega_c^{-k}$ of $R^k f_* \mathcal{O}_M$ lies in $\operatorname{Ker} d_i$ *if and only if at each supersingular point x, the inequalities*

$$\operatorname{ord}_x \left(\sum_{0 \leq j \leq l} \binom{k-j}{l-j} B^{l-j} \frac{(-1)^j}{j!} \theta^j(f) \right) \geq k - 2l$$

hold for $l = 0, \ldots, i$. If $i < k$, the image of such a section in $R^{k-i} f_* \Omega_M^{i+1}$ is represented by the section

$$-(i+1) \left(\sum_{0 \leq j \leq i+1} \binom{k-j}{i+1-j} B^{i+1-j} \frac{(-1)^j}{j!} \theta^j(f) \right) \frac{dq}{q} \omega_c^{2i-k}$$

of $\Omega_I^1 \otimes \omega^{2i-k}$.

b) The stalk of $\operatorname{Ker} d_i$ *at a supersingular point x is spanned by $t^k V_x \omega_c^{-k}$ and the sections $s = f\omega_c^{-k}$ with $\operatorname{ord}_x(f) \geq k + i(p-1)$. If f is a function near x with $\operatorname{ord}_x(f) = l \geq k + (i-1)(p-1)$ and f is an eigenvector for the $(\mathbf{Z}/p\mathbf{Z})^\times$ action, then the image under d_i of $f\omega_c^{-k}$ in $R^{k+1-i} f_* \Omega_M^i$ is represented by*

$$-i \binom{k-l}{i} B^i f \frac{dq}{q} \omega_c^{2i-2-k}.$$

c) The image of d_i in $R^{k+1-i} f_* \Omega_M^i \cong \underline{C}_k^i$ consists of those classes represented by sections of $\Omega_I^1 \otimes \omega^{2i-2-k}$ vanishing to order at least $(i-1)$.

Proof: By Proposition 5.4, $s = f\omega_c^{-k}$ lies in Ker d_i if and only if there exists a section $z = \sum_j f_j \omega_c^j \eta_c^{k-j}$ of $\underline{H}^k_{dR}(M/I)$ with $f_0 = f$ and $\nabla(z) \in \Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)$. But

$$\nabla(z) = \sum_j (df_j + (j+1) f_{j+1} \frac{dq}{q}) \omega_c^j \eta_c^{k-j}$$

so $\nabla(z) \in \Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)$ if and only if $f_j = \frac{(-1)^j}{j!}\theta^j(f)$ for $j \leq i$. Then the conditions 3.11 for z to be regular are exactly the stated inequalities on f. Conversely, if the inequalities hold, then setting $f_j = \frac{(-1)^j}{j!}\theta^j(f)$ for $j \leq i$ we can choose functions $f_{i+1}, \ldots, f_k$ so that $z = \sum f_j \omega_c^j \eta_c^{k-j}$ is a regular section of $\underline{H}^k_{dR}(M/I)$ with $\nabla(z) \in \Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)$ and so $s \in$ Ker d_i.

Again by Proposition 5.4, if $s = f\omega_c^{-k}$ is a section of Ker d_i and $z = \sum_j f_j \omega_c^j \eta_c^{k-j}$ is a section of $\underline{H}^k_{dR}(M/I)$ with $f_0 = f$ and $\nabla(z) \in \Omega^1_I \otimes F^i \underline{H}^k_{dR}(M/I)$, then the image of s in $R^{k-i} f_* \Omega^{i+1}_M$ is represented by the section

$$(df_i + (i+1) f_{i+1} \frac{dq}{q}) \omega_c^{2i-k}.$$

We have already seen that $df_i = d\frac{(-1)^i}{i!}\theta^i(f) = \frac{(-1)^i}{i!}\theta^{i+1}(f)\frac{dq}{q}$; on the other hand, the conditions 3.11 for z to be regular imply that f_{i+1} is congruent to

$$-\sum_{j<i+1} \binom{k-j}{i+1-j} B^{i+1-j} \frac{(-1)^j}{j!}\theta^j(f)$$

modulo functions vanishing to order $k - 2i - 2$ at each supersingular point. This implies that $(df_i + (i+1) f_{i+1} \frac{dq}{q}) \omega_c^{2i-k}$ is congruent to

$$-(i+1) \left(\sum_{j \leq i+1} \binom{k-j}{i+1-j} B^{i+1-j} \frac{(-1)^j}{j!}\theta^j(f) \right) \frac{dq}{q} \omega_c^{2i-k}$$

modulo sections of $\Omega^1_I \otimes \omega^{2i-k}$ vanishing to order at least $p-2$ at each supersingular point. These two sections of $\Omega^1_I \otimes \omega^{2i-k}$ thus define the same section of $R^{k-i} f_* \Omega^{i+1}_M = \underline{C}^{i+1}_k$. This completes the proof of part a) of the proposition.

Now suppose $\mathrm{ord}_x(f) = l \geq k + (i-1)(p-1)$ at each supersingular point x. Then the inequalities of part a) are clearly satisfied for $l \leq i-1$ (using Lemma 3.5) so $s = f\omega_c^{-k}$ is a section of Ker d_{i-1}. To compute the image of s under d_i, consider an eigenuniformiser t at a supersingular point x, as above. Then $B = ct^{-p-1} + \cdots$ for some non-zero c in the residue field at x and $dq/q = c^{-1}t^p dt + \cdots$ (since the residue of Bdq/q is 1); also $f = bt^l + \cdots$. The leading term of

$$-i\left(\sum_{j\leq i}\binom{k-j}{i-j}B^{i-j}\frac{(-1)^j}{j!}\theta^j(f)\right)$$

is then

$$\begin{aligned}
-ic^i b\sum\binom{k-j}{i-j}&\frac{(-1)^j}{j!}l(l-1)\cdots(l-j+1)t^{l-i(p+1)}\\
&= -ic^i b\sum(-1)^j\binom{k-j}{i-j}\binom{l}{j}t^{l-i(p+1)}\\
&= -ic^i b\binom{k-l}{i}t^{l-i(p+1)}.
\end{aligned}$$

Taking $l = k+p(i-1)+1, \ldots, k+i(p-1)-1$, $d_i(s)$ gives non-zero sections of $R^{k+1-i}f_*\Omega^i_M$ vanishing to orders $i-1, \ldots, p-3$. This shows that the image of d_i is at least as large as claimed. It also verifies the formula in part b) for the image of a section $f\omega_c^{-k}$, since the last displayed expression is congruent to $-i\binom{k-l}{i}B^i f$ modulo functions vanishing to order $k-2i$ at x.

Note also that if for $1 \leq i \leq k-1$, V_i denotes the subspace of the stalk $(R^k f_*\mathcal{O}_M)_x$ generated over the ground field by $t^l\omega_c^{-k}$ where $l = k+p(i-1)+1, \ldots, k+i(p-1)-1$, then we have just seen that V_i lies in the kernel of d_{i-1} and d_i is injective on V_i. Write V_k for the subspace generated by $(t^k V_x\omega_c^{-k})$ and $f\omega_c^{-k}$ where $\mathrm{ord}_x(f) \geq pk$. Since $(R^k f_*\mathcal{O}_M)_x = V_1 \oplus \cdots \oplus V_k$, to show that the image of d_i is no larger than claimed, and that the kernel of d_i is as claimed, it suffices to prove that V_k is in the kernel of d_i for $i = 1, \ldots, k$.

We have already seen that if $\mathrm{ord}_x(f) \geq pk$, then $f\omega_c^{-k}$ is in the kernel of the d_i. Next, note that $t^k W_x \omega_c^k$ is exactly the subspace of the stalk of $R^k f_* \mathcal{O}_M$ which is invariant under the $(\mathbf{Z}/p\mathbf{Z})^\times$ action, and d_i is $(\mathbf{Z}/p\mathbf{Z})^\times$-equivariant. But for $i = 1, \ldots, k$, by 4.1 we have $(R^{k+1-i} f_* \Omega_M^i)_x(\chi^0) = 0$. Thus $t^k W_x \omega_c^k$ is in the kernel of each d_i. But the d_i are $\mathcal{O}_I^p$-linear, so we also have $t^k V_x \omega_c^{-k} \subseteq \mathrm{Ker}\ d_i$. This completes the proof of the proposition. $\square$

6. Skyscraper contributions to $T^{i,j}$ In this section we will use the 5 term exact sequences

$$0 \to R^{k-i} f_* B_M^{i+1} \to R^{k+1-i} f_* Z_M^i \to R^{k+1-i} f_* \Omega_M^i \to R^{k+1-i} f_* B_M^{i+1} \to R^{k+2-i} f_* Z_M^i \to 0 \tag{6.1}$$

and

$$0 \to R^{k+1-i} f_* B_M^i \to R^{k+1-i} f_* Z_M^i \to R^{k+1-i} f_* \Omega_M^i \to R^{k+2-i} f_* B_M^i \to R^{k+2-i} f_* Z_M^i \to 0 \tag{6.2}$$

to analyze the sheaves $R^{k+2-i} f_* Z_M^i$ and $R^{k+2-i} f_* B_M^i$. As we have seen, they are skyscrapers supported at the supersingular points and using them we will prove the lower bounds on the T^{ij} asserted in Theorem 2.3b. In order to get more precise information on their stalks at the supersingular points, we will have to analyze the cokernels of the maps

$$R^{k+1-i} f_* Z_M^i \xrightarrow{1_i} R^{k+1-i} f_* \Omega_M^i$$

and

$$R^{k+1-i} f_* Z_M^i \xrightarrow{\mathcal{C}_i} R^{k+1-i} f_* \Omega_M^i$$

induced by the natural inclusion $Z_X^i \subseteq \Omega_X^i$ and the Cartier operator $\mathcal{C} : Z_X^i \to \Omega_X^i$ respectively. We will eventually do this by using the relation between the two filtrations on the deRham cohomology sheaves $\underline{H}_{dR}^{k+1}(M)$. In this direction, we first prove two lemmas about the maps appearing in the Gauss-Manin sequence 5.3.

Lemma 6.3. *For $1 \leq i \leq k+1$, the composition*

$$\Omega^1 \otimes F^{i-1}\underline{H}_{dR}^k(M/I) \to F^i\underline{H}_{dR}^{k+1}(M) \to \mathrm{Gr}^{\,i}\, \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i}f_*\Omega_M^i/\,\mathrm{Im}\ d_i$$

is surjective. It sends the section $\sigma_{i-1}\omega_c^{i-1}\eta_K^{k+1-i} + \cdots + \sigma_k\omega_c^k$ to the class represented by $\sigma_{i-1}\omega_c^{2i-2-k}$. The composition

$$\begin{aligned}\Omega^1 \otimes F_{i-1}\underline{H}_{dR}^k(M/I) &\to F_i\underline{H}_{dR}^{k+1}(M) \\ &\to \mathrm{Gr}_{\,i}\, \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i}f_*\mathcal{H}_M^i \xrightarrow[\sim]{\mathcal{C}} R^{k+1-i}f_*\Omega_M^i\end{aligned}$$

is surjective. It sends the section $\tau_{i-1}\eta_K^{i-1}\eta_c^{k+1-i} + \cdots + \tau_0\eta_c^k$ to the class represented by $\mathcal{C}_I((-B)^{i-1}\tau_{i-1})\omega_c^{2i-2-k}$.

Proof: In both cases, the surjectivity is clear from the formula for the map and our description of regular sections of $\underline{H}_{dR}^k(M/I)$. To check the first formula, consider the commutative diagram

$$\begin{array}{ccc} \Omega^1 \otimes F^{i-1}\underline{H}_{dR}^k(M/I) & \to & F^i\underline{H}_{dR}^{k+1}(M) \\ \downarrow & & \downarrow \\ \Omega_I^1 \otimes \mathrm{Gr}^{\,i-1}\, \underline{H}_{dR}^k(M/I) & & \mathrm{Gr}^{\,i}\, \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i}f_*\Omega_M^i/\,\mathrm{Im}\ d_i \\ \downarrow \wr & & \uparrow \\ \Omega^1 \otimes R^{k+1-i}f_*\Omega_{M/I}^{i-1} & \to & R^{k+1-i}f_*\Omega_M^i \end{array}$$

Here the top row comes from the Gauss-Manin sequence 5.3, the bottom row comes from the cohomology of the exact sequence of relative differentials, and the vertical maps are the natural projections. The isomorphism between $R^{k+1-i}f_*\Omega_M^i$ and a quotient of $\Omega_I^1 \otimes \omega^{2i-2-k}$ is induced by the bottom row, and we have seen (after Lemma 3.7) that the image of

$\sigma\omega_c^{i-1}\eta_K^{k+1-i}$ under $\Omega^1_I \otimes F^{i-1}\underline{H}^k_{dR}(M/I) \to \Omega^1_I \otimes R^{k+1-i}f_*\Omega^{i-1}_{M/I} \cong \Omega^1_I \otimes \omega^{2i-2-k}$ is just $\sigma\omega_c^{2i-2-k}$. This proves the first formula.

To check the second formula, consider the commutative diagram

$$\begin{array}{ccccc}
\Omega^1 \otimes F_{i-1}\underline{H}^k_{dR}(M/I) & & \to & & F_i\underline{H}^{k+1}_{dR}(M) \\
\downarrow & & & & \downarrow \\
\Omega^1 \otimes \mathrm{Gr}_{\,i-1}\,\underline{H}^k_{dR}(M/I) & & \to & & \mathrm{Gr}_{\,i}\,\underline{H}^{k+1}_{dR}(M) \cong R^{k+1-i}f_*\mathcal{H}^i_M \\
\downarrow & & & & \downarrow \wr\, \mathcal{C}_X \\
\Omega^1 \otimes R^{k+1-i}f'_*\Omega^{i-1}_{M'/I} & \to & R^{k+1-i}f'_*\Omega^i_{M'} & \longleftarrow & R^{k+1-i}f_*\Omega^i_M
\end{array}$$

where to simplify we have omitted F_{I*} or F_{X*} in several places. In the top square, the vertical maps are the natural projections and the horizontal maps come from the exact sequence of relative differentials. We have seen (again, after Lemma 3.7) that the image of $\tau_{i-1}\eta_K^{i-1}\eta_c^{k+1-i} + \ldots + \tau_0\eta_K^k$ under the top left vertical map is $(-B)^{i-1}\tau_{i-1}\omega_c^{p(2i-2-k)}$. In the bottom square, the right vertical map is induced by $\mathcal{C}_X$ and the horizontal maps are induced by base change and the wedge product of forms. A discussion similar to that in the proof of 4.2 shows that the lower square commutes where the left vertical map is the one sending $\tau\omega_c^{p(2i-2-k)}$ to $\mathcal{C}_I(\tau)\omega_c'^{(2i-2-k)}$. This proves the desired formula. $\square$

As a corollary, we have that

$$\Omega^1_I \otimes F^{i-1}\underline{H}^k_{dR}(M/I) \to F^i\underline{H}^{k+1}_{dR}(M)$$

and

$$\Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I) \to F_i\underline{H}^{k+1}_{dR}(M)$$

are surjective for all i. In particular, if we write $(F^i \cap F_i)\underline{H}^{k+1}_{dR}(M)$ for $(F^i\underline{H}^{k+1}_{dR}(M)) \cap (F_i\underline{H}^{k+1}_{dR}(M))$, then a section of $\underline{H}^{k+1}_{dR}(M)$ lies in $(F^i \cap F_i)\underline{H}^{k+1}_{dR}(M)$ if and only if it has a lift to a section $s \in \Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I)$ such that there exists a section $z \in \underline{H}^k_{dR}(M/I)$ with $s - \nabla(z) \in \Omega^1_I \otimes F^{i-1}\underline{H}^k_{dR}(M/I)$. The following lemma gives one possible sufficient condition for such a z to exist.

Lemma 6.4. *Suppose* $s = \sigma_0\eta_K^k + \cdots + \sigma_k\omega_c^k$ *is a section of* $\Omega_I^1 \otimes \underline{H}_{dR}^k(M/I)$ *such that for some* l *and all* $j \leq l$, σ_j *vanishes to order at least* $k - 2j + (p-1)(l-j)$. *Then there exists a section* $z \in \underline{H}_{dR}^k(M/I)$ *such that* $s - \nabla(z) \in \Omega_I^1 \otimes F^l\underline{H}_{dR}^k(M/I)$.

Proof: Consider the largest integer m such that s lies in $\Omega_I^1 \otimes F^m\underline{H}_{dR}^k(M/I)$; if $m \geq l$ there is nothing to prove. If $m < l$, set $z = x\omega_c^{m+1}\eta_K^{k-m-1}$ with

$$x = \frac{\sigma_m}{(m+1)\frac{dq}{q}}.$$

Then x vanishes at each supersingular point to order at least $k - 2m + (p-1)(l-m) - p > k - 2(m+1)$ so z is a regular section. Moreover,

$$\begin{aligned} s - \nabla(z) = &\left(\sigma_{m+1} - dx + (k-2m-2)xB\frac{dq}{q}\right)\omega_c^{m+1}\eta_K^{k-m-1} \\ &+ \left(\sigma_{m+2} - (k-m-1)x(B^2\frac{dq}{q} + dB)\right)\omega_c^{m+2}\eta_K^{k-m-2} \\ &+ \sum_{j=m+3}^{k} \sigma_j\omega_c^j\eta_K^{k-j} \end{aligned}$$

and Lemma 3.4 shows that $s - \nabla(z)$ again satisfies the hypotheses of this lemma, but lies in $\Omega_I^1 \otimes F^{m+1}\underline{H}_{dR}^k(M/I)$. Thus the lemma follows by induction. $\square$

The following proposition, which gives some information on the relationship between the two filtrations on the deRham cohomology sheaves of M, is the key point in analyzing the images of 1_i and $\mathcal{C}_i$.

Proposition 6.5. *a) For* $1 \leq i \leq (k+2)/2$, *the map*

$$(F^i \cap F_i)\underline{H}_{dR}^{k+1}(M) \to \operatorname{Gr}^i \underline{H}_{dR}^{k+1}(M)$$

is zero and the map

$$(F^i \cap F_i)\underline{H}_{dR}^{k+1}(M) \to \operatorname{Gr}_i \underline{H}_{dR}^{k+1}(M)$$

is surjective.

b) For $(k+3)/2 \leq i \leq k$ the image of the map

$$(F^i \cap F_i)\underline{H}_{dR}^{k+1}(M) \to \mathrm{Gr}^{\,i}\, \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i} f_* \Omega_M^i / \operatorname{Im} d_i$$

is contained in a subsheaf of codimension at least $k+1-i$ at each supersingular point. Precisely, the image is contained in the subsheaf represented by sections $\sigma\omega_c^{2i-2-k}$ of $\Omega_I^1 \otimes \omega^{2i-2-k}$ such that for $1 \leq j \leq k+1-i$, at each supersingular point $B^j\sigma$ is the sum of an exact differential and a differential vanishing to order at least $-pj$.

c) For $(k+3)/2 \leq i \leq k+1$ the image of the map

$$(F^i \cap F_i)\underline{H}_{dR}^{k+1}(M) \to \mathrm{Gr}_{\,i}\, \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i} f_* \Omega_M^i$$

contains the subsheaf of sections vanishing to order at least $2i-k-3$ at each supersingular point.

Remark: Using the fact that B is an eigenvector for the action of $(\mathbf{Z}/p\mathbf{Z})^\times$, the condition on σ in part b) can be rephrased as follows: at each supersingular point x, if t is an eigenuniformiser at x and $\sigma = \sum_{n \geq k+2-2i} a_n t^n \, dt$, then $a_0 = \cdots = a_{k-i} = 0$.

Proof: We are going to use relative deRham cohomology to construct sections and impose restrictions. We begin with parts a) and c) which follow fairly easily from the two lemmas. Indeed, using Lemma 6.3, a section of $\mathrm{Gr}_{\,i}\, \underline{H}_{dR}^{k+1}(M)$ can be lifted to $s = \tau\eta_K^{i-1}\eta_c^{k+1-i}$ in $\Omega_I^1 \otimes F_{i-1}\underline{H}_{dR}^k(M/I)$ with τ vanishing to order at least $k+(p-1)(k+2-i)$ at each supersingular point. Rewriting s in the ω_c, η_K basis:

$$s = \tau(\eta_K - B\omega_c)^{k+1-i}\eta_K^{i-1}$$

$$= \sum \sigma_j \omega_c^j \eta_K^{k-j}$$

where $\sigma_j = (-B)^j \binom{k+1-i}{j}\tau$, we have that σ_j vanishes to order at least $k - 2j + (p - 1)(k + 2 - i - j)$ for all $j \leq k + 2 - i$. By Lemma 6.4, there exists $z \in \underline{H}^k_{dR}(M/I)$ so that $s - \nabla(z)$ lies in $\Omega^1_I \otimes F^{k+2-i}\underline{H}^k_{dR}(M/I)$ and thus the image of s in $\underline{H}^{k+1}_{dR}(M)$ lies in $F^{k+3-i}\underline{H}^{k+1}_{dR}(M)$.

This shows that $F_i\underline{H}^{k+1}_{dR}(M) \subseteq F^{k+3-i}\underline{H}^{k+1}_{dR}(M)$, and so $(F^i \cap F_i)\underline{H}^{k+1}_{dR}(M) \to \mathrm{Gr}_i\,\underline{H}^{k+1}_{dR}(M)$ is surjective if $i \leq (k+3)/2$ and $(F^i \cap F_i)\underline{H}^{k+1}_{dR}(M) \to \mathrm{Gr}^i\,\underline{H}^{k+1}_{dR}(M)$ is zero if $i \leq (k+2)/2$. This proves part a) of the Proposition and one case of c). To prove all of c), we argue similarly: a section of $\mathrm{Gr}_i\,\underline{H}^{k+1}_{dR}(M)$ which vanishes to order $2i - k - 3$ can be lifted to a section $s = \tau\eta_K^{i-1}\eta_c^{k+1-i}$ with τ vanishing to order at least $(p+1)(i-1) - 1 \geq k + (p-1)(i-1)$. Rewriting as above and using Lemma 6.4 provides a z so that $s - \nabla(z) \in \Omega^1_I \otimes F^{i-1}\underline{H}^k_{dR}(M/I)$.

Part b) will require more work. Suppose that $(k+3)/2 \leq i \leq k$ and that we are given a section in the image of $(F^i \cap F_i)\underline{H}^{k+1}_{dR}(M) \to \mathrm{Gr}^i\,H^{k+1}_{dR}(M)$ near some supersingular point x. By the remarks before Lemma 6.4, the section can be lifted to a section

$$s = \sigma_{i-1}\omega_c^{i-1}\eta_K^{k+1-i} + \cdots + \sigma_k\omega_c^k$$

of $\Omega^1_I \otimes F^{i-1}\underline{H}^k_{dR}(M/I)$ such that there exists a section

$$z = \sum g_j \eta_K^j \eta_c^{k-j}$$

of $\underline{H}^k_{dR}(M/I)$ with $s - \nabla(z) \in \Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I)$. We will show that the existence of z places restrictions on $\sigma = \sigma_{i-1}$. Rewriting s in the (η_c, η_K) basis, we have

$$s = \sum_{j=0}^{k} \left(\binom{i-1}{k-j} (-1)^{k+1-i-j} B^{1-i}\sigma + \tau_j \right) \eta_K^j \eta_c^{k-j}$$

with

$$\tau_j = \sum_{l \geq i} \binom{l}{k-j} (-1)^{k-j-l} B^{-l} \sigma_l;$$

note that $\mathrm{ord}_x(\tau_j) \geq k + (p-1)i$ at each supersingular point x for all j. Using that $s - \nabla(z) \in \Omega^1_I \otimes F_{i-1} \underline{H}^k_{dR}(M/I)$, we have that

$$\begin{aligned}
(-1)^{i-1} B^{1-i} \sigma + \tau_k &= \frac{d(B^k g_k)}{B^k} \\
(-1)^{i-2} \binom{i-1}{1} B^{1-i} \sigma + \tau_{k-1} &= \frac{d(B^{k-1} g_{k-1})}{B^{k-1}} - k g_k \left(B \frac{dq}{q} + \frac{dB}{B}\right) \\
&\vdots \\
(-1)^{k-1} \binom{i-1}{k-i} B^{1-i} \sigma + \tau_i &= \frac{d(B^i g_i)}{B^i} - (i+1) g_{i+1} \left(B \frac{dq}{q} + \frac{dB}{B}\right)
\end{aligned}$$

or, clearing denominators,

$$\begin{aligned}
(-1)^{i-1} B^{k+1-i} \sigma + B^k \tau_k &= d(B^k g_k) \\
(-1)^{i-2} \binom{i-1}{1} B^{k-i} \sigma + B^{k-1} \tau_{k-1} &= d(B^{k-1} g_{k-1}) - k B^k g_k \left(\frac{dq}{q} + B^{-2} dB\right) \\
&\vdots \\
(-1)^{k-1} \binom{i-1}{k-i} B \sigma + B^i \tau_i &= d(B^i g_i) - (i+1) B^{i+1} g_{i+1} \left(\frac{dq}{q} + B^{-2} dB\right).
\end{aligned} \tag{6.6}$$

We will show that these equations force the desired restrictions on σ. Let us refer to the equation containing $B^\ell \tau_\ell$ as 6.6_ℓ, and let us set $g_{k+1} = 0$ by convention.

Let t be an eigenuniformiser at x and let $W = W_x$ and $V = V_x$ be the subspaces of $\mathcal{O}_{I,x}$ defined in Section 5: W is the space of $(\mathbf{Z}/p\mathbf{Z})^\times$-invariants and $V = \mathcal{O}^p_{I,x} W$. We will use the following facts:

(i) $\mathrm{ord}_x(B^{\ell+1-i}\sigma) \geq k - 2i + 2 - (\ell + 1 - i)(p+1) > -p(\ell + 2 - i)$

(ii) $\mathrm{ord}_x(B^\ell \tau_\ell) \geq k + (p-1)i - \ell(p+1) > -p(\ell + 1 - i)$

(iii) $\mathrm{ord}_x(B^\ell g_\ell) \geq p(k - 2\ell)$

(iv) $(dq/q + B^{-2}dB) \in t^{3p-2}W dt$

(i)-(iii) follow from the regularity conditions for sections of $\underline{H}^k_{dR}(M/I)$ and (iv) follows from Lemma 3.4. To finish the proof, it will suffice to check the following claim: for $i \le \ell \le k$ we have

$$B^{\ell+1}g_{\ell+1}(\frac{dq}{q} + B^{-2}dB) \in t^{p(k-2\ell+1)-2}V dt + t^{-p(\ell+1-i)}\mathcal{O}_{I,x}dt.$$

Indeed, the smallest valuation of a nonexact differential in $t^{p(k-2\ell+1)-2}V dt$ is $p(k-2\ell+p-1)-1$ which is $> -p$. Thus 6.6_ℓ and (ii) show that $B^{\ell+1-i}\sigma$ is the sum of an exact differential and one vanishing to order at least $-p(\ell+1-i)$.

We verify the claim by descending induction on ℓ, the case $\ell = k$ being trivial as $g_{k+1} = 0$. So suppose

$$B^{\ell+1}g_{\ell+1}(\frac{dq}{q} + B^{-2}dB) \in t^{p(k-2\ell+1)-2}V dt + t^{-p(\ell+1-i)}\mathcal{O}_{I,x}dt.$$

Then 6.6_ℓ, (i), and (ii) imply that

$$d(B^\ell g_\ell) \in t^{p(k-2\ell+1)-2}V dt + t^{-p(\ell+2-i)}\mathcal{O}_{I,x}dt$$

and so by (iii)

$$\begin{aligned} B^\ell g_\ell &\in t^{p(k-2\ell)}\mathcal{O}^p_{I,x} + t^{p(k-2\ell+1)-1}V + t^{-p(\ell+2-i)+1}\mathcal{O}_{I,x} \\ &\subseteq t^{p(k-2\ell)}V + t^{-p(\ell+2-i)+1}\mathcal{O}_{I,x}. \end{aligned}$$

Finally, using (iv), we have

$$B^\ell g_\ell(\frac{dq}{q} + B^{-2}dB) \in t^{p(k-2\ell+3)-2}V dt + t^{-p(\ell-1-i)-1}\mathcal{O}_{I,x}dt$$

which is more than enough to finish the induction. This completes the proof of Proposition 6.5. $\square$

We can now compute the dimensions of the stalks of $R^{k+2-i}f_*Z^i_M$ and $R^{k+2-i}f_*B^i_M$. We recall that by Proposition 4.1, for $1 \le i \le k$, $R^{k+1-i}f_*\Omega^i_M = \underline{C}^i_k$ where $\underline{C}^i_k$ is the quotient of $\Omega^1_I \otimes \omega^{2i-2-k}$ by its subsheaf of sections vanishing to order at least $p-2$ at each supersingular point; we have

$$\dim_{\mathbf{F}_p}(R^{k+1-i}f_*\Omega^i_M)_x(\chi^b) = \begin{cases} 1 & \text{if } b \not\equiv 0 \pmod{p-1} \\ 0 & \text{if } b \equiv 0 \pmod{p-1}. \end{cases}$$

Similarly, using Theorem 4.5, we have

$$\dim_{\mathbf{F}_p}(R^1 f_* B^{k+1}_M)_x(\chi^b) = \begin{cases} 1 & \text{if } b \equiv 1, \ldots, k-1 \pmod{p-1} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.7. *For each supersingular point x of I, and for each i and a with $1 \le i \le k+1$, $0 < a < p-1$ we have*

$$\dim_{\mathbf{F}}(R^{k+2-i}f_*Z^i_M)_x(\chi^a) = \begin{cases} 0 & \text{if } i \le a+1 \\ i-a-1 & \text{if } a+1 \le i \le (k+a+2)/2 \\ k+1-i & \text{if } (k+a+2)/2 \le i \end{cases}$$

and

$$\dim_{\mathbf{F}}(R^{k+2-i}f_*B^i_M)_x(\chi^a) = \begin{cases} 0 & \text{if } i \le a+1 \\ i-a-1 & \text{if } a+1 \le i \le (k+a+3)/2 \\ k+2-i & \text{if } (k+a+3)/2 \le i. \end{cases}$$

Proof: Note first of all that we have a map

$$R^{k+1-i}f_*Z^i_M \to (F^i \cap F_i)\underline{H}^{k+1}_{dR}(M)$$

which sends a Čech $(k+1-i)$-cocycle λ with coefficients in Z^i_X to the hypercocycle $(0, \ldots, \lambda, \ldots, 0)$. One checks easily that this map is well-defined and surjective. Moreover, the map $\mathcal{C}_i$ obviously factors as

$$R^{k+1-i}f_*Z^i_M \to (F^i \cap F_i)\underline{H}^{k+1}_{dR}(M) \to \mathrm{Gr}_i\, \underline{H}^{k+1}_{dR}(M) \cong R^{k+1-i}f_*\Omega^i_M$$

and the composition

$$R^{k+1-i}f_*Z^i_M \xrightarrow{1_i} R^{k+1-i}f_*\Omega^i_M \to R^{k+1-i}f_*\Omega^i_M/\operatorname{Im} d_i$$

factors as

$$R^{k+1-i}f_*Z^i_M \to (F^i \cap F_i)\underline{H}^{k+1}_{dR}(M) \to \operatorname{Gr}^i \underline{H}^{k+1}_{dR}(M) \cong R^{k+1-i}f_*\Omega^i_M/\operatorname{Im} d_i.$$

As the image of 1_i certainly contains the image of d_i, we can compute the images of 1_i and $\mathcal{C}_i$ by using Proposition 6.5.

Indeed, Proposition 6.5 plus the computation of $R^{k+1-i}f_*\Omega^i_M$ as $\underline{C}^i_k$ implies that the image of

$$R^{k+1-i}f_*\Omega^i_M \xrightarrow{\delta^i} R^{k+2-i}f_*B^i_M$$

is supported at the supersingular points and its stalk at each supersingular point is an $\mathbf{F}$ vector space of dimension 0 if $1 \le i \le (k+2)/2$ and of dimension at most $2i-k-3$ if $(k+3)/2 \le i \le k+1$. Similarly, using 5.5c) (on the image of d_i) and 6.5, we have that the image of

$$R^{k+1-i}f_*\Omega^i_M \xrightarrow{d^i} R^{k+1-i}f_*B^{i+1}_M$$

is supported at the supersingular points and its stalk at each supersingular point is an $\mathbf{F}$ vector space of dimension $i-1$ if $1 \le i \le (k+2)/2$ and of dimension at least $k+1-i$ if $(k+3)/2 \le i \le k+1$.

On the other hand, the sequences 6.1 and 6.2 imply that at each supersingular point the sum (over i) of the dimensions of the stalk of Im δ^i must be equal to the sum of the dimensions of the stalk of Im d^i. So at each supersingular x,

$$\sum_{i=1}^{(k+2)/2} i-1+\sum_{i=(k+3)/2}^{k+1} k+1-i \le \sum_{i=1}^{k+1} \dim \operatorname{Im}(d^i)_x = \sum_{i=1}^{k+1} \dim \operatorname{Im}(\delta^i)_x \le \sum_{i=(k+3)/2}^{k+1} 2i-k-3.$$

But the ends of this inequality are equal (to $k^2/4$ is k is even and to $(k^2-1)/4$ if k is odd). Thus our inequalities on the dimensions of the stalks of the images of d^i and δ^i are in fact equalities.

To finish the proof, we just need to compute the multiplicities with which the characters χ^a occur in the stalks of the images of d^i and δ^i, and use this information in the sequences 6.1 and 6.2. Note that at each supersingular point x, the elements of the stalk of $R^{k+1-i}f_*\Omega^i_M = \underline{C}^i_k$ which are eigenvectors for $(\mathbf{Z}/p\mathbf{Z})^\times$ with character χ^a are represented by a section of $\Omega^1_I \otimes \omega^{2i-2-k}$ vanishing to order $a-1$. We find that at each supersingular point, the characters occuring in the stalk of the image of d^i are χ^j $j = 1, \ldots, i-1$ (each with multiplicity 1) if $i \leq (k+2)/2$, and χ^j, $j = 2i-1-k, \ldots, i-1$ (again with multiplicity 1) if $i \geq (k+3)/2$; on the other hand, the image of δ^i is 0 if $i \leq (k+2)/2$ and the characters occuring in the stalk of its image are χ^j, $j = 1, \ldots, 2i-k-3$ (with multiplicity 1) if $i \geq (k+3)/2$.

Putting all this into 6.1 and 6.2 and using, for example, that $R^{k+1}f_*B^1_M = 0$ (by Theorem 4.5), one finds that the multiplicity of χ^a $(0 < a < p-1)$ in $(R^{k+2-i}f_*Z^i_M)_x$ is

$$\begin{cases} 0 & \text{if } i \leq a+1 \\ i-a-1 & \text{if } a+1 \leq i \leq (k+a+2)/2 \\ k+1-i & \text{if } (k+a+2)/2 \leq i. \end{cases}$$

Similarly, the multiplicity of χ^a in $(R^{k+2-i}f_*B^i_M)_x$ is

$$\begin{cases} 0 & \text{if } i \leq a+1 \\ i-a-1 & \text{if } a+1 \leq i \leq (k+a+3)/2 \\ k+2-i & \text{if } (k+a+3)/2 \leq i. \end{cases}$$

This completes the proof of Theorem 6.7. □

Remark: The proof of Theorem 6.7 also allows one to find the dimension of the stalk of $(R^{k+2-i}f_*Z^i_M)_x(\chi^a)$ when $N \leq 4$. Indeed, let l_i denote the integer defined in [U2], Theorem 7.1 (depending also on p, N, k, and a), and let

$$\phi_1(j) = \begin{cases} 1 & \text{if } a < 2j - k - 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(j) = \begin{cases} 1 & \text{if } a > 2j - k - 2 \text{ and } a < j \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_1(j)$ (resp. $\phi_2(j)$) is the dimension of the χ^a part of the stalk at x of the cokernel of $\mathcal{C}_j$ (resp. 1_j), and we have

$$\dim_{\mathbf{F}}(R^{k+2-i}f_* Z^i_M)_x(\chi^a) = \left(\sum_{j=1}^{i-1} l_j\left(\phi_2(j) - \phi_1(j)\right)\right) - l_i\phi_1(i).$$

Note that when $i = a+2$, the right hand side is just l_{i-1}.

We can now use the skyscrapers $R^{k+2-i}f_* Z^i_M$ to produce classes in logarithmic cohomology:

Proof of 2.3b: By the Leray spectral sequence, $H^{k+2-i}(M, Z^i)$ maps surjectively to $H^0(I, R^{k+2-i}f_* Z^i_M)$. Since the number of supersingular points on I is $(p-1)w$, Theorem 6.7 shows that

$$\dim_{\mathbf{F}} H^{k+2-i}(M, Z^i) \geq d(a,i) = \begin{cases} 0 & \text{if } i \leq a+1 \\ (p-1)w(i-a-1) & \text{if } a+1 \leq i \leq (k+a+2)/2 \\ (p-1)w(k+1-i) & \text{if } (k+a+2)/2 \leq i \end{cases}.$$

By Proposition 4.6e), dim $U_1^{i,k+2-i} \geq d(a,i)$.

On the other hand, Milne's duality theorem ([M], 1.11) implies that the dimension of the group representing $H^{k+1-i}(M_a, \Omega^i_{log})$ is equal to the dimension of the group representing $H^{i+1}(M_{-a}, \Omega^{k+1-i}_{log})$. Thus by Proposition 4.6c), the dimension of $U_1^{i,k+2-i}$ is also at least $d(a', i)$. Note that for a fixed i and a, at most one of $d(i,a)$ and $d(a',i)$ is non-zero, so

$$\dim U_1^{i,k+2-i} \geq \max(d(a,i), d(a',i)) = d(a,i) + d(a',i).$$

Applying 4.6e) again, we have $T^{i-1,k+2-i} = \dim U^{i,k+2-i} \geq d(a,i) + d(a',i)$, and this completes the proof of Theorem 2.3b ∎

Remarks: 1) It seems possible to construct directly classes in $H^1(I, R^{k+1-i}f_*\Omega^i_{M,log})(\chi^a)$ which account for the $d(a', i)$ term in the inequality of the theorem, thereby avoiding the use of Milne duality.

2) By introducing extra level structure and passing to invariants, the proof of 2.3b) and the remark after 6.7 give a lower bound on the $T^{i-1,k+2-i}$ also in the case $N \leq 4$. In particular, one finds that for all N, if $i = a + 2$ then $T^{i-1,k+2-i} \geq l_{i-1}$ and if $i = k + a - p$ then $T^{i-1,k+2-i} \geq l_{i+1}$. (Here the l_i are the numbers defined in [U2], 7.1.)

7. The $\mathcal{C}$ construction In the previous section we showed that the group representing $H^{k+1-i}(M_a, \Omega^i_{log})$ is positive-dimensional for certain values of a and i. In Section 8 we will calculate this group for the remaining a and i by using the exact sequence of sheaves

$$0 \to \Omega^i_{X,log} \to Z^i_X \xrightarrow{1-\mathcal{C}} \Omega^i_X \to 0$$

(for the étale topology on X). To do this we need to find the kernel of

$$1 - \mathcal{C} : H^{k+1-i}(M_a, Z^i) \to H^{k+1-i}(M_a, \Omega^i)$$

and this will evidently require some information on $H^{k+1-i}(M_a, Z^i)$. This group is isomorphic to $H^0(I, R^{k+1-i}f_*Z^i_M)(\chi^a)$, but unfortunately, so far we have no reasonable description of the sheaves $R^{k+1-i}f_*Z^i_M$. In this section we will approximate them by sheaves with which we can effectively calculate.

First, we will need a construction analogous to the d-construction 5.5 involving the conjugate filtration and the Cartier operator. This construction does not seem to come from any spectral sequence. Consider the composition

$$R^0f_*\Omega^{k+1} \to \underline{H}^{k+1}_{dR}(M) \to \mathrm{Gr}_{k+1}\,\underline{H}^{k+1}_{dR}(M) \cong R^0f_*\mathcal{H}^{k+1}_M \xrightarrow[\sim]{\mathcal{C}} R^0f_*\Omega^{k+1}_M$$

where the last map is induced by the Cartier operator. This composition is easily seen to be just $R^0 f_*\mathcal{C} : R^0 f_*\Omega_M^{k+1} \to R^0 f_*\Omega_M^{k+1}$ and its kernel is $R^0 f_* B_M^{k+1}$. Set $C_{k+1} = R^0 f_* B_M^{k+1}$ and define inductively

$$C_i = \operatorname{Ker}\left(C_{i+1} \to F_i \underline{H}_{dR}^{k+1}(M) \to \operatorname{Gr}_i \underline{H}_{dR}^{k+1}(M) \cong R^{k+1-i} f_* \mathcal{H}_M^i \xrightarrow[\sim]{\mathcal{C}} R^{k+1-i} f_* \Omega_M^i\right)$$

for $k \geq i \geq 1$. Since the $R^{k+1-i} f_* \Omega_M^i$ are supported at the supersingular points, the sheaves C_i are locally free $\mathcal{O}_I^p$-modules of rank $p-1$ and they are all isomorphic to $R^0 f_* B_M^{k+1} \cong B_I^1(kS)$ away from the supersingular points. Note that C_i is just the preimage in $R^0 f_* \Omega_M^{k+1}$ of $(F^{k+1} \cap F_{i-1}) \underline{H}_{dR}^{k+1}(M)$.

Recall that each supersingular point x we have defined a subspace $V_x \subseteq \mathcal{O}_{I,x}$ as $(\mathbf{Z}/p\mathbf{Z})^\times$-invariant functions times p-powers.

Proposition 7.1. (The $\mathcal{C}$ construction) *For $1 \leq i \leq k$, we have:*

a) A section $s = \sigma\omega_c^k$ of $R^0 f_* \Omega_M^{k+1}$ lies in C_{i+1} if and only if there exists a function f such that $\sigma = ((-1)^{k-i} i!/k!) d\theta^{k-i}(f)$ and such that at each supersingular point x the inequalities

$$\operatorname{ord}_x\left(\sum_{k \geq j \geq l} \binom{j}{l} B^{-j} \frac{(-1)^{j-i} i!}{j!} \theta^{j-i}(f)\right) \geq pk - l(p-1)$$

hold for $l = i, \ldots, k$. The image of such a section in $R^{k+1-i} f_* \Omega_M^i$ is represented by the section $-i\mathcal{C}(f\, dq/q)\omega_c^{2i-2-k}$ of $\Omega_I^1 \otimes \omega^{2i-k}$.

b) The stalk of C_1 at a supersingular point x consists of sections $\sigma\omega_c^k$ such that σ is exact and its expansion at x lies in $t^{-k+p-2} V_x dt + d\theta^k(t^{pk}\mathcal{O}_{I,x})dt$, where t is an eigenuniformiser at x. The stalk of C_2 is spanned over the residue field by C_1 and the sections $d\theta^{k-1}(t^{pb-1})$ for $b = k, \ldots, k-1+(p-2)$. For $2 \leq i \leq k$, the stalk of C_{i+1} is spanned over the residue field by C_i and the sections $d\theta^{k-i}(t^{pb-1})$ for $b = k-i, \ldots, k-i+(p-1-i)$.

c) The map $C_2 \to R^k f_* \Omega^1$ is onto. If $i \geq 2$, the image of $C_{i+1} \to R^{k+1-i} f_* \Omega^i_M \cong \underline{C}^i_k$ consists of those classes represented by sections of $\Omega^1_I \otimes \omega^{2i-2-k}$ vanishing to order at least $(i-2)$.

Proof: Note that we can view s as a section of $\Omega^1_I \otimes \underline{H}^k_{dR}(M/I)$ via the map

$$R^0 f_* \Omega^{k+1}_M \cong \Omega^1_I \otimes \omega^k \hookrightarrow \Omega^1_I \otimes \underline{H}^k_{dR}(M/I).$$

As remarked after Lemma 6.3, we have that s lies in C_{i+1} if and only if there exists a section $z = \sum f_j \omega_c^j \eta_c^{k-j}$ of $\underline{H}^k_{dR}(M/I)$ such that $s - \nabla(z) \in \Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I)$. But

$$\nabla(z) = \sum \left(df_j + (j+1) f_{j+1} \frac{dq}{q} \right) \omega_c^j \eta_c^{k-j}$$

so $s - \nabla(z) \in \Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I)$ if and only if $f_j = \frac{(-1)^{j-i} i!}{j!} \theta^{j-i}(f_i)$ for $j \geq i$, and $\sigma = df_k = \frac{(-1)^{k-i} i!}{k!} d\theta^{k-i}(f_i)$. The inequalities of part a) are then just the conditions 3.12 on $f_i, \ldots, f_k$ for z to be regular. Conversely, if a function f as in the proposition exists, then defining f_j $j \geq i$ by the formula $f_j = \frac{(-1)^{j-i} i!}{j!} \theta^{j-i}(f)$, we can choose functions $f_0, \ldots, f_{i-1}$ so that $z = \sum f_j \omega_c^j \eta_c^{k-j}$ is a regular section of $\underline{H}^k_{dR}(M/I)$ and $s - \nabla(z) \in \Omega^1_I \otimes F_{i-1}\underline{H}^k_{dR}(M/I)$, and so $s \in C_{i+1}$.

By Lemma 6.3, the image of such an s in $R^{k+1-i} f_* \Omega^i_M$ is represented by the section

$$-\mathcal{C} \left(df_{i-1} + i f_i \frac{dq}{q} \right) \omega_c^{2i-2-k} = -i\mathcal{C} \left(f_i \frac{dq}{q} \right) \omega_c^{2i-2-k}$$

of $\Omega^1_I \otimes \omega^{2i-2-k}$. This completes the proof of part a).

Note that if $1 \leq i \leq k$ and f_i is a function vanishing to order at least $-k + (k-i)(p+1) + 1$ at each supersingular point, then the inequalities of part a) are trivially satisfied and $s = d\theta^{k-i}(f_i)\omega_c^k$ is a section of C_{i+1}. The image of s in $R^{k+1-i} f_* \Omega^i_M$ is represented by $-i\mathcal{C}(f_i\,dq/q)\omega_c^{2i-2-k}$ and (since $-k+(k-i)(p+1)+1 = p(k-i)+1-i$) the order of vanishing of $\mathcal{C}(f_i \frac{dq}{q})$ at a supersingular point can be any integer $\geq k-i$ if $i \geq 2$ and any integer $\geq k+1-i = k$ if $i=1$. Thus, the image of $C_{i+1} \to R^{k+1-i}f_*\Omega^i_M$ contains all sections vanishing to order at least $k-i+2i-2-k = i-2$ if $i \geq 2$ and all sections vanishing to order at least 0 if $i=1$. This proves that the image is at least as large as claimed in c). Note also that by taking f_i to have the form t^{pb-1} where $b = k-i, \ldots, k-i+(p-1-i)$ if $i \geq 2$ or $b = k, \ldots, k-1+(p-2)$ if $i=1$, we get sections $s = \sigma\omega_c^k$ of C_{i+1} not lying in C_i; the valuations of these σ run through all integers v satisfying $v \equiv i-2-k \pmod p$ and $-k \leq v \leq p(p-1-i)+i-2-k$.

Next we note that if t is an eigenuniformiser at x, then $t^{-k+p-2}W_x dt\,\omega_c^k$ is the $(\mathbf{Z}/p\mathbf{Z})^\times$-invariant subspace of the stalk of $R^0f_*\Omega^{k+1}$ at x. Since $(R^{k+1-i}f_*\Omega^i_M)_x(\chi^0) = 0$ for $i = 1, \ldots, k$, these sections lie in the kernel of each of the maps $C_{i+1} \to R^{k+1-i}f_*\Omega^i_M$, i.e., in C_1. Also, these maps are all $\mathcal{O}_I^p$-linear, so we have $t^{-k+p-2}V_x dt\,\omega_c^k \subseteq C_1$. Moreover, if f_0 is a function vanishing to order at least pk, then again using a), we find that $d\theta^k(f_0)\omega_c^k$ is a section of C_1. Thus if σ is exact and its expansion at x lies in $V_1 = t^{-k+p-2}V_x dt + d\theta^k(t^{pk}\mathcal{O}_{I,x})dt$, then $\sigma\omega_c^k$ is a section of C_1. Note that V_1 contains differentials of all valuations v satisfying $v \equiv 0, 1, \ldots, p-2-k \pmod p$, $v \geq 0$, and all valuations v satisfying $v \equiv i-2-k \pmod p$, $v \geq p(p-i)+i-2-k$, with $1 \leq i \leq k$.

Reviewing the last 2 paragraphs, we see that every exact differential σ can be written as a sum of differentials $d\theta^{k-i}(t^{pb-1})$ with b as above and a differential $\sigma' \in V_1$. This proves that the $d\theta^{k-i}(t^{pb-1})$ span C_{i+1}/C_i and that $C_1 = V_1\omega_c^k$, i.e., part b) of the proposition. It also shows that the image of $C_{i+1} \to R^{k+1-i}f_*\Omega^i_M$ contains exactly the sections already mentioned above; this is exactly the claim of part c). $\square$

Figure 2 below may help to understand the discussion in the remainder of this section. The sheaves $\mathcal{F}_{i,j}$ and D_j will be defined below.

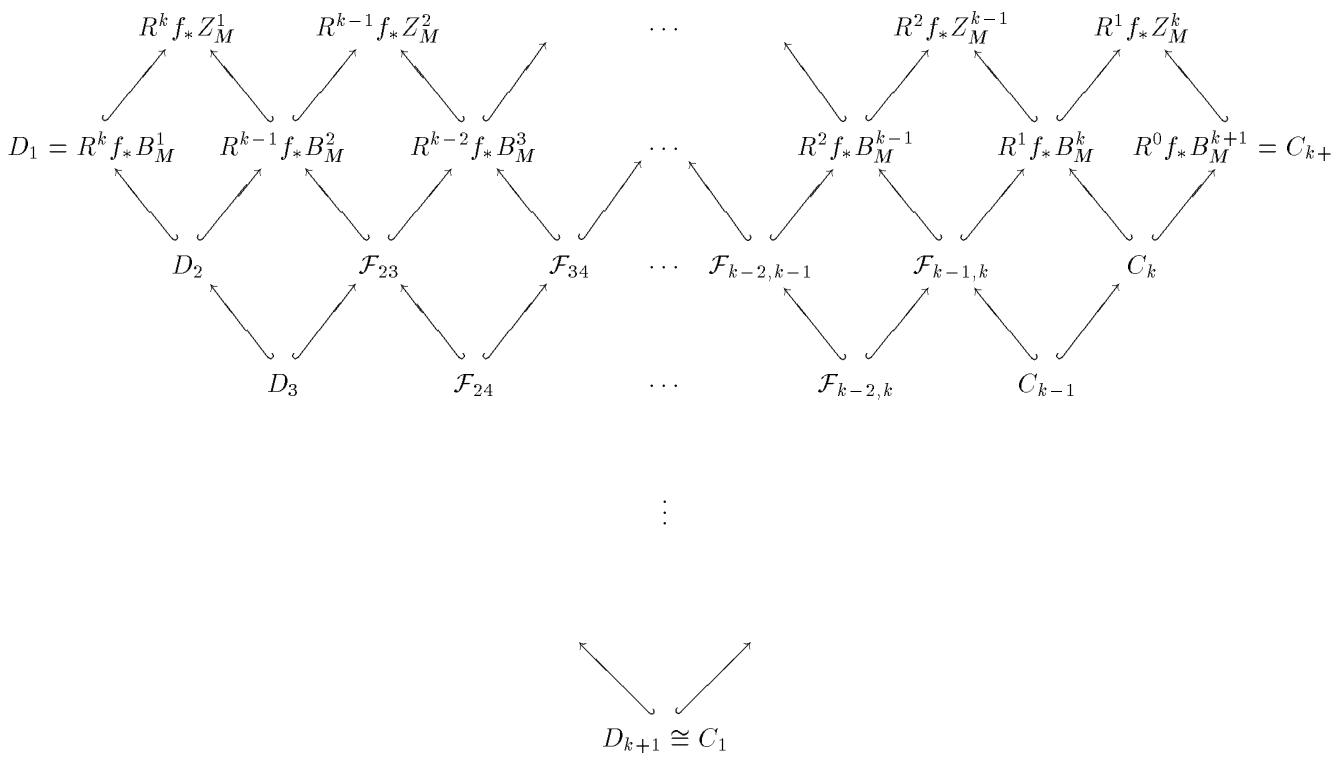

Figure 2

Recall the maps d_i (from a certain subsheaf of $R^k f_* \mathcal{O}_M$ to $R^{k+1-i} f_* \Omega_M^i$). We can put the C_j and the Ker d_i in a uniform framework as follows. We have injections $C_{k+1} = R^0 f_* B_M^{k+1} \xrightarrow{\delta} R^1 f_* Z_M^k$ and $R^1 f_* B_M^k \xrightarrow{n} R^1 f_* Z_M^k$ (6.1 and 6.2 for $i = k$). If s is a section of C_{k+1}, $\delta(s)$ lies in the image of n if and only if its image under the composition

$$R^0 f_* B_M^{k+1} \xrightarrow{\delta} R^1 f_* Z_M^k \xrightarrow{\mathcal{C}} R^1 f_* \Omega_M^k$$

is 0. It is not hard to check that this condition is exactly the one defining C_k, i.e., $C_k = \delta^{-1}(\mathrm{Im}\ \delta \cap \mathrm{Im}\ n)$. Similarly, C_{k-1} is preimage in C_k of the intersection of the image of

$$C_k \hookrightarrow R^1 f_* B_M^k \hookrightarrow R^2 f_* Z_M^{k-1}$$

and

$$R^2 f_* B_M^{k-1} \hookrightarrow R^2 f_* Z_M^{k-1}.$$

More generally, if define $\mathcal{F}_{ij}$ as the intersection

$$\big((\cdots((R^{k+1-i} f_* B_M^i \cap R^{k-i} f_* B_M^{i+1}) \cap R^{k-1-i} f_* B_M^{i+2}) \cap \cdots\big) \cap R^{k+1-j} f_* B_M^j$$

then $C_j = \mathcal{F}_{j,k+1}$. It follows easily from the discussion of the d_j in Section 5 (i.e., the d-construction) that $\mathcal{F}_{1,j}$ is the image under $R^k f_* \mathcal{O}_M \to R^k f_* B_M^1$ of Ker d_{j-1}. We write D_j for $\mathcal{F}_{1,j}$. If $f\omega_c^{-k}$ is a section of $R^k f_* \mathcal{O}_M$ we will also sometimes abusively write $f\omega_c^{-k}$ for its image in $R^k f_* B_M^1$.

It will be useful later to have an explicit description of the isomorphism $C_1 \cong D_{k+1}$.

Proposition 7.2. *Viewing D_{k+1} as a subsheaf of $R^k f_* B_M^1 \cong \mathcal{O}_I(-kS)/\mathcal{O}_I(-kS)^p$ and C_1 as a subsheaf of $R^0 f_* B_M^{k+1} \cong B_I^1(kS)$, the isomorphism $D_{k+1} \to C_1$ is given explicitly by*

$$f\omega_c^{-k} \mapsto (-1)^k \frac{1}{k!} \Theta^k(df)\omega_c^k.$$

Proof: $f\omega_c^{-k}$ lies in D_{k+1} if and only if there exists a section

$$z = \sum f_j \omega_c^j \eta_c^{k-j}$$

of $\underline{H}_{dR}^k(M/I)$ with $f_0 = f$ such that $\nabla(z) \in \Omega_I^1 \otimes F^k \underline{H}_{dR}^k(M/I)$. In this case, the image of $f\omega_c^{-k}$ in C_1 is just the image of $\nabla(z)$ under $\Omega_I^1 \otimes F^k \underline{H}_{dR}^k(M/I) \cong R^0 f_* \Omega_M^{k+1}$, namely $df_k \omega_c^k$. Now

$$\nabla(z) = \sum (df_j + (j+1) f_{j+1} \frac{dq}{q}) \omega_c^j \eta_c^{k-j}.$$

and this lies in $F^k \underline{H}^k_{dR}(M/I)$ if and only if for $1 \leq j \leq k$, $f_j = \frac{-df_{j-1}}{j\frac{dq}{q}} = \frac{-1}{j}\theta(f_{j-1})$. Thus $df_k = (-1)^k \frac{1}{k!} d\theta^k(f) = (-1)^k \frac{1}{k!}\Theta^k(df)$ and the isomorphism is as claimed. $\square$

Next we introduce sheaves which approximate $R^{k+1-i}f_*Z^i_M$ and $R^{k+1-i}f_*B^i_M$: let $\mathcal{Z}^i$ be the subsheaf of $R^{k+1-i}f_*Z^i_M$ generated by C_{i+1} and D_i and let $\mathcal{B}^i$ be the subsheaf of $R^{k+1-i}f_*B^i_M$ generated by C_i and D_i (cf. Figure 2 above). By definition, a section of $\mathcal{Z}^i$ can be written locally as the sum of the image in $R^{k+1-i}f_*Z^i_M$ of a section $f\omega_c^{-k}$ of D_i and a section $dg\omega_c^k$ of C_{i+1}; we will use the notation

$$(f, dg) \tag{7.3}$$

to denote such a section and we will employ a similar notation for $\mathcal{B}^i$. The next proposition says that the sheaves $\mathcal{Z}^i$ and $\mathcal{B}^{i+1}$ are good approximations of $R^{k+1-i}f_*Z^i_M$ and $R^{k-i}f_*B^{i+1}_M$ for the characters we are interested in.

Proposition 7.4. *The quotients $R^{k+1-i}f_*Z^i_M/\mathcal{Z}^i$ and $R^{k-i}f_*B^{i+1}_M/\mathcal{B}^{i+i}$ are skyscrapers supported at the supersingular points, and at each supersingular point x we have*

$$(R^{k+1-i}f_*Z^i_M/\mathcal{Z}^i)_x(\chi^a) = (R^{k-i}f_*B^{i+1}_M/\mathcal{B}^{i+1})_x(\chi^a) = 0$$

for $i-1 \leq a \leq p-1$.

Proof: Let $\mathcal{C}_j$ denote the map $R^{k+1-j}f_*Z^j_M \to R^{k+1-j}f_*\Omega^j_M$ induced by the Cartier operator and let c_j denote that map $C_{j+1} \to R^{k+1-j}f_*\Omega^j_M$ given by the $\mathcal{C}$ construction. Then looking at Figure 2 shows that $R^{k+1-i}f_*Z^i_M/\mathcal{Z}^i$ and $R^{k-i}f_*B^{i+1}_M/\mathcal{B}^{i+1}$ can be imbedded in an extension of the skyscrapers

$$\frac{\text{Im } \mathcal{C}_j}{\text{Im } c_j}$$

for $j \le i$. But we have calculated Im $\mathcal{C}_j$ and Im c_j in 6.5 and 7.1 and we have

$$\left(\frac{\text{Im } \mathcal{C}_j}{\text{Im } c_j}\right)_x (\chi^a) = 0$$

if $a > j-2$ or $a < \max(1, 2j-k-2)$. This yields the Proposition. $\square$

8. Cohomology of logarithmic differentials Our goal in this section is to prove Theorem 2.4. The main point is a calculation of $H^{k+1-i}(M_a, \Omega^i_{log})$, for certain i and a, in terms of the cohomology of Igusa curves. Since $R^j f_* \Omega^i_{M,log} = 0$ for $j \le k-i$ by 4.6a), we have

$$H^{k+1-i}(M_a, \Omega^i_{log}) \cong H^0(I, R^{k+1-i} f_* \Omega^i_{M,log})(\chi^a)$$

and since $R^{k-i} f_* \Omega^i_M = 0$ by 4.1,

$$R^{k+1-i} f_* \Omega^i_{M,log} \cong \text{Ker } \left(1 - \mathcal{C} : R^{k+1-i} f_* Z^i_M \to R^{k+1-i} f_* \Omega^i_M\right).$$

Using the sheaf $\mathcal{Z}^i$ of the previous section we will identify Ker $1 - \mathcal{C}$ with a certain sheaf of functions. We will then compute the group of global sections of this sheaf.

We fix data k, a, and i satisfying the standing hypotheses as usual, and we assume from now until the end of the section that a and i satisfy $i - 1 \le a$ and $k - i \le p - 1 - a$. Let

$$c_1 = \binom{k+1-i+a}{i}, \qquad c_2 = \binom{a}{i},$$

and $b = a + k - 2i$. It will be convenient to look at the various eigenspaces separately. We will be a little sloppy and write $\mathcal{F}(\chi^b)$ for certain sheaves $\mathcal{F}$ on I (when we really mean $(g_*\mathcal{F})(\chi^b)$, where $g : I \to X_1(N)$ is the canonical map) and speak as if this were a sheaf on I. As most of the work will take place at supersingular points, where the stalks of $g_*\mathcal{F}$ and $\mathcal{F}$ can be identified, this abuse should be harmless.

Let $\mathbf{F}(I)$ be the function field of I over $\mathbf{F}$ and consider the constant sheaf $\underline{\mathbf{F}(I)}$ on I; let $\underline{\mathbf{F}(I)}^0$ be the subsheaf of rational functions h on I such that $h\,dq/q$ is an exact rational differential; the sheaf $\underline{\mathbf{F}(I)}^0$ is a free module of rank $p-1$ over the sheaf $\underline{\mathbf{F}(I)}^p$ of p-powers of rational functions and $\underline{\mathbf{F}(I)} \cong \underline{\mathbf{F}(I)}^0 \oplus \underline{\mathbf{F}(I)}^p$ as $\underline{\mathbf{F}(I)}^p$ modules. Sections of $\underline{\mathbf{F}(I)}^0$ are also characterized as functions in the image of θ.

Now under the hypotheses on i and a, Proposition 7.4 says that $\mathcal{Z}^i = R^{k+1-i}f_*Z^i_M$. We define a map $\phi : \mathcal{Z}^i \to \underline{\mathbf{F}(I)}^0$ as follows: To each local section s of $\mathcal{Z}^i$ which is the sum of the image of a section $f\omega_c^{-k}$ of D_i and a section $dg\omega_c^k$ of C_{i+1} (so $s = (f, dg)$ in the notation of 7.3), we associate the rational function $h = \phi(f, dg)$ defined by

$$\phi(f, dg) = \frac{1}{i!}\theta^i(f) + \frac{(-1)^k k!}{i!}\theta^{p-1-(k-i)}(g)$$

where θ is the operator defined in Section 3. Since the intersection of D_i and C_{i+1} is $D_{k+1} = C_1$, Proposition 7.2, which describes the isomorphism between D_{k+1} and C_1, proves that h depends only on s, not on its expression as (f, dg).

We note that away from the supersingular points, ϕ defines an isomorphism between $\mathcal{Z}^i(\chi^a)$ and $B^1_I(\chi^b)$. Indeed, on the ordinary locus $I^h = I \setminus S$, $R^{k+1-i}f_*\Omega^i_M = 0$, all of the D_i are equal and they are isomorphic to B^1_I via $f\omega_c^{-k} \mapsto df$, and all of the C_{i+1} are equal and isomorphic to B^1_I via $dg\,\omega_c^k \mapsto dg$. Since θ is an automorphism of B^1_I on I^h, the map $(f, dg) \mapsto dh$ gives the desired isomorphism. (The twist comes from the facts that ω_c is in the χ^1 eigenspace and θ sends the χ^a eigenspace to the χ^{a-2} eigenspace.)

Proposition 8.1. *The map $\phi : \mathcal{Z}^i(\chi^a) \to \underline{\mathbf{F}(I)}^0$ is an injection.*

Proof: As noted above, ϕ is certainly injective off the supersingular points. Fix a supersingular point x and an eigenuniformiser t at x. We have defined subspaces $W_x \subseteq \mathcal{O}_{I,x}$ as the $(\mathbf{Z}/p\mathbf{Z})^\times$-invariant functions and $V_x \subseteq \mathcal{O}_{I,x}$ as $W_x\mathcal{O}_{I,x}^p$. Let us recall the description of the stalks of $D_i(\chi^a)_x$ and $C_{i+1}(\chi^a)_x$: by 5.5, the former consists of the images in $R^kf_*B_M^1$ of sections $f\omega_c^{-k}$ of $R^kf_*\mathcal{O}_M$ with $f \in t^{k+(p-1)(i-1)+a}W_x$; by 7.1, the latter is generated by $C_1(\chi^a)_x$ and sections $d\theta^{k-j}(t^{pl-1})\omega_c^k$ with $1 \le j \le i$ and $l = k-2j+a+1$.

Let (f, dg) be an element of the stalk of $\mathcal{Z}^i(\chi^a)$ at x and suppose that it goes to zero in $\mathcal{F}_x$, i.e., that $h = \phi(f, dg) = 0$. Without loss of generality, we can suppose that $f\omega_c^{-k}$ and $dg\omega_c^k$ both lie in the χ^a eigenspace. Moreover, using that $C_1 \cong D_{k+1}$ and rewriting (f, dg), we can also suppose that the section dg is an $\mathbf{F}$-linear combination of terms $d\theta^{k-j}(t^{pl-1})$ with $1 \le j \le i$ and $l = k-2j+a+1$. Since $h \in \underline{\mathbf{F}(I)}^0$, $dh = 0$ if and only if $h = 0$. Now the contribution of dg to dh is $((-1)^k i!/k!)\Theta^{-(k-i)}dg$ and a short calculation shows that this differential vanishes to order at least $(p-1)(k-2i+a+1)+k-2i+a-1$ at x. If dh is zero, then the contribution of f, namely $\frac{1}{i!}d\theta^i f$, must also vanish to at least this order. Now since $f \in t^{k+(p-1)(i-1)+a}W_x$, $l = \mathrm{ord}_x(f)$ has the form $l = k+a+(i-1+m)(p-1)$ for some $m \ge 0$. Then

$$\begin{aligned}\mathrm{ord}_x(d\theta^i f) &\ge k+a+(i-1+m)(p-1)-i(p+1)-1\\ &= k-2i+a-1+(m-1)(p-1)\end{aligned}$$

with equality if and only if $l \not\equiv 0,\ldots,i \pmod p$. Thus if $l \not\equiv 0,\ldots,i \pmod p$, we must have $m \ge k-2i+a+2$; on the other hand the smallest m such that $l \equiv 0,\ldots,i \pmod p$ is $m = k-2i+a+1$. In any case $f\omega_c^{-k} \in D_{k+1}$, since under the hypotheses on a and i, $i-1-m \ge k-1$ and so $l \ge k+a+(k-1)(p-1)$. But then (f, dg) can be written as $(0, dg')$ for a suitable g'. As ϕ is obviously injective on sections of $\mathcal{Z}^i$ of the form $(0, dg')$, we have $(f, dg) = 0$ and this proves that ϕ is injective. $\square$

Now define a subsheaf

$$\mathcal{F} = \mathcal{F}_{i,a} \subseteq \underline{\mathbf{F}(I)}^0(\chi^b)$$

by requiring that $h \in \underline{\mathbf{F}(I)}^0(\chi^b)$ lies in $\mathcal{F}$ if and only if h is regular at ordinary points, h vanishes at each cusp, and at each supersingular point, h lies in

$$c_1 h_x - c_2 h_x^p + \mathcal{O}_{I,x}$$

for some h_x with $\mathrm{ord}_x(h_x) \geq -p$. Note that h determines h_x up to the addition of a function regular at x.

Recall that for $i \leq a+1$, we have an equality $(R^{k+1-i} f_* Z_M^i)(\chi^a) = \mathcal{Z}^i(\chi^b)$. The following theorem is the key point of this section.

Theorem 8.2. *The map ϕ defines an isomorphism of sheaves*

$$\left(\mathrm{Ker}\ 1 - \mathcal{C} : R^{k+1-i} f_* Z_M^i \to R^{k+1-i} f_* \Omega_M^i\right)(\chi^a) \xrightarrow{\sim} \mathcal{F}_{i,a}.$$

Proof: We have already seen that on I^h, $\mathcal{F}_{i,a} = B_I^1(\chi^b)$, and since $R^{k+1-i} f_* \Omega_M^i$ is zero on I^h, $\mathrm{Ker}\,(1-\mathcal{C}) = \mathcal{Z}^i$ there; thus ϕ induces the desired isomorphism away from the supersingular points. Also, we have that that ϕ is injective on all of I, so to prove the theorem we need to check that at each supersingular point ϕ takes sections of $\mathrm{Ker}\,(1-\mathcal{C})(\chi^a)$ to $\mathcal{F}_{i,a}$, and that all sections of $\mathcal{F}_{i,a}$ are in the image of ϕ restricted to $\mathrm{Ker}\,(1-\mathcal{C})(\chi^a)$. For the rest of the discussion we fix a supersingular point x and an eigenuniformiser t there.

Fix a section (f, dg) of $\mathrm{Ker}\,(1-\mathcal{C})(\chi^a)$. We will examine the images of sections $f\omega_c^{-k}$ and $dg\omega_c^k$ in $R^{k+1-i} f_* \Omega_M^i$ and relate this to their contributions to $h = \phi(f, dg)$. Note that D_i goes to zero under the map $\mathcal{C} : \mathcal{Z}^i \to R^{k+1-i} f_* \Omega_M^i$ and C_{i+1} goes to zero under

$1 : \mathscr{Z}^i \to R^{k+1-i}f_*\Omega^i_M$. We also recall (4.1) that the stalk $(R^{k+1-i}f_*\Omega^i_M)_x(\chi^a)$ is one dimensional for all a with $1 \leq a \leq p-2$.

Consider a section $f\omega_c^{-k}$ of D_i. By the d-construction, if $\mathrm{ord}_x(f) = l$, then the image of $f\omega_c^{-k}$ in $R^{k+1-i}f_*\Omega^i_M$ is represented by the section

$$-i\binom{k-l}{i}B^i f\frac{dq}{q}\omega_c^{2i-2-k}$$

If $a = i-1$, this class is zero (for either $l = k + p(i-1)$ and the binomial coefficient is zero, or $l \geq k + p(i-1) + (p-1)$ and the section vanishes to order at least $p-2$, thus giving zero in $R^{k+1-i}f_*\Omega^i_M$). If $a > i-1$, we get a non-zero contribution to $R^{k+1-i}f_*\Omega^i_M$ if and only if l has its minimum possible value, namely $k + (p-1)(i-1) + a$.

Next, we want to compute the contribution of $f\omega_c^{-k}$ to h, which is

$$\frac{1}{i!}\theta^i(f),$$

up to functions regular at x. We have $\mathrm{ord}_x(\theta^i(f)) \geq l - i(p+1) \geq k - 2i + a - (p-1)$ and

$$\frac{1}{i!}\theta^i(f) \equiv \binom{l}{i}B^i f$$

modulo functions vanishing to order $l-i(p+1)+(p-1) \geq k-2i+a$. But $k-2i+a \geq 0$ unless $a = i-1$, $i = k$, and $l = k + p(k-1)$, so aside from this case, we have $\mathrm{ord}_x(\phi(f,0)) > -p$ and

$$\phi(f,0) \equiv \binom{l}{i}B^i f$$

modulo regular functions. In the case $a = i-1$, $i = k$, it may happen that $\phi(f,0)$ has a pole of order p at x, and we can read off the leading term from the congruence above, but $d\phi(f,0)$ is regular (since $\phi(f,0)$ is a section of $\underline{\mathbf{F}(I)}^0$). Thus in all cases, we have determined the contribution of f to h up to the addition of a function regular at x.

Now consider sections $dg\omega_c^k$ of $C_{i+1}(\chi^a)$. By the $\mathcal{C}$-construction, there exists a function g_i satisfying certain inequalities such that $dg = \frac{(-1)^{k-i}i!}{k!} d\theta^{k-i}(g_i)$; the image of $dg\,\omega_c^k$ in $R^{k+1-i}f_*\Omega_M^i$ under $\mathcal{C}$ is then represented by the section

$$-i\mathcal{C}(g_i \frac{dq}{q})\omega_c^{2i-2-k}.$$

Since all sections of $C_1(\chi^a)$ are already accounted for in the image of $D_{k+1}(\chi^a)$, we can take dg to be a linear combination of $d\theta^{k-j}(t^{pl-1})\omega_c^k$ with $1 \leq j \leq i$ and $l = k-2j+a+1$. Then g_i is a linear combination of $\theta^{i-j}t^{pl-1}$ and the image of $dg\omega_c^k$ in $R^{k+1-i}f_*\Omega_M^i$ is non-zero if and only if the coefficient of $d\theta^{k-i}t^{p(k-2i+a+1)-1}$ (the term with $j = i$) is non-zero. We note that each of the functions $\theta^{i-j}t^{pl-1}$ is regular at x, except in the case $j = i$, $i = k$, $a = i-1$, in which case it has a simple pole. But if $i = k$, $a = i-1$, and $(1-\mathcal{C})(f, dg) = 0$, then $dg\omega_c^k$ must also go to zero in $R^{k+1-i}f_*\Omega_M^i$ since $f\omega_c^{-k}$ goes to zero automatically. This implies that that the $j = i$ term cannot occur in g_i, so g_i is regular at x in all cases.

On the other hand, the contribution of $dg\,\omega_c^k$ to h is

$$\frac{(-1)^k k!}{i!}\theta^{p-1-(k-i)}(g) = (-1)^i\theta^{p-1}(g_i).$$

If $g_i = t^{p(k-2i+a+1)-1}$, then $\mathrm{ord}_x(\theta^{p-1}(g_i)) = p(k-2i+a+1-p)$ and if $g_i = \theta^{i-j}(t^{pl-1})$ with $j < i$, then $\theta^{p-1}g_i$ is regular at x.

At this point we can show that the image of ϕ lies in $\mathcal{F}$. First we treat the cases $i = k$, $a = k-1$ and $i = 1$, $a = p-k$, (the $**$ region) which are easier. In the first case, we have $c_1 \neq 0$, $c_2 = 0$ and in the second we have $c_1 = 0$, $c_2 \neq 0$; in both cases, $b \equiv -1 \pmod{p-1}$. In both cases, $\mathcal{F}$ is just the subsheaf of $\underline{\mathbf{F}(I)}^0(\chi^{-1})$ consisting of functions h which vanish at the cusps, are regular at ordinary points, and which satisfy $\mathrm{ord}_x(h) \geq -p$

at each supersingular point. (In the second case we cannot have $\mathrm{ord}_x(h) = -p^2$, since $\mathrm{ord}_x(dh_x) \geq 0$ and this implies $\mathrm{ord}_x(h_x) = \mathrm{ord}_x(\theta^{p-1}h_x) \geq -p^2+2$.) But we have seen above that if $i = k$, $a = k-1$, then $\phi(f,0)$ is regular at supersingular points x and $\mathrm{ord}_x(\phi(0,dg)) \geq -p$. If $i=1$ and $a = p-k$, then $\mathrm{ord}_x(\phi(f,0)) \geq -p$ and the assumption that $(1-\mathcal{C})(f,dg) = 0$ implies that $\phi(0,dg)$ is regular at x. Thus the image of ϕ does indeed lie in $\mathcal{F}$ in these two cases.

We now consider the more interesting cases where neither $i=k$, $a=k-1$, nor $i=1$, $a=p-k$ holds. By Lemma 3.6, we have

$$\mathcal{C}(g_i\frac{dq}{q}) = (g_i - \theta^{p-1}(g_i))^{1/p}\frac{dq}{q}$$

and so the assumption that $(1-\mathcal{C})(f,dg) = 0$ implies that

$$-i\binom{k-l}{i}B^i f + i(g_i - \theta^{p-1}(g_i))^{1/p}$$

vanishes to order at least $k-2i$ at x. In fact, this function lies in the χ^{k-2i+a} eigenspace for the $(\mathbf{Z}/p\mathbf{Z})^\times$ action, so it must vanish to order $\geq k-2i+a$, i.e., it must be regular at x (since $a > i-1$ or $i < k$). Eliminating the $-i$ and raising to the p-th power, we have that

$$\binom{k-l}{i}B^{pi}f^p - \big(g_i - \theta^{p-1}(g_i)\big)$$

is regular at x. Since g_i is regular, so is

$$\binom{k-l}{i}B^{pi}f^p + \theta^{p-1}(g_i).$$

Thus h can be written near x as the sum of

$$\binom{l}{i}B^i f - (-1)^i\binom{k-l}{i}B^{pi}f^p \tag{8.3}$$

and a regular function.

Now l, the valuation of f, has the form $k + (p-1)(i-1) + a + m(p-1)$ for some non-negative integer m. If $m > 0$, then both terms of 8.3 are regular. On the other hand, if $m = 0$, then $l \equiv k + 1 - i + a \pmod{p}$. Thus in all cases, h can be written near x as

$$\begin{aligned}\binom{k+1-i+a}{i} h_x - (-1)^i \binom{i-1-a}{i} h_x^p &= \binom{k+1-i+a}{i} h_x - \binom{a}{i} h_x^p \\ &= c_1 h_x - c_2 h_x^p\end{aligned}$$

where h_x is a rational function near x with $\mathrm{ord}_x(h_x) \geq -p$. This proves that ϕ maps $\mathrm{Ker}\,(1-\mathcal{C})(\chi^a)$ into $\mathcal{F}_{i,a}$

To see that ϕ is surjective, take an arbitrary element h of the stalk $\mathcal{F}_x$. Reviewing the discussion of the images of (f, dg) under $1 - \mathcal{C}$ and ϕ and their relation, we see that it is possible to write down a section (f, dg) of $\mathcal{Z}^i$ killed by $1 - \mathcal{C}$ and differing from h by a function regular at x. Now using the d-construction to modify f by functions with valuations $k + a + (i - 1 + m)(p - 1)$, $m = 1, \ldots, k - 2i + a + 1$, we can arrange that h and $\phi(f, dg)$ differ by a function vanishing to order at least $p(k - 2i + a + 1)$. But then by the $\mathcal{C}$ construction, if we set $g_i = h - \phi(f, dg)$, the $g_i\, dq/q$ is exact and $\mathrm{ord}_x(g_i) \geq -k + (k-i)(p+1) + 1$ so $d\theta^{k-i} g_i$ is a section of C_{i+1} killed by $1 - \mathcal{C}$. Modifying dg by adding $d\theta^{k-i} g_i$, we have $h = \phi(f, dg)$. This proves that ϕ maps $\mathrm{Ker}\,(1-\mathcal{C})(\chi^a)$ onto $\mathcal{F}_{i,a}$ and thus induces an isomorphism. $\square$

Next we analyze the global sections of $\mathcal{F}_{i,a}$.

Theorem 8.4. *Assume the standing hypotheses, fix an integer i with $1 \leq i \leq k$, and suppose $i - 1 \leq a$ and $k - i \leq p - 1 - a$. Then we have an exact sequence*

$$H^0(I, \Omega^1_I(C))(\chi^{b+2})^{c_1\mathcal{C} - c_2 = 0} \to H^0(I, \mathcal{F}_{i,a}) \to H^1(I, \mathcal{O}_I)(\chi^b)^{c_1 - c_2 F = 0}.$$

Suppose the ground field $\mathbf{F}$ *is algebraically closed. Then the right hand map is surjective; the left hand map is injective if* $b \not\equiv 0 \pmod{p-1}$ *or* $c_2 = 0$*; when* $c_2 \neq 0$ *and* $b \equiv 0 \pmod{p-1}$ *it has as its kernel a group of order* p *generated by* $x\,dq/q$ *where* x *satisfies* $x^{p-1} = c_1/c_2$.

Remark: Note that when a and i satisfy the strict inequalities $i-1 < a$ and $k-i < p-1-a$, the constants c_1 and c_2 are not 0. On the other hand, when $p-1-a = k-i$, then $c_1 = 0$ and (when $\mathbf{F}$ is algebraically closed) $H^0(I, \mathcal{F}_{i,a})$ is isomorphic to $H^1(I, \mathcal{O})(\chi^b)^{F=0}$; when $a = i-1$, we have $c_2 = 0$ and (when $\mathbf{F}$ is algebraically closed) $H^0(I, \mathcal{F}_{i,a})$ is isomorphic to $H^0(I, \Omega^1)(\chi^b)^{\mathcal{C}=0}$.

Proof: We will need the old-fashioned but eminently useful description of H^1 of a coherent sheaf on a curve in terms of "répartitions" (i.e., something like adèles). For any divisor D on I, we have

$$H^1(I, \mathcal{O}(D)) \cong \frac{\mathbf{A}_I}{\mathcal{O}(D)(0)\mathbf{F}(I)}$$

where $\mathbf{A}_I$ is the subgroup of $(h_x) \in \prod_x \mathbf{F}(I)$ with almost all h_x in the local ring at x, $\mathcal{O}(D)(0)$ is the subgroup with all h_x in the stalk of $\mathcal{O}(D)$ at x, and $\mathbf{F}(I)$ is the function field of I, imbedded diagonally. ($\mathbf{A}_I$ is not quite the ring of adéles, since we have not taken completions.) In terms of this isomorphism, the Serre duality pairing is

$$\langle (h_x), \sigma \rangle = \sum_x \operatorname{Res}_x h_x \sigma.$$

(See [S], Chap. II, No. 5 for proofs.) We note also it is clear from this isomorphism that if D' is a second divisor with $D' - D$ effective, then a class in $H^1(I, \mathcal{O}(D))$ has a representative (h_x) with $\operatorname{ord}_x(h_x) \geq -\operatorname{ord}_x(D')$ if and only it goes to zero under the natural projection $H^1(I, \mathcal{O}(D)) \to H^1(I, \mathcal{O}(D'))$.

We now define the maps in the statement of the theorem. Suppose $h \in H^0(I, \mathcal{F})$, so h is a rational function on I (in the χ^b eigenspace for the $(\mathbf{Z}/p\mathbf{Z})^\times$ action) with $h\,dq/q$ exact. Furthermore, h is regular at ordinary points, vanishes at the cusps, and there are functions h_x at each supersingular point x with $\mathrm{ord}_x(h_x) \geq -p$ such that h lies in

$$c_1 h_x - c_2 h_x^p + \mathcal{O}_{I,x}.$$

As we remarked above, h_x is determined by h up to the addition of functions regular at x. Defining h_x to be 0 if x is not a supersingular point, we get a well-defined class

$$(h_x) \in \; H^1(I, \mathcal{O})(\chi^b)$$

and this class is clearly in the kernel of $c_1 - c_2 F$. The kernel of the map $H^0(I, \mathcal{F}) \to H^1(I, \mathcal{O})(\chi^b)$ consists of sections h which can be written globally as $c_1 g - c_2 g^p$ where g is a function regular at ordinary points and at the cusps, and with $\mathrm{ord}_x(g) \geq -p$ at supersingular points. (*A priori*, we have only that $h = c_1 g - c_2 g^p + c_3$ for some constant c_3. But since h vanishes at each cusp, expanding h in a power series at a cusp rational over $\mathbf{F}$, we see that c_3 has the form $c_1 a_0 - c_2 a_0^p$; modifying g by a_0, we can assume $c_3 = 0$.) Thus $g\,dq/q \in H^0(I, \Omega^1_I(C))(\chi^{b+2})$; since $h\,dq/q$ is exact we have

$$0 = \mathcal{C}\left((c_1 g - c_2 g^p)\frac{dq}{q}\right) = c_1 \mathcal{C}(g\frac{dq}{q}) - c_2 g \frac{dq}{q}$$

and so $g\,dq/q$ lies in the kernel of $c_1\mathcal{C} - c_2$. Conversely, if $\sigma \in H^0(I, \Omega^1(C))(\chi^{b+2})$ and σ is in the kernel of $c_1\mathcal{C} - c_2$, then setting $g = \sigma/(dq/q)$, we have that $h = c_1 g - c_2 g^p$ is a global section of $\mathcal{F}$ which obviously goes to zero in $H^1(I, \mathcal{O})$. The establishes the existence of the exact sequence of the theorem.

Now suppose the ground field $\mathbf{F}$ is algebraically closed. The kernel of the map

$$H^0(I, \Omega^1(C))(\chi^{b+2})^{c_1\mathcal{C}-c_2=0} \to H^0(I, \mathcal{F})$$

is clearly the set of differentials $x\,dq/q$ in the χ^{b+2} eigenspace with $c_1x - c_2x^p = 0$. Since dq/q lies in the χ^2 eigenspace, if $b \equiv 0 \pmod{p-1}$ and neither c_1 nor c_2 are zero, the kernel is a group of order p; otherwise it is the trivial group. It remains to prove the surjectivity of

$$H^0(I, \mathcal{F}) \to H^1(I, \mathcal{O})(\chi^b)^{c_1-c_2F=0}.$$

Let (h_x) be a répartition representing a class in the target. Since $H^1(I, \mathcal{O}(pS)) = 0$ (for degree reasons), we can change the representative so that $\mathrm{ord}_x(h_x) \geq -p$ at each supersingular point x and $h_x = 0$ at all ordinary points and cusps. Then there exists a rational function h with $h - c_1h_x + c_2h_x^p$ regular at each x. However, $h\,dq/q$ may not be exact so we do not yet have a section of $\mathcal{F}$. We do have that $\mathcal{C}(h\,dq/q)$ is regular at supersingular and ordinary points and has at worst simple poles at the cusps (this follows by looking at the local conditions on h). We need the following lemma.

Lemma 8.5. *Suppose that V is a finite dimensional vector space over a perfect field $\mathbf{F}$ of characteristic p and $T : V \to V$ is a p-linear or p^{-1}-linear endomorphism. If c_1 and c_2 are non-zero elements of $\mathbf{F}$, then the kernels of the maps $c_1 - c_2T$ and $c_1T - c_2 : V \to V$ are finite groups. If $\mathbf{F}$ is algebraically closed, then these maps are surjective.*

Proof: This is a twisted version of a well-known fact: if V and T are as in the statement and $\mathbf{F}$ is algebraically closed, then V can be written as a direct sum $V = V_n \oplus V_s$ of T-stable subspaces such that T is nilpotent on V_n and such that V_s has a basis of vectors fixed by T. The lemma follows easily from this. $\square$

Assuming for the moment that c_1 and c_2 are non-zero, we apply the lemma with $V = H^0(I, \Omega^1_I(C))(\chi^{b+2})$ and $T = \mathcal{C}$. This furnishes us with a differential σ solving the equation

$$c_1\mathcal{C}(\sigma) - c_2\sigma = h\frac{dq}{q}.$$

Setting $g = \sigma/(dq/q)$, we have that $\mathrm{ord}_x(g) \geq -p$ at supersingular points and g is regular elsewhere. Replacing h with $h - c_1 g + c_2 g^p$ we have that h satisfies the local conditions at the supersingular points to be in $\mathcal{F}$, and it is regular at ordinary points and cusps. Moreover, $h\,dq/q$ is exact so h must vanish at the cusps and it clearly maps to the given class in $H^1(I, \mathcal{O})$. Thus the map is onto if c_1 and c_2 are non-zero. If $c_2 = 0$ then $H^1(I, \mathcal{O})^{c_1 - c_2 F = 0} = 0$ and there is nothing to prove, so assume $c_1 = 0 \neq c_2$. Then $h' = \theta^{p-1}h$ satisfies $\mathrm{ord}_x(h') > -p^2$, h and h' differ by the p-th power of a rational function (so h' maps to the same class in $H^1(I, \mathcal{O})$ as h does), and since $\theta h = \theta h'$, the polar part of h' at each supersingular x is a p-th power. Thus h' is the required section of $\mathcal{F}$. This completes the proof of Theorem 8.4 $\square$

Proof of Theorem 2.4: We start with a). The hypotheses then imply that neither c_1 nor c_2 is zero. The finiteness of $H^{k+1-i}(M_a, \Omega^i_{log})$ is immediate from Theorem 8.2 and 8.4 combined with Lemma 8.5. When $\mathbf{F}$ is algebraically closed, there is an obvious three step filtration on $H^{k+1-i}(M_a, \Omega^i_{log})$ whose graded pieces are

$$H^0(I, \Omega^1)(\chi^{b+2})^{c_1\mathcal{C} - c_2 = 0}, \qquad \left(\frac{H^0(I, \Omega^1(C))}{\mathbf{F}\,dq/q + H^0(I, \Omega^1)}(\chi^{b+2})\right)^{c_1\mathcal{C} - c_2 = 0},$$

and

$$H^1(I, \Omega^0_{log})(\chi^b)^{c_1 - c_2 F = 0}.$$

Multiplication by a solution x to $x^{p-1} = c_2/c_1$ gives isomorphisms with the three groups in the statement of the theorem. Since for algebraically closed $\mathbf{F}$, $U_1^{i,k+1-i}(\mathbf{F})$, the set of $\mathbf{F}$-rational points of the representing group, is equal to $H^{k+1-i}(M_a, \Omega^i_{log})$ and we have shown this group is finite, we have dim $U_1^{i,k+1-i} = 0$. By Proposition 4.6d), this implies $T^{i-1,k+2-i} = 0$.

For b), we treat the case $i = a+1$, the other being similar. When $\mathbf{F}$ is algebraically closed, the isomorphism $H^{k+1-i}(M_a, \Omega^i_{log}) \cong H^0(I, B^1_I)(\chi^{b+2})$ follows immediately from 8.2 and 8.4. Moreover, it implies that dim $U_1^{i,k+1-i} = 0$ if and only if $H^0(I, B^1_I)(\chi^{b+2}) = 0$. Thus by Proposition 4.6c) and e), $T^{i-1,k+2-i} = 0$ if and only if $H^0(I, B^1_I)(\chi^{b+2}) = 0$. It is well-known that this condition is equivalent to the vanishing of $H^1_{cris}(I/W)(\chi^{b+2})_{(0,1)}$. $\square$

9. Cohomology of exact differentials In this section we will prove Theorem 2.6. To that end, recall the sheaves of higher exact differential forms B^i_n on X: we set $B^i_1 = B^i$ and define the higher $B^i_n \subseteq Z^i_X$ inductively via the exact sequences

$$0 \to B^i \to B^i_{n+1} \xrightarrow{\mathcal{C}} B^i_n \to 0. \tag{9.1}$$

Similarly, we have sheaves $B^1_{n,I}$ of higher exact differentials on I. As in Section 8, the key point is a sheaf computation:

Theorem 9.2. *Assume the standing hypotheses, fix an integer i with $0 \leq i \leq k$, and set $b = a + k - 2i$. Suppose $i \leq a$ and $k - i \leq p - 1 - a$. Then for all $n \geq 1$ we have isomorphisms of sheaves*

$$R^{k-i} f_* B^{i+1}_{n,M}(\chi^a) \cong B^1_{n,I}(\chi^b)$$

compatible with the Cartier operators.

Proof: To establish this isomorphism we will use the d- and $\mathcal{C}$-constructions, as in the proof of 8.2. First of all, note that we have an exact sequence

$$0 \to B^{i+1}_{n,X} \xrightarrow{(\mathcal{C},1)} B^{i+1}_{n-1,X} \oplus Z^{i+1}_X \xrightarrow{1-\mathcal{C}} \Omega^{i+1}_X \to 0;$$

injectivity on the right follows from that of $1 : B^{i+1}_{n,X} \to Z^{i+1}_X$, surjectivity on the left follows from that of $\mathcal{C} : Z^{i+1}_X \to \Omega^{i+1}_X$, and exactness in the middle is just the definition of $B^{i+1}_{n,X}$. Taking cohomology, we have

$$0 \to R^{k-i} f_* B^{i+1}_{n,M} \xrightarrow{(\mathcal{C},1)} R^{k-i} f_* B^{i+1}_{n-1,M} \oplus R^{k-i} f_* Z^{i+1}_M \xrightarrow{1-\mathcal{C}} R^{k-i} f_* \Omega^{i+1}_M.$$

If s is a section of $R^{k-i} f_* B^{i+1}_{n,M}$ we deduce n sections $s_1, \ldots, s_n$ of $R^{k-i} f_* Z^{i+1}_M$: by definition s_j is the image of s under the map

$$R^{k-i} f_* B^{i+1}_{n,M} \xrightarrow{\mathcal{C}^{n-j}} R^{k-i} f_* B^{i+1}_{j,M} \xrightarrow{1} R^{k-i} f_* Z^{i+1}_M.$$

Defining $s_0 = 0$ by convention, the s_j satisfy $\mathcal{C}(s_j) - 1(s_{j-1}) = 0$ in $R^{k-i} f_* \Omega^{i+1}_M$. Conversely, each system $s_1, \ldots, s_n$ of sections of $R^{k-i} f_* Z^{i+1}_M$ satisfying these equations determines uniquely a section s of $R^{k-i} f_* B^{i+1}_{n,M}$.

Now for the a and i under consideration, 7.4 gives us an isomorphism $R^{k-i} f_* Z^{i+1}_M \cong \mathcal{Z}^{i+1} = D_{i+1} + C_{i+2}$. For each s_j we have sections $(f_j \omega_c^{-k}, dg_j \omega_c^k)$ of $D_{i+1} \oplus C_{i+2}$ whose image in $R^{k-i} f_* Z^{i+1}_M$ is s_j. The pair $(f_j \omega_c^{-k}, dg_j \omega_c^k)$ is not uniquely determined by s_j, but by 7.2, the function h_j defined by

$$h_j \frac{dq}{q} = d\phi(f, dg) = \frac{1}{i!} \Theta^i(df_j) + \frac{(-1)^k k!}{i!} \Theta^{(p-1)-(k-i)}(dg_j)$$

is well-defined. Let c' be the constant

$$c' = \binom{k-i+a}{i+1} \Big/ \binom{a}{i+1}$$

and define a rational differential τ by

$$\tau = \left(h_1^{p^{n-1}} + c' h_2^{p^{n-2}} + c'^2 h_3^{p^{n-3}} + \cdots + c'^{(n-1)} h_n\right) \frac{dq}{q}.$$

Since $h_i\, dq/q$ is exact, $\mathcal{C}^n(\tau) = 0$ and (supposing for the moment that τ is a regular differential), we have $\tau \in B^1_{n,I}(\chi^b)$; thus we have defined a homomorphism $R^{k-i} f_* B^{i+1}_{n,M} \xrightarrow{\psi_n} B^1_{n,I}$. Moreover, the diagram

$$\begin{array}{ccccccccc}
0 \to & R^{k-i} f_* B^{i+1}_M & \to & R^{k-i} f_* B^{i+1}_{n+1,M} & \xrightarrow{\mathcal{C}} & R^{k-i} f_* B^{i+1}_{n,M} & \to 0 \\
 & \Big\downarrow c'^n \psi_1 & & \Big\downarrow \psi_{n+1} & & \Big\downarrow \psi_n & \\
0 \to & B^1_I & \to & B^1_{n+1,I} & \xrightarrow{\mathcal{C}} & B^1_{n,I} & \to 0
\end{array}$$

has exact rows and commutes. By induction and the 5-lemma, to prove that each ψ_n is an isomorphism, it suffices to treat the case $n = 1$.

Summing up, to finish the proof of the theorem we have to show that the rational differential $\tau = \psi_n(s)$ is regular, and that $\psi_1 : R^{k-i} f_* B^{i+1}_M(\chi^a) \to B^1_I(\chi^b)$ is an isomorphism. We first prove that τ is regular. From the definition, it is clear that τ is regular off the supersingular points; fix one supersingular point x and an eigenuniformiser t there. Calculating as in the proof of Theorem 8.2 (using the d- and $\mathcal{C}$-constructions) we find that if the image of s_j under $1 : R^{k-i} f_* Z^{i+1}_M \to R^{k-i} f_* \Omega^{i+1}_M$ is represented in the stalk at x by

$$\alpha_j t^{k-2i-p+a-1} \frac{dq}{q} \omega_c^{2i-k}$$

(with $\alpha_j \in \mathbf{F}$) and if the image of s_j under $\mathcal{C} : R^{k-i} f_* Z^{i+1}_M \to R^{k-i} f_* \Omega^{i+1}_M$ is represented by

$$\beta_j t^{k-2i-p+a-1} \frac{dq}{q} \omega_c^{2i-k}$$

($\beta_j \in \mathbf{F}$), then the function h_j is congruent to

$$(-1)^{i+1} \beta_j^p t^{p(k-2i-p+a-1)} - c' \alpha_j t^{k-2i-p+a-1}$$

modulo regular functions. Since the constraint $1(s_j) - \mathcal{C}(s_{j+1}) = 0$ is exactly the condition that $\alpha_j = \beta_{j+1}$, we have that τ is indeed regular.

It remains to check that ψ_1 is an isomorphism. Injectivity follows from that of ϕ which was proven in 8.1. Finally, let us check that ψ_1 is surjective. Note that by using sections $\Theta^{k-i}(dg)\omega_c^k$ of C_{i+1} with $\mathrm{ord}_x(dg) \geq -i + p(k-i)$ we can obtain any element $\tau \in B^1_{I,x}$ with $\mathrm{ord}_x(\tau) \geq -i + p(k-i)$. To show surjectivity in the required eigenspaces, it thus suffices to produce a τ in the image with $\mathrm{ord}_x(\tau) = l$ for all l satisfying $l + 1 \equiv a + k - 2i \pmod{p-1}$ and $0 \leq l < -i + p(k-i)$. On the other hand, by using sections $f\omega_c^{-k}$ of D_{i+1} with $\mathrm{ord}_x(f) \geq k + (p-1)i + a$ we can produce sections τ of $B^1_{I,x}$ with any valuation $\geq k - 2i$ and $\not\equiv -1, -2, \ldots, -1-i \pmod{p}$. A short computation shows that we obtain all the needed sections. This completes the proof of the theorem. □

Proof of Theorem 2.6: Theorem 2.3a says that T^{ij} vanishes unless $i + j = k + 1$. Either the hypotheses of a) and Theorem 2.4a) or the hypotheses of b) and Theorem 2.4b), imply that $T^{i-1,k+2-i} = T^{i,k+1-i} = 0$. As we remarked in Section 2, the proof of [I-R], IV.4.5 and the vanishing of these T's then shows that the three W-modules $H^{k+1-i}(M_a, ZW\Omega^i)$, $H^{k+1-i}(M_a, W\Omega^i)$, and $H^{k+1-i}(M_a, BW\Omega^{i+1})$ are isomorphic to direct factors of $H^{k+1}_{cris}(M_a/W)$ (the parts of slopes i, $[i, i+1)$ and $(i, i+1)$ respectively) and we have an isomorphism

$$H^{k+1-i}(M_a, W\Omega^i) \cong H^{k+1-i}(M_a, ZW\Omega^i) \oplus H^{k+1-i}(M_a, BW\Omega^{i+1})$$

compatible with the actions of F and V. Also, $H^{k+2-i}(M_a, W\Omega^i) = 0$ as it is a direct factor of $H^{k+2}_{cris}(M_a/W)$ which is 0. Since F is an automorphism of $H^{k+1-i}(M_a, ZW\Omega^i)$, we have

$$H^{k+1-i}(M_a, BW\Omega^{i+1})/F \cong H^{k+1-i}(M_a, W\Omega^i)/F \cong H^{k+1-i}(M_a, W\Omega^i/F).$$

By [I], II.2.2.2, we have

$$H^{k+1-i}(M_a, W\Omega^i/F) \cong \varprojlim_n H^{k+1-i}(M_a, B_n^{i+1})$$

where the inverse limit is taken with respect to the maps induced by the Cartier operator. Similarly, we have an isomorphism

$$H^1(I, BW\Omega^1)/F \cong H^1(I, W\Omega^1)/F \cong \varprojlim_n H^1(I, B_n^1).$$

By Theorem 6.7, for the i and a in question, $R^{k+1-i}f_*B_M^{i+1}(\chi^a) = 0$, and using 9.1 and induction we have also $R^{k+1-i}f_*B_{M,n}^{i+1}(\chi^a) = 0$ for all $n \geq 1$. Thus $H^{k+1-i}(M_a, B_n^{i+1}) \cong H^1(I, R^{k-i}f_*B_n^{i+1})(\chi^a)$ and the theorem follows immediately from Theorem 9.2. $\square$

[**E**] Ekedahl, T.: Diagonal complexes and F-gauge structures. Paris: Hermann (Travaux en cours 18) 1986

[**H**] Hida, H.: Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. Invent. Math. **85** (1986) 545-613

[**I1**] Illusie, L.: Complexe de deRham-Witt et cohomologie cristalline. Ann. Sci. Ec. Norm. Super. **12** (1979) 501-661

[**I2**] Illusie, L.: Finiteness, duality, and Künneth theorems in the cohomology of the deRham Witt complex. In: Raynaud, M. and Shiota, T. (Eds.) Algebraic Geometry Tokyo-Kyoto (Lect. Notes in Math. 1016.) pp. 20-72 Berlin Heidelberg New York: Springer 1982

[**I-R**] Illusie, L. and Raynaud, M.: Les suites spectrales associées au complexe de deRham-Witt. Inst. Hautes Etudes Sci. Publ. Math. **57** (1983) 73-212

[**K1**] Katz, N.: p-adic properties of modular schemes and modular forms. In: Kuyk, W. and Serre, J.-P.(Eds.) Modular Functions of One Variable III (Lect. Notes in Math. 350.) pp. 69-190 Berlin Heidelberg New York: Springer 1973

[**K2**] Katz, N.: A result on modular forms in characteristic p. In: Serre, J.-P. and Zagier, D.B. (Eds.) Modular Functions of One Variable V (Lect. Notes in Math. 601.) pp. 53-61 Berlin Heidelberg New York: Springer 1977

[**K3**] Katz, N.: Slope Filtration of F-crystals. Astérisque **63** (1979) 113-164

[**K-M**] Katz, N. and Mazur, B.: Arithmetic Moduli of Elliptic Curves. Princeton: Princeton University Press 1985

[**K-O**] Katz, N. and Oda, T.: On the differentiation of deRham cohomology classes with respect to parameters. J. Math. Kyoto Univ. **8** (1968) 199-213

[**M**] Milne, J.S.: Values of zeta functions of varieties over finite fields. Amer. J. Math. **108** (1986) 297-360

[**Mi**] Miyake, T.: Modular Forms. Berlin Heidelberg New York: Springer 1989

[**O**] Oda, T.: The first deRham cohomology group and Dieudonné modules. Ann. Sci. Ec. Norm. Super.(4) **2** (1969) 63-135

[**S**] Serre, J.-P.: Groupes Algebriques et Corps de Classes. Paris: Hermann 1959

[**Sh**] Shimura, G.: Introduction to the Arithmetic Theory of Automorphic Functions. Princeton: Princeton University Press 1971

[**U1**] Ulmer, D.L.: p-descent in characteristic p. Duke Math. J. **62** (1991) 237-265

[**U2**] Ulmer, D.L.: On the Fourier coefficients of modular forms. Preprint (To appear in l'Annales Sci. ENS) (1993)

[**U3**] Ulmer, D.L.: Slopes of modular forms. In: Childress, N. and Jones, J. (Eds.) Arithmetic Geometry (Contemporary Mathematics **174**). pp. 167-183 Providence: American Mathematical Society 1994