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Slopes of Modular Forms

DOUGLAS L. ULMER

Abstract. We formulate a strategy for proving that the number of modular eigenforms of weight w and level pN whose eigenvalue for the Hecke operator U_p has a fixed p-adic valuation is bounded independently of w. We also carry out the first step in this program.

1. Introduction

In this note, we continue our study of the p-adic valuations of eigenvalues of Hecke operators on modular forms on $\Gamma_1(pN)$. Our aim is to discuss the possibility of proving a "control theorem" in the style of Hida. Such a theorem would assert roughly that the number of normalized eigenforms of weight w and level pN whose eigenvalue for U_p has some fixed valuation λ is bounded independently of w. This assertion is known when $\lambda = 0$ and it is the starting point for one possible approach to Hida's theory of p-adic analytic families of *ordinary* modular eigenforms (cf. [**H**, Ch. 7]). Our motivation for studying this question comes from the possibility of generalizing Hida's theory to the non-ordinary case, i.e., to $\lambda > 0$.

Here we take the first step toward a control theorem by extending the results of [**U1**], which were valid only for small weights, to all weights. As we will explain below, the truth of a control theorem is then equivalent to a suitable extension of the results of [**U2**] to all weights. We also discuss the relationship between our results and recent conjectures of Gouvêa and Mazur. It is interesting to note that their conjectures (which predict a very simple periodicity property, as a function of the weight, for the number of eigenforms with a given λ) imply the existence of many cohomology classes of a type usually considered to be pathological.

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2. Slopes of modular forms

We begin by introducing spaces of modular forms with p-adic coefficients and their slope decompositions. Our main objects of study are the dimensions of the factors in this decomposition.

Fix a positive integer M and a non-negative integer k. Then we have the complex vector space $S_{k+2}(\Gamma_1(M))$ of cusp forms of weight $k+2$ for the congruence subgroup $\Gamma_1(M)$ of $\mathrm{SL}_2(\mathbf{Z})$. This space carries an action of Hecke operators T_ℓ for all primes $\ell \not| M$, U_ℓ for $\ell | M$, and $\langle d\rangle_M$ for $d \in (\mathbf{Z}/M\mathbf{Z})^\times$ ([Sh], Ch. 3 or [Mi], 4.5). If $M = M_1 M_2$ with $(M_1, M_2) = 1$, then the operators $\langle d\rangle_M$ can be factored $\langle d\rangle_M = \langle d\rangle_{M_1}\langle d\rangle_{M_2}$ and $\langle d\rangle_{M_i}$ depends only on the image of d in $(\mathbf{Z}/M_i\mathbf{Z})^\times$. We have a direct sum decomposition

$$S_{k+2}(\Gamma_1(M)) \cong \bigoplus_{\psi:(\mathbf{Z}/M\mathbf{Z})^\times\to\mathbf{C}} S_{k+2}(\Gamma_0(M),\psi)$$

where $S_{k+2}(\Gamma_0(M),\psi)$ is the subspace on which the $\langle d\rangle_M$ act via the character ψ. This decomposition is preserved by the Hecke operators.

We need a rational version of these decompositions. Let

$$S_{k+2}(\Gamma_1(M);\mathbf{Q}) \subseteq S_{k+2}(\Gamma_1(M))$$

be the subspace of modular forms all of whose Fourier coefficients at the standard cusp ∞ are rational numbers. A fundamental theorem asserts that this space is a $\mathbf{Q}$-structure on $S_{k+2}(\Gamma_1(M))$, i.e.,

$$S_{k+2}(\Gamma_1(M);\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C} \cong S_{k+2}(\Gamma_1(M)).$$

(This even holds with $\mathbf{Q}$ replaced with $\mathbf{Z}$; cf. [Sh] 3.52.) Moreover, the Hecke operators preserve this structure. For any commutative $\mathbf{Q}$-algebra R, define

$$S_{k+2}(\Gamma_1(M);R) = S_{k+2}(\Gamma_1(M);\mathbf{Q}) \otimes_{\mathbf{Q}} R;$$

we will be interested mostly in the case where R is a field of p-adic numbers.

If $M = M_1M_2$ is a factorization with $(M_1, M_2) = 1$ and R contains the $\phi(M_1)$-th roots of unity $\mu_{\phi(M_1)}$, then we have a direct sum decomposition

$$S_{k+2}(\Gamma_1(M);R) \cong \bigoplus_{\psi:(\mathbf{Z}/M_1\mathbf{Z})^\times\to R} S_{k+2}(\Gamma_1(M);R)(\psi)$$

where $f \in S_{k+2}(\Gamma_1(M); R)$ lies in $S_{k+2}(\Gamma_1(M); R)(\psi)$ if and only if $\langle d\rangle_{M_1} f = \psi(d) f$ for all $d \in (\mathbf{Z}/M_1\mathbf{Z})^\times$.

Now fix a prime number p, a non-negative integer k, and an integer N prime to p. Let R be $\mathbf{Q}_p$, the p-adic numbers. We can apply the above constructions with $M_1 = p$, $M_2 = N$, and we get a direct sum decomposition:

$$S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p) \cong \bigoplus_{a=0}^{p-2} S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a)$$

where $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p$ is the Teichmüller character (characterized by $\chi(d) \equiv d \pmod{p}$). (Note that we are not decomposing for characters modulo N.) For any integer a, define

$$S(k, N, a) = S_{k+2}(\Gamma_1(pN); \mathbf{Q}_p)(\chi^a).$$

We will sometimes abbreviate this space to just S when k, N, and a are fixed.

We want to consider the action of U_p on $S = S(k, N, a)$ when $0 \le a < p - 1$. This action is semi-simple and when $a \ne 0$, its eigenvalues α are algebraic integers all of whose complex embeddings satisfy $\alpha\overline{\alpha} = p^{(k+1)}$. When $a = 0$, the eigenvalues of U_p coming from forms new at p satisfy $\alpha^2 = \zeta p^k$ (where ζ is a root of unity) and the other eigenvalues satisfy $\alpha\overline{\alpha} = p^{(k+1)}$ ([Mi], 4.6.17). Let v be the valuation of $\overline{\mathbf{Q}_p}$ normalized so that $v(p) = 1$. Then if α is an eigenvalue of U_p on S, we have $0 \le v(\alpha) \le k + 1$. We define $v(\alpha)$ to be the *slope* of the eigenvalue, and if $f \in S \otimes \overline{\mathbf{Q}_p}$ is an eigenvector for U_p with eigenvalue α, we will also call $v(\alpha)$ the slope of f.

The characteristic polynomial of U_p is a polynomial over $\mathbf{Q}_p$ and all of the roots of each of its irreducible factors are conjugate over $\mathbf{Q}_p$ and thus have the same valuation. Taking the generalized eigenspaces corresponding to each irreducible factor and collecting together those with a fixed slope gives a unique $\mathbf{Q}_p$-rational decomposition

$$S \cong \bigoplus_{\lambda} S_\lambda$$

(compatible with the Hecke operators) such that all of the eigenvalues of U_p on S_λ have slope λ.

We want to study the numbers

$$d_\lambda = d(k, N, a)_\lambda = \dim S(k, N, a)_\lambda$$

as functions of k and a. Gouvêa and Mazur have conjectured the following periodicity property: if $p > 3$, $k > 2\lambda$, and $n \ge \lambda$, then

$$d(k, N, 0)_\lambda \stackrel{?}{=} d(k + p^n(p-1), N, 0)$$

[**G-M**, Conjecture 1]. It is reasonable to suppose that more generally,

$$d(k, N, a)_\lambda \stackrel{?}{=} d(k + p^n(p-1), N, a)$$

for all a; in private communication they have in fact made even stronger variations of their basic conjecture. The displayed assertions are theorems when $\lambda = 0$, by work of Hida. Note also that the second assertion certainly imples that for a fixed N and λ, $d(k, N, a)_\lambda$ is bounded independently of k and a. It is this weaker assertion (refered to as a "control theorem") that we wish to discuss for the moment.

It will be convenient to package the numbers d_λ into Newton polygons. Recall that if

$$H(T) = 1 + a_1 T + \cdots + a_d T^d = \prod_{i=1}^{d} (1 - \alpha_i T)$$

is a polynomial with $\overline{\mathbf{Q}_p}$ coefficients and with the factors α_i ordered so that $v(\alpha_i) \leq v(\alpha_{i+1})$, then the *Newton polygon* of H with respect to v is defined to be the graph of the continuous, piecewise linear, convex function f on $[0, d]$ with $f(0) = 0$ and $f'(x) = v(\alpha_i)$ for all $x \in (i-1, i)$. This polygon can be seen to be part of the boundary of the convex hull in the plane of the points $(0, 0)$ and $(i, v(a_i))$ for $i = 1, \ldots, d$. In particular, if the $v(a_i) \in \mathbf{Z}$, the break points of the Newton polygon (i.e., the points on the polygon where f changes slope) have integer coordinates.

For fixed k, N, and a, knowledge of the d_λ is clearly equivalent to knowledge of the Newton polygon of the characteristic polynomial

$$H(T) = \det(1 - U_p T | S(k, N, a)).$$

Given non-negative real numbers $l_0, \ldots, l_n$, define the associated *Hodge polygon* to be the graph of the continuous, piecewise linear, convex function f on $[0, l_0 + \cdots + l_n]$ with $f(0) = 0$ and $f'(x) = i$ for all $x \in (l_0 + \cdots + l_{i-1}, l_0 + \cdots + l_i)$. We want to consider the highest Hodge polygon (i.e., polygon with integral slopes) lying on or below the Newton polygon of H. Generally, given any Newton polygon, define its associated *contact polygon* as the highest Hodge polygon lying on or below it and having the same endpoints. From its definition, the contact polygon has the following properties: the Newton and contact polygons must meet at some point along each edge of the contact polygon; and, if $A_1, \ldots, A_n$ are the break points of the Newton polygon which lie on the contact polygon (in order), then for each j, either a) the Newton and contact polygons coincide along the segment $\overline{A_j A_{j+1}}$ (which thus has integral slope) or b) the contact polygon has two edges between A_j and A_{j+1}, with slopes i and $i+1$ for some integer i and the slopes of the Newton polygon between A_j and A_{j+1} all lie in the interval $(i, i+1)$. (Both properties hold since if they failed, we could raise some edge of the contact polygon.) Figure 1 illustrates the two possibilities.

Note that in the second case, if l_i and l_{i+1} are the lengths of the two edges of the contact polygon between A_j and A_{j+1} and if $v(\alpha_1), \ldots, v(\alpha_k)$ are the slopes of the Newton polygon between A_j and A_{j+1} (with multiplicities) we have $l_i + l_{i+1} = k$ and $il_i + (i+1)l_{i+1} = \sum v(\alpha_j)$. Thus the following formula gives the lengths m_i of the sides of the contact polygon in terms of the slopes of the Newton polygon:

$$m_i = \sum_{v(\alpha_j)\in(i-1,i)} (v(\alpha_j) - (i-1)) + \sum_{v(\alpha_j)=i} 1 + \sum_{v(\alpha_j)\in(i,i+1)} (i+1-v(\alpha_j)).$$

Since all the $v(a_i)$ are integers, the break points of the Newton polygon occur at points with integer coordinates and so the m_i are also integers.

Figure 1

Knowledge of the contact polygon attached to $H(T)$ gives some control on the dimensions d_λ. Indeed, if $m_0, \ldots, m_{k+1}$ are the lengths of the contact polygon, we have

$$d_i \leq m_i$$

and

$$\sum_{i\leq\lambda\leq i+1} d_\lambda \leq m_i + m_{i+1}.$$

Also,

$$\sum_{j\leq i} m_j \leq \sum_{\lambda<i+1} d_\lambda \leq \sum_{\lambda\leq i+1} d_\lambda \leq \sum_{j\leq i+1} m_j$$

so a control theorem for a given p and N is equivalent to the assertion that the corresponding m_i are bounded independently of k and a.

In the next section, we will review some work of Crew and Ekedahl which allows one to relate the m_i to cohomological invariants and in Sections 4 and 5, we compute some of these invariants. This amounts to extending in some form the Newton-Hodge inequalities of [**U1**] to all weights k. (The results of [**U1**] were proven only for $k < p$ and the extension to all k gives a slightly less precise result.) The remaining invariant was studied in [**U2**], again for $k < p$, and we hope to extend those computations to all k.

3. Contact polygons and cohomology

Let $\mathbf{F}$ be a perfect field of characteristic p and X a smooth projective variety over $\mathbf{F}$. The action of Frobenius on the (crystalline or ℓ-adic) cohomology of X gives rise to a characteristic polynomial $H(T)$ with $\overline{\mathbf{Q}}$-coefficients and thus to a Newton polygon which encodes the p-adic valuations of the roots of H. This is in general of course a rather delicate invariant of X and the associated contact polygon is also subtle. However, in some cases good control on the contact polygon can be obtained. We want to use this to study the polygons attached to modular forms.

We need to work in a more general setting, so let $\mathbf{F}$ and X be as above and assume further that $\Pi \in \mathbf{Q}_p[\mathrm{Aut}_{\mathbf{F}} X]$ is a projector, i.e., $\Pi^2 = \Pi$. The pair (X, Π) is then a (Chow) motive with $\mathbf{Q}_p$-coefficients defined over $\mathbf{F}$, which we denote by M. For each n, the group algebra element Π defines a projector on $H^n_{cris}(X/W) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (where $W = W(\mathbf{F})$ is the Witt ring of $\mathbf{F}$) and we set

$$H^n_{cris}(M/W) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \Pi\left(H^n_{cris}(X/W) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\right).$$

The absolute Frobenius of $\mathbf{F}$ acts on $H^n_{cris}(X/W)$ and commutes with Π; extending scalars to K, the fraction field of the Witt ring of the algebraic closure of $\mathbf{F}$, we get a finite dimensional K vector space with a σ-linear automorphism (where σ is the automorphism of K induced by the absolute Frobenius of $\mathbf{F}$), i.e., an F-isocrystal.

Now finite dimensional K vector spaces with σ-linear automorphisms F are completely classified by their slopes, which are defined as follows. There exist non-negative rational numbers $r_1/s_1, \dots, r_k/s_k$ (written in lowest terms and with 0 written as $0/1$) such that

$$V \cong \bigoplus_i K[T]/(T^{s_i} - p^{r_i})$$

with $F(T^j) = T^{j+1}$ and extended by semi-linearity. The r_i/s_i are unique up to permutation and by definition they are the *slopes* of V, with each r_i/s_i counted with multiplicity s_i. If V happens to come by extension of scalars from a $\mathrm{Frac}(W(\mathbf{F}_{p^f}))$ vector space V_0, so that F^f is a linear automorphism of V_0, then the slopes and multiplicities of V as defined here coincide with the valuations of the inverse roots of the characteristic polynomial $\det(1 - F^f T|V_0)$, computed with respect to the valuation with $v(p) = 1/f$.

In the next section we will find a pair (X, Π) over $\mathbf{F}_p$ so that the slopes of $H^n_{cris}(M/W) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ for some fixed n are the slopes of U_p on a suitable combination of the $S(k, N, a)$. Thus we can apply all the techniques of p-adic cohomology to study these slopes. In fact we need to do something slightly more, namely to find a projector $\Pi \in \mathbf{Z}_p[\mathrm{Aut}_{\mathbf{F}}]$ (i.e., Π should have p-integral coefficients). Then Π acts on $H^n_{cris}(X/W)$ itself and we define $H^n_{cris}(M/W) = \Pi H^n_{cris}(X/W)$.

Return now to a general (X, Π), but assume that Π has $\mathbf{Z}_p$ coefficients. Let m_{ij} be the i-th contact number associated with the slopes of $H^{i+j}_{cris}(M/W)$. We want to review some work of Crew and Ekedahl which allows one to compute the m_{ij} in terms of two other (hopefully more tractable) cohomological invariants. More precisely, there exist non-negative integers h^{ij}_W and T^{ij} to be discussed below such that

$$m_{ij} = h^{ij}_W - T^{ij} + 2T^{i-1,j+1} - T^{i-2,j+2}. \tag{3.1}$$

The h^{ij}_W are called Hodge-Witt numbers and are closely related to the usual Hodge numbers $h^{ij} = \dim_{\mathbf{F}} \Pi H^j(X, \Omega^i)$ (which make sense because Π has p-integral coefficients). The T^{ij} are a measure of the non-finitely generated torsion in the deRham-Witt cohomology of (X, Π).

The precise definitions of the h^{ij}_W and T^{ij} are rather technical, so for the moment we will just give some of the properties these numbers enjoy and which make it possible to compute them in some cases. We will give a precise definition of T^{ij} in Section 5 and then formula 3.1 can be taken as the definition of the h^{ij}_W.

First of all, $T^{ij} \geq 0$ and $T^{ij} = 0$ if $i < 0$ or $i \geq \dim X - 1$ or $j \leq 1$ or $j > \dim X$. Since $\sum_{i+j=n} m_{ij} = b_n = \dim_K H^n_{cris}(M/W) \otimes K$ by definition, the formula 3.1 gives that

$$\sum_{i+j=n} h^{i,j}_W = b_n.$$

If we define the Hodge-Witt polygon to be the Hodge polygon associated to the $h^{i,j}_W$ (where $n = i + j$ is fixed), then 3.1 can be interpreted as saying that $T^{i,j}$ is the number of units the slope $i + 1$ edge of the Hodge-Witt polygon should be raised in order that it coincide with the slope $i + 1$ edge of the contact polygon along some interval. In particular, the vanishing of the T^{ij} for (i, j) outside the range $0 \leq i \leq \dim X - 1$, $1 \leq j \leq \dim X$ and the formula for the sum of the h^{ij}_W imply that the Hodge-Witt and contact (and Newton) polygons have the same endpoints.

Secondly, there are two relations between the h^{ij}_W and the h^{ij}, namely *Crew's formula:*

$$\sum_j (-1)^j h^{ij}_W = \sum_j (-1)^j h^{ij} = \chi(M, \Omega^i)$$

(which we will use below) and *Ekedahl's inequality:*

$$h^{ij}_W \leq h^{ij}.$$

The latter, together with 3.1, implies that when $\sum_{i+j=n} h^{ij} = b_n$ we have $h^{ij}_W = h^{ij}$ for $i + j = n$.

The interested reader can find more details about the slopes associated to an F-crystal in [**Ma**, Ch. 2]. The work of Crew and Ekedahl, as well as results of Illusie and Raynaud upon which it is based, is beautifully surveyed in Illusie's article [**I**]; we will use it as a general reference for these ideas. See also the related work of Milne [**M**] which among other things gives another proof of Crew's formula.

4. Slopes of a modular variety

In this section we are going to define a modular variety $\tilde{X}$ and a projector Π such that the slopes of Frobenius on its cohomology are the slopes of U_p on certain spaces of modular forms. The arguments are essentially a simple modification of those of Section 2 of [**U1**], so we will be brief.

Fix a prime $p > 2$, an integer $N \geq 5$ and prime to p, and a positive integer k. Let $I = Ig_1(pN)$ be the Igusa curve of level pN, i.e., the complete modular curve parameterizing elliptic curves with a $\Gamma_1(N)$ structure and an Igusa structure of level p. Forgetting the Igusa structure induces a map $I \to X_1(N)$ of degree $p-1$ which is unramified off the points representing supersingular elliptic curves and is totally ramified over the supersingular points. Let $C \subseteq I$ be the reduced subscheme of cusps, i.e., the points representing singular elliptic curves and let $S \subseteq I$ be the reduced subscheme of supersingular points.

Let $\pi_N : \mathcal{E}_N \to X_1(N)$ be the universal generalized elliptic curve over $X_1(N)$ and $\pi : \mathcal{E} \to I$ its pull back to I. Define X to be the k-fold fiber product

$$\mathcal{E} \times_I \cdots \times_I \mathcal{E}.$$

When $k > 1$, this variety has singularitites coming from the product of double points in the fibers of π over C. There is a canonical disingularization $\tilde{X} \to X$ defined by Deligne which is described in some detail in [**U1**], Section 4; it is obtained from X by blowing up points in the fibers over C. In particular, away from C, $f : \tilde{X} \to I$ is still the k-fold fiber product of a smooth elliptic curve.

The group $H = (\mathbf{Z}/N\mathbf{Z} \rtimes \mu_2)^k$ acts on X, with the factors $\mathbf{Z}/N\mathbf{Z}$ acting by translation by the canonical point of order N and the factors μ_2 acting by inversion; this action lifts to $\tilde{X}$. Also, $(\mathbf{Z}/p\mathbf{Z})^\times$ acts on I via its natural action on Igusa structures and on $\tilde{X}$; the map f is equivariant for these actions. Define a character $\epsilon : H \to \{\pm 1\}$ by requiring that ϵ be 1 on the factors $\mathbf{Z}/N\mathbf{Z}$ and the identity $\mu_2 \xrightarrow{\sim} \{\pm 1\}$ on the factors μ_2, and let Π be the corresponding projector in $\mathbf{Z}[1/2N][\mathrm{Aut}\,\tilde{X}]$. For any $\mathbf{Z}[1/2N][H]$-module V, we write $V(\epsilon)$ for ΠV.

Now define polynomials encoding the eigenvalues of U_p as follows:

$$\begin{aligned} H_k(T) = &\prod_{\psi:(\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}} \det\left(1 - T_p T + \psi(p)p^{k+1}T^2 | S_{k+2}(\Gamma_1(N), \psi)\right) \\ &\times \prod_{\substack{\psi:(\mathbf{Z}/pN\mathbf{Z})^\times \to \mathbf{C} \\ p | cond(\psi)}} \det\left(1 - U_p T | S_{k+2}(\Gamma_1(pN), \psi)\right). \end{aligned}$$

(This is essentially the characteristic polynomial of U_p on $S_{k+2}(\Gamma_1(pN))$, except that we have removed the factors coming from eigenforms which are new at p but whose character has conductor prime to p; these forms all have slope $k/2$.)

To state the main result of this section, let c be the number of cusps on the modular curve $X_1(N)$, and set $c_1 = (p-1)c$; then c_1 is the number of cusps on I, i.e., the degree of the divisor C. We also introduce multiplicities

$$e_{kj} = \binom{k}{j} - \binom{k}{j-1}.$$

THEOREM 4.1. *Let p be a prime number, $N \geq 5$ an integer relatively prime to p, and k a non-negative integer. Define the variety $\tilde{X}$, the character ϵ and the polynomials H_j as above. Then*

a)

$$\det(1 - \mathrm{Fr}\, T | (H^{k+1}_{cris}(\tilde{X}/W) \otimes \mathbf{Q}_p)(\epsilon)) = H_k(T) \prod_{1 \leq j \leq k/2} \left(H_{k-2j}(p^j T) P_{k,j}(T)\right)^{e_{kj}}$$

where $P_{kj}(T)$ is a product $\prod(1-\alpha T)$ of degree c_1 if $k-2j \neq 0$ or degree $c_1 - 1$ if $k - 2j = 0$ all of whose inverse roots α have the form ζp^{k+1-j} where ζ is a root of unity.

b)

$$\det(1 - \mathrm{Fr}\, T | (H^{k}_{cris}(\tilde{X}/W) \otimes \mathbf{Q}_p)(\epsilon)) = \begin{cases} (1 - p^{k/2}T)^{e_{k,k/2}} & \textit{if } k \textit{ is even} \\ 0 & \textit{otherwise.} \end{cases}$$

c)

$$H^i_{cris}(\tilde{X}/W)(\epsilon) = 0 \quad \textit{if } i \neq k, k+1.$$

The proof of this theorem is a minor modification of that of Corollary 2.2 of [**U1**] and will be left to the reader. The main difference is that since we are not using the symmetric group in our projector, the computation using the Künneth formula, the coherent cohomology of abelian varieties, and linear algebra which takes place in Proposition 2.1b of [**U1**] has a different outcome. The new computation is, however, every bit as standard as the old. Not using the symmetric group has the advantage that our projector has p-integral coefficients for all k. For simplicity, we have not broken down the result according to characters of $(\mathbf{Z}/p\mathbf{Z})^\times$.

REMARKS. 1) Note that since $\tilde{X}$ is defined over $\mathbf{F}_p$, the valuations of the inverse roots of the polynomials of the Theorem are the slopes of $(\tilde{X}, \Pi)$ as defined in the last section.
2) If G_k denotes the polynomial on the right hand side of part a) of the Theorem, then knowledge of the valuations of the inverse roots of G_j for $j \leq k$ is clearly equivalent to knowledge of the valuations of the inverse roots of the H_j for $j \leq k$. In particular, the mixing together of different weights caused by not using the

symmetric group does not prevent us from obtaining results on the slopes for individual weights.

Corollary 4.2. *We have* $m_{ij} = 0$ *if* $i + j \neq k$ *or* $k + 1$, *or if* $i + j = k$ *and* $i \neq k/2$. *Also,* $m_{k/2,k/2} = e_{k/2,k/2} = \binom{k}{k/2} - \binom{k}{k/2-1}$.

5. Hodge-Witt numbers of a modular variety

In this section we will compute the Hodge-Witt numbers h_W^{ij} of the pair $(\tilde{X}, \Pi)$ defined in the previous section.

To state the result, let g be the genus of $X_1(N)$, c the number of cusps on this curve and set $w = (g - 1) + c/2$; then w is the degree of the invertible sheaf $\omega = (R^1 \pi_{N*} \mathcal{O}_{\mathcal{E}_N})^{-1}$ on $X_1(N)$. Also set $w_1 = (p - 1)w$ and $c_1 = (p - 1)c$. These are the degree of the pull back of ω to I and the number of cusps on I respectively. The genus g_1 of I satisfies $2g_1 - 2 = pw_1 - c_1$. Define integers $l_{k,i}$ by the formulae

$$l_{k,0} = l_{k,k+1} = g_1 - 1 + kw_1 + \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$l_{k,i} = (p - 2)w_1 \qquad 1 \leq i \leq k$$

Theorem 5.1. *Let* p *be an odd prime number,* $N \geq 5$ *an integer prime to* p, *and* k *a non-negative integer. The Hodge-Witt numbers of the pair* $(\tilde{X}, \Pi)$ *defined above are as follows: for* $0 \leq i \leq k + 1$

$$h_W^{i,k+1-i} = \sum_{j \leq \min(i,k+1-i,k/2)} l_{k-2j,i-j} e_{kj} + \begin{cases} c_1 e_{k,k+1-i} & \text{if } i \geq \frac{k+3}{2} \\ (c_1 - 1) e_{k,k+1-i} & \text{if } i = \frac{k+2}{2} \\ 0 & \text{if } i \leq \frac{k+1}{2}, \end{cases}$$

$$h_W^{k/2,k/2} = \begin{cases} e_{k/2,k/2} & \text{if } k \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

and all other $h_W^{ij} = 0$.

Remarks. 1) If M_k denotes the pair $(\tilde{X}_k, \Pi_k)$ attached to modular forms of weight $k + 2$ and level pN studied in [**U1**] (so $k < p$), then we have that the non-zero Hodge-Witt (and Hodge) numbers of M_k are $h_W^{i,k+1-i} = l_{k,i}$ if $k > 0$; when $k = 0$, $h_W^{01} = h_W^{10} = g_1$ and $h_W^{00} = 1$. So, ignoring the terms involving c_1, the theorem says that the motive $M = (\tilde{X}, \Pi)$ "looks like" the direct sum of the M_{k-2j}, each with multiplicity e_{kj}. This of course should not be taken too seriously, since M_{k-2j} does not exist as a motive defined by a p-integral projector as soon as $k - 2j \geq p$.

2) The variety $\tilde{X}$ and the projector Π can be lifted to $\mathbf{Z}_p$ and the Hodge-Witt numbers of M are equal to the Hodge numbers of the generic fiber of this lifting. This equality of of Hodge-Witt and Hodge numbers is known to fail for a general smooth and proper variety over $\mathbf{Z}_p$, so it would be interesting to have some *a*

priori criterion that could be applied to the case at hand.
3) On the other hand, the Hodge-Witt numbers of $(\tilde{X}, \Pi)$ are definitely not equal to the Hodge numbers of this pair over $\mathbf{F}_p$. This fails as soon as $k \geq 2p-1$.

Corollary 5.2. *The Newton polygon of the polynomial*

$$\prod_{0 \leq j \leq k} H_{k-2j}(p^j T)^{e_{kj}}$$

lies on or above the Hodge polygon associated to the integers

$$l_i = \sum_{j \leq \min(i, k+1-i, k/2)} l_{k-2j, i-j} e_{kj} \qquad 0 \leq i \leq k+1$$

and these two polygons have the same endpoints.

Proof. By the definition (3.1) of the Hodge-Witt numbers, the Newton polygon of M lies on or above the Hodge polygon associated to the Hodge-Witt numbers. By 4.1, the Newton polygon has an edge of slope i and length at least $c_1 e_{k,k+1-i}$ for each $i \geq (k+3)/2$, and an edge of slope $(k+2)/2$ and length at least $(c_1 - 1)e_{k,k/2}$ when k is even. Removing these edges and reducing the Hodge-Witt numbers by a corresponding amount gives the result.

Remark. Corollary 5.2 would follow from an extension of the Newton-Hodge inequalities of [**U1**] (i.e., Theorem 1.2) to all weights, using the evident extension of the formulae for the Hodge numbers. Using other methods, Arthur Ogus has sketched a proof of such an extension [**O**, 8.4].

Proof of 5.1. The strategy will be first to show that the $T^{ij} = 0$ for $i + j \neq k+1$, then to use the definition of the h_W^{ij} (3.1) and the computation of the m_{ij} (4.2) to get the h_W^{ij} for all $i + j \neq k+1$. Then Crew's formula reduces the problem to computing the Euler characteristics $\chi(M, \Omega^i)$, which is easy.

At this point, we must give the definition of the T^{ij}! The usual definition is in terms of certain non-finitely generated torsion in deRham-Witt cohomology (hence the name torsion numbers). By a Theorem of Illusie and Raynaud, the T^{ij} are also equal to another cohomological invariant which is much easier to compute. We will take this latter invariant as our definition. For more details on what follows, see [**I**] and the references there.

Let $W_n\Omega^{\cdot}_{\tilde{X}}$ be the deRham-Witt complex of length n on $\tilde{X}$. This is a complex of étale sheaves of $\mathbf{Z}/p^n\mathbf{Z}$-modules, we have $W_1\Omega^{\cdot}_{\tilde{X}} = \Omega^{\cdot}_{\tilde{X}}$, and $W_n\Omega^0_{\tilde{X}} = W_n\mathcal{O}$ is the sheaf of Witt vectors introduced by Serre. Let $W_n\Omega^i_{log}$ be the subsheaf of $W_n\Omega^i_{\tilde{X}}$ generated additively by differentials

$$d\underline{x}_1/\underline{x}_1 \wedge \cdots \wedge d\underline{x}_i/\underline{x}_i$$

where $x_j \in \mathcal{O}^\times$ and $\underline{x}_j \in W_n\mathcal{O}$ is its multiplicative representative. There are exact sequences

$$0 \to W_m\Omega^i_{log} \to W_{m+n}\Omega^i_{log} \to W_n\Omega^i_{log} \to 0 \tag{5.3}$$

and we have $W_1\Omega^i_{log} = \Omega^i_{log}$, the usual sheaf of logarithmic differentials. (So if $Z^i_{\tilde{X}}$ denotes the sheaf of closed differentials, then

$$\Omega^i_{log} = \ker\left(1 - \mathcal{C} : Z^i_{\tilde{X}} \to \Omega^i_{\tilde{X}}\right).$$

Here of course 1 denotes the natural inclusion $Z^i_{\tilde{X}} \to \Omega^i_{\tilde{X}}$ and $\mathcal{C}$ is the Cartier operator.)

Now consider, on the category of perfect $\mathbf{F}_p$-algebras equipped with the étale topology, the sheaf associated to the presheaf

$$A \longmapsto \Pi H^j_{\acute{e}t}(\tilde{X} \times \operatorname{Spec} A, W_n\Omega^i_{log}).$$

This sheaf turns out to be represented by a commutative algebraic perfect group scheme of finite dimension and this dimension is by definition $T^{i-1,j}$. Moreover, the value of the sheaf on an algebraically closed field is equal to the value of the presheaf, so using the exact sequence 5.3, to show $T^{ij} = 0$ it suffices to show that $\Pi H^j(\tilde{X} \times \operatorname{Spec}\mathbf{F}, \Omega^{i-1}_{log}) = 0$ for all algebraically closed fields $\mathbf{F}$ of characteristic p.

Proposition 5.4. *$T^{ij} = 0$ whenever $i + j \neq k + 1$.*

Proof. First some reductions: since $T^{ij} = T^{k-1-i,k+3-j}$ ([**I**, 4.4.5]), it suffices to prove that $T^{ij} = 0$ for $i+j > k+1$ and to prove this, it suffices to prove that $H^j(M \otimes \overline{\mathbf{F}_p}, \Omega^i_{log}) = 0$ for $i+j > k+2$. For the rest of the proof, we replace M with $M \otimes \overline{\mathbf{F}}_p$, i.e., with $(\tilde{X} \otimes \overline{\mathbf{F}_p}, \Pi)$.

The next point is that $H^j(M, \Omega^i_{\tilde{X}}) = 0$ unless $i + j = k$, $k+1$, or $k+2$. The detailed proof of this requires a detour through the cohomology of the log schemes $X^\times$ and $I^\times$, as in [**U1**], Section 4, which we will omit. The main point is to use the exact sequence

$$0 \to f^*\Omega^1_{I^\times} \otimes \Omega^{i-1}_{X^\times/I^\times} \to \Omega^i_{X^\times} \to \Omega^i_{X^\times/I^\times} \to 0$$

together with the Leray spectral sequence and the calculation

$$\Pi R^j f_*\Omega^i_{X^\times/I^\times} = \begin{cases} \left(\omega^{2i-k}\right)^{\oplus\binom{k}{i}} & \text{if } i+j=k \\ 0 & \text{otherwise} \end{cases}$$

and then compare the Hodge cohomology of $X^\times$ and $\tilde{X}$. This last comparison requires checking that Proposition 4.1 of [**U1**] still holds using the projector of this paper in place of that of [**U1**]; the crucial point here is the vanishing of the Hodge cohomology groups of the toric varieties $\tilde{Q}_r$ and $\tilde{P}_r$ after applying a suitable projector [**U1**, 4.5]. Essentially the same proof works with the projector considered here.

Now let $Z^i_{\tilde{X}}$ and $B^i_{\tilde{X}}$ denote the sheaves of closed and exact differentials respectively. Taking cohomology of the sequences

$$0 \to Z^i_{\tilde{X}} \to \Omega^i_{\tilde{X}} \xrightarrow{d} B^{i+1}_{\tilde{X}} \to 0$$

and

$$0 \to B^i_{\tilde{X}} \to Z^i_{\tilde{X}} \xrightarrow{\mathcal{C}} \Omega^i_{\tilde{X}} \to 0$$

and using that $Z^{k+1}_{\tilde{X}} = \Omega^{k+1}_{\tilde{X}}$, we have also that $H^j(M, Z^i_{\tilde{X}}) = H^j(M, B^i_{\tilde{X}}) = 0$ if $i + j > k + 3$. Using the sequence

$$0 \to \Omega^i_{log} \to Z^i_{\tilde{X}} \xrightarrow{1-\mathcal{C}} \Omega^i_{\tilde{X}} \to 0$$

we already have that $H^j(M, \Omega^i_{log}) = 0$ for $i + j > k + 3$. To finish, it suffices to prove that

a) $1 - \mathcal{C} : H^{k+2-i}(M, Z^i_{\tilde{X}}) \to H^{k+2-i}(M, \Omega^i_{\tilde{X}})$ is onto, and
b) $H^{k+3-i}(M, Z^i_{\tilde{X}}) = 0$

for all i.

I claim that both a) and b) follow from the assertion that the natural map

$$H^{k+2-i}(M, Z^i_{\tilde{X}}) \xrightarrow{n} H^{k+2-i}(M, \Omega^i_{\tilde{X}}) \tag{5.5}$$

is onto for all i. Indeed, assume by induction that b) holds for some i. Then 5.5 implies that $H^{k+2-i}(M, B^{i+1}_{\tilde{X}}) = 0$ which implies that

$$H^{k+1-i}(M, Z^{i+1}_{\tilde{X}}) \xrightarrow{\mathcal{C}} H^{k+1-i}(M, \Omega^{i+1}_{\tilde{X}})$$

is onto. By Lemma 5.7 below, this implies a) for $i + 1$. It also implies the vanishing of $H^{k+2-i}(M, Z^{i+1}_{\tilde{X}})$, i.e., b) for $i + 1$. To start the induction, note that both a) and b) hold for $i = 0$.

Now we have a commutative diagram

$$\begin{array}{ccc} H^1(I, \Pi R^{k+1-i} f_* Z^i_{\tilde{X}}) & \xrightarrow{n} & H^1(I, \Pi R^{k+1-i} f_* \Omega^i_{\tilde{X}}) \\ \downarrow & & \downarrow \\ H^{k+2-i}(M, Z^i_{\tilde{X}}) & \xrightarrow{n} & H^{k+2-i}(M, \Omega^i_{\tilde{X}}) \end{array}$$

and the right hand vertical map is an isomorphism, so it will suffice to prove that the top map is surjective. But this cohomology is on a curve I, so it suffices to show that the sheaf

$$\mathrm{Coker}\left(\Pi R^{k+1-i} f_* Z^i_{\tilde{X}} \xrightarrow{n} \Pi R^{k+1-i} f_* \Omega^i_{\tilde{X}}\right) \tag{5.6}$$

is supported on points.

To see this, we use deRham cohomology; although we will not explicitly use the results of [**U2**] here, Sections 3 and 5 of that paper may help to unwind what follows. Recall the relative deRham cohomology sheaves $\underline{H}^j_{dR}(M/I) = \Pi R^j f_*(\Omega^{\cdot}_{\tilde{X}/I})$ and the deRham cohomology sheaves $\underline{H}^j_{dR}(M) = \Pi R^j f_*(\Omega^{\cdot}_{\tilde{X}})$, with their Hodge filtrations $F^{\cdot}$ and their conjugate filtrations $F_{\cdot}$. The relative Hodge to deRham spectral sequence $\Pi R^j f_* \Omega^i_{\tilde{X}/I} \Rightarrow \underline{H}^{i+j}_{dR}(M/I)$ degenerates and we have $\underline{H}^j_{dR}(M/I) = 0$ unless $j = k$. Moreover, over $U = I - \{C \cup S\}$, we have canonical isomorphisms

$$\begin{aligned}\underline{H}^k_{dR}(M/I) &\cong H^1_{dR}(\mathcal{E}/I)^{\otimes^k} \\ &\cong \left(F^1 H^1_{dR}(\mathcal{E}/I) \oplus F_0 H^1_{dR}(\mathcal{E}/I)\right)^{\otimes^k} \\ &\cong \left(\omega \oplus \omega^{-1}\right)^{\otimes^k}\end{aligned}$$

Over U, the invertible sheaf ω is trivial with a canonical nowhere vanishing section ω_c; also there is a distinguished nowhere vanishing 1-form dq/q on U. In terms of ω_c and dq/q, the Gauss-Manin connection ∇ on $\underline{H}^k_{dR}(M/I)$ is the k-th tensor power of the connection on $\omega \oplus \omega^{-1}$ defined by $\nabla(\omega_c) = (dq/q)\omega_c^{-1}$ and $\nabla({\omega_c}^{-1}) = 0$. Using the exact sequence

$$0 \to \underline{H}^k_{dR}(M) \to \underline{H}^k_{dR}(M/I) \xrightarrow{\nabla} \Omega^1_I \otimes \underline{H}^k_{dR}(M/I) \to \underline{H}^{k+1}_{dR}(M) \to 0$$

we can compute $\underline{H}^{k+1}_{dR}(M)$ and its Hodge and conjugate filtrations completely explicitly and see that the Hodge to deRham spectral sequence $\Pi R^j f_* \Omega^i_{\tilde{X}} \Rightarrow \underline{H}^{i+j}_{dR}(M)$ degenerates and that $\Pi R^j f_* \Omega^i_{\tilde{X}} = (F^i \cap F_i)(\underline{H}^{i+j}_{dR}(M))$ (again, all over U). But we have another commutative diagram over U

$$\begin{array}{ccc} \Pi R^{k+1-i} f_* Z^i_{\tilde{X}} & \xrightarrow{n} & \Pi R^{k+1-i} f_* \Omega^i_{\tilde{X}} \\ \downarrow & & \downarrow \\ (F^i \cap F_i)(\underline{H}^{i+j}_{dR}(M)) & \longrightarrow & \operatorname{gr}^i \underline{H}^{k+1}_{dR}(M) \end{array}$$

with the bottom and right arrows isomorphisms and the left arrow a surjection. This shows that the top arrow is surjective, so the sheaf 5.6 is supported on (at worst) the cusps and supersingular points.

To finish the proof of Proposition 5.4, we need to prove the lemma alluded to above.

LEMMA 5.7. *Suppose* $\mathbf{F}$ *is an algebraically closed field of characteristic* p, V *and* W *are finite dimensional* $\mathbf{F}$*-vector spaces, and* f *and* g *are maps* $V \to W$ *with* f *linear and* g p^{-1}*-linear (i.e.,* $g(a^p v) = a g(v)$ *for* $a \in \mathbf{F}$ *and* $v \in V$*). Then*

$$V \xrightarrow{f-g} W$$

is surjective if g *is.*

PROOF. Choose a splitting h of g, i.e., a p-linear map $h : W \to V$ such that $g \circ h = \mathrm{id}_W$. Then the composition

$$W \xrightarrow{h} V \xrightarrow{f-g} W$$

is the difference of a p-linear endomorphism of W and the identity of W and as such is well-known to be surjective.

This completes the proof of Proposition 5.4.

Using the Proposition, we can compute the h_W^{ij} for all $i+j \neq k+1$. Indeed, by Corollary 4.2, we know the m_{ij} when $i+j \neq k+1$, and using formula 3.1 we find that when $i+j \neq k+1$, $h_W^{ij}=0$ unless k is even and $i=j=k/2$, in which case $h_W^{k/2,k/2}=e_{k,k/2}$. Applying Crew's formula, we have

$$h_W^{i,k+1-i} = \begin{cases} (-1)^{k+1-i}\chi(M,\Omega^i_{\tilde{X}}) & \text{if } i \neq k/2 \\ (-1)^{k+1-i}\chi(M,\Omega^i_{\tilde{X}}) + e_{k,k/2} & \text{if } i = k/2. \end{cases}$$

On the other hand, using the exact sequence of relative differentials and the computation alluded to in the proof of Proposition 5.4, we have

$$\chi(M,\Omega^i_{\tilde{X}}) = (-1)^{k+1-i}\binom{k}{i-1}\chi(I,\Omega^1_I\otimes\omega^{2i-2-k}) + (-1)^{k-i}\binom{k}{i}\chi(I,\omega^{2i-k}).$$

Now using that the degree of ω is w_1 and doing a bit of rearranging, we get Theorem 5.1.

6. A remark on other cases of the main theorems

Our main results, namely the calculation of the slopes and the Hodge numbers associated to the pair $(\tilde{X},\Pi)$, can be carried out in many other cases. Indeed, we can consider modular forms on $\Gamma_1(p^nN)$ with p odd, N prime to p, and $n \geq 0$. We need only assume that if $p^n \geq 3$, and if $p^n = 3$ then $N \geq 5$. Then Theorems 4.1 and 5.1 are true as stated, provided we make suitable modifications to the definitions of the Hecke polynomials $H_k(T)$ and the integers l_{kj}. Also, for any direct factor of $(\mathbf{Z}/p^nN\mathbf{Z})^\times$ of order prime to p, we can decompose 4.1 and 5.1 for characters of this factor. This is most useful for the factor $(\mathbf{Z}/p\mathbf{Z})^\times$.

7. What remains to be done

As we saw in Section 2, in order to prove a control theorem, i.e., that the number of eigenforms of weight k, level pN and slope λ is bounded independently of k, we need to show that certain contact numbers are bounded independently of k. In this section, we boil this down to an assertion about the torsion numbers T^{ij}.

Changing notation, let m_{ki} $0 \leq i \leq k+1$ be the contact numbers associated to the prime p and the polynomial H_k defined in Section 3. Thus the Hodge polygon associated to $m_{k0},\ldots,m_{k,k+1}$ is the highest polygon with integral slopes below the Newton polygon of the action of U_p on $S_{k+2}(\Gamma_1(pN))$ modulo the subspace of forms which are new at p but whose character is trivial at p.

A control theorem is then equivalent to the assertion that for fixed i, m_{ki} is bounded independently of k. Clearly we can assume that $k \geq 2i$. Let M_{ki} $0 \leq i \leq k+1$ be the contact numbers associated to $\left(H^{k+1}_{cris}(\tilde{X}_k/W)\otimes\mathbf{Q}_p\right)(\epsilon)$, as in Section 4 (where the subscript on $\tilde{X}_k$ is to emphasize the dependence on k). Then using Theorem 4.1, as soon as $k \geq 2i$, we have

$$M_{ki} = \sum_{0 \leq j \leq i} m_{k-2j,i-j} e_{kj}.$$

Inverting this formula, we have that

$$m_{ki} = \sum_{0 \leq j \leq i} (-1)^j \binom{k-j}{j} M_{k-2j,i-j}. \tag{7.1}$$

Now for each k, let T_k^i be the torsion number $T^{i,k+1-i}$ associated to the pair $(\tilde{X}_k, \Pi)$. Also recall the integers l_{ki} introduced above Theorem 5.1. Then 7.1 becomes

$$m_{ki} = l_{ki} - \sum_{0 \leq j \leq i} (-1)^j \binom{k-j}{j} \left(T_k^{i-j} - 2T_k^{i-j-1} + T_k^{i-j-2} \right).$$

Summing up:

THEOREM 7.2. *For a fixed integer λ, if the integers*

$$l_{ki} - \sum_{0 \leq j \leq i} (-1)^j \binom{k-j}{j} \left(T_k^{i-j} - 2T_k^{i-j-1} + T_k^{i-j-2} \right)$$

are bounded independently of k for all $i \leq \lambda$, then the number of eigenforms of weight $k+2$, level pN, and slope $\leq \lambda$ is bounded independently of k. Conversely, if the number of such forms is bounded, then the integers above are also bounded for all $i \leq \lambda - 1$.

In fact, slightly more is true, namely that a control theorem slopes $\leq \lambda'$ where λ' is a real number $< \lambda$ follows from the boundedness of the integers of the theorem for $i \leq \lambda - 1$. For example, to obtain a control theorem for slopes $\leq \lambda' < 1$, it suffices to prove that

$$T_k^0 \geq k w_1 - b$$

for all k and some constant b. Using the techniques of [**U2**], it is not hard to show that at least

$$T_k^0 \geq k(1 - 1/p) w_1 - b.$$

In a sense we do not want to make precise, this inequality comes already from working modulo p, i.e., with the Ω^i_{log}; working modulo p^n, i.e., with the $W_n\Omega^i_{log}$ should allow one to prove that

$$T_k^0 \geq k(1 - 1/p^n) w_1 - b$$

and so to obtain the desired result, one should work with all of the $W_n\Omega^i_{log}$.

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Department of Mathematics, University of Arizona, Tucson, AZ 85721

E-mail address: ulmer@math.arizona.edu