WORKSHOP FOR UNIVERSITY OF ARIZONA'S SONIA KOVALEVSKY DAY 2013: THE TOWER OF HANOI

TOVA BROWN & MEGAN MCCORMICK

The Summary

The Tower of Hanoi is an old mathematical puzzle (with an even older legend behind it, about an Indian temple and the end of the world). Here is the puzzle: Given a tower of some number of discs, stacked in decreasing size on one of three pegs, try to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a larger disc onto a smaller. We will solve this puzzle hands-on and using a mathematical idea called "recurrence relations."

The Puzzle

There are 3 pegs and 8 discs with different diameters. The discs are all stacked on one of the pegs. In the stack, they are ordered by size: smallest on top, largest on the bottom. The goal is to move this tower onto one of two free pegs, by the following rules:

- Only one disc may be moved at a time,
- A larger disc may never be stacked onto a smaller disc.

We will let the girls experiment, each with their own Tower of Hanoi set, to see what they come up with. Along the way, there are two main questions we will ask:

Question 1. *Is there a solution to this puzzle? In other words, is there a way to transfer all the discs to another peg according to the given rules?*

Question 2. *How many moves are necessary and sufficient in order to solve this problem?*

The first question can probably be solved simply by experimenting with the pieces. To answer the second question, it is helpful to re-phrase the puzzle in terms of some mathematical ideas and solve it that way.

The Math

Step 1: Generalize. Instead of 8 discs, what if we ask the question with n discs? Generalizing this way actually allows us to simplify the problem by looking at small cases first.

$\underline{n = 1}$ is completely trivial (only 1 move)!

$\underline{n = 2}$ it's fairly easy to see it takes 3 moves.

$\underline{n = 3}$ experimenting, we can see that it takes 7 moves.

Step 2: Introduce notation. Let $T_n :=$ the minimum number of moves necessary to move n blocks according to the rules given. It's fairly easy to see that in our small cases above, we couldn't have solved the puzzle in fewer moves. Therefore, we know that

$$T_0 = 0 \qquad T_1 = 1 \qquad T_2 = 3 \qquad T_3 = 7$$

Step 3: Find a pattern. By experimenting again, we can determine that one way to move n discs is to

- move $n-1$ discs onto the middle peg,
- move the last disc onto the desired peg,
- move the $n-1$ discs onto the last disc.

These three steps require T_{n-1}, 1, and T_{n-1} moves, respectively. We can thus conclude that

$$T_n \leq 2T_{n-1} + 1 \quad \text{for } n > 0.$$

Can we do any better than this? No, since each of the steps listed above are certainly necessary. We could move the last disc more than once, if we're not being careful. Then

$$T_n \geq 2T_{n-1} + 1 \quad \text{for } n > 0.$$

Putting these together, plus our initial condition, we see that

$$\begin{aligned} &T_0 = 0 \\ &T_n = 2T_{n-1} + 1 \quad \text{for } n \geq 1. \end{aligned}$$

What we've found at this point is called a "recurrence relation." We can use it to compute T_n for any n that we like. For instance, how many moves does it take to transfer 6 discs? (63 moves) This is a lot of work, so let's see if we can do better.

Step 4: Solve the recurrence. This step is optional, depending on the level of our students and whether we have extra time or not. To solve the recurrence, we first look at small cases to see if we can discern a pattern:

$$T_3 = 2 \cdot 3 + 1 = 7 \qquad T_4 = 2 \cdot 7 + 1 = 15 \qquad T_5 = 2 \cdot 15 + 1 = 31 \qquad T_6 = 2 \cdot 31 + 1 = 63$$

For $n \leq 6$ at least, we see that

$$T_n = 2^n - 1.$$

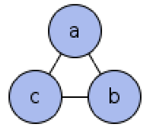
To prove that this is true for *all* n, we have to use a method called "mathematical induction." One way to think about this method is as a ladder: in order to prove that we can climb a ladder to any height we'd like, we show that we can climb onto the bottom step of the ladder and that once we're on some step of the ladder we can always climb up one more step.

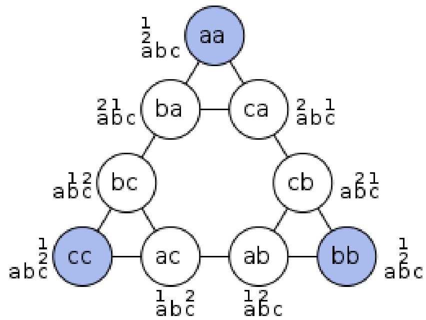
We already proved that we can climb onto the bottom step of our mathematical ladder (in fact any of the bottom 6 steps!) by our cases above. Now suppose we've climbed onto the $n-1$ step. In other words, suppose that $T_{n-1} = 2^{n-1} - 1$. Then

$$T_n = 2T_{n-1} + 1 = 2(2^{n-1} - 1) + 1 = 2^n - 1.$$

Connection to Graph Theory

We include the following two diagrams of a graphical interpretation of the game. The images are from the Wikipedia page on Tower of Hanoi.





Sources

The material for this workshop is adapted from §1.1 of *Concrete Mathematcs: A Foundation for Computer Science* by Graham, Knuth, Patashnik.