THE CONNECTION BETWEEN MAP ENUMERATION AND MATRIX INTEGRALS

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Outline

1. Introduction to analytic combinatorics

2. Map enumeration

3. Matrix integral calculations

Analytic Combinatorics

- Define a collection of objects, sorted by a parameter, for example

$$n = \#(\text{vertices}).$$

- Count them using this sorting parameter:

$$a_n = \#(\text{objects in collection with n vertices}).$$

- Define a generating function:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

- Study $f(z)$ as an analytic function.

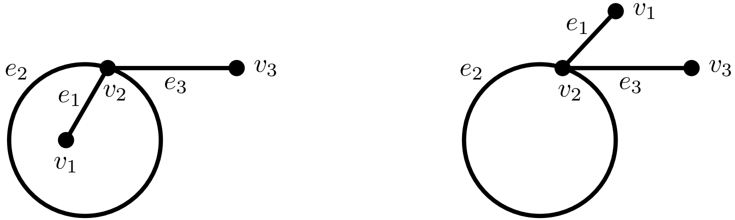
Defining my collection of objects: Graphs

Definition

A graph is a collection of vertices, edges, and incidence relations.

Example:

$$V = \{v_1, v_2, v_3\}$$
$$E = \{e_1, e_2, e_3\}$$
$$I = \{e_1 = (v_1 v_2), e_2 = (v_2 v_2), e_3 = (v_2 v_3)\}$$



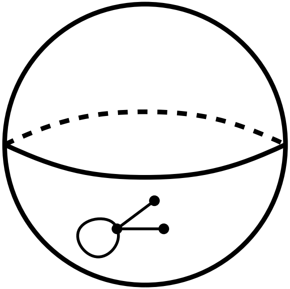
DEFINING MY COLLECTION OF OBJECTS: MAPS

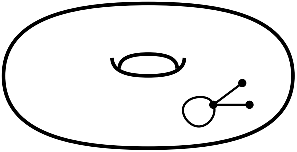
DEFINITION

A map is a graph which is topologically embedded into a surface so that

- the images of the edges do not intersect and
- dissecting the surface along the edges decomposes it into a union of open cells.

Examples:





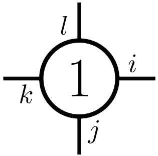
DEFINING MY COLLECTION OF OBJECTS: DIAGRAMS

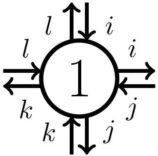
DEFINITION

A 4-valent diagram on n vertices has

- n vertices, labeled by numbers $1, 2, \ldots, n$,
- a labeling of the 4 edges incident to vertex σ by the letters i, j, k, l in cyclic clockwise order around the vertex, and
- a pairing of all the labeled edges into pairs.

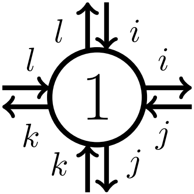
The environment of vertex 1: a diagram (left) and a ribbon diagram (right).



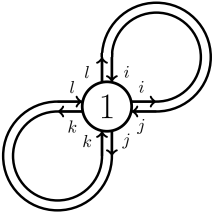


All 4-valent 1-vertex ribbon diagrams

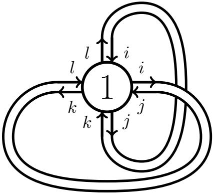
Counting all the diagrams = counting all possible pairings of edges



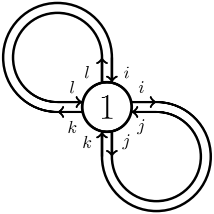
(A)



(B) pairs (i,i), (l,j), (j,l), (k,k)



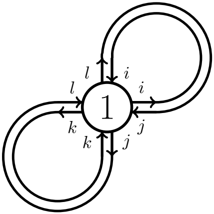
(A) pairs (i,l), (k,j), (j,i), (l,k)



(B) pairs (i,k), (j,j), (k,i), (l,l)

CONSIDERING DIAGRAMS AS MAPS

Consider how to embed this graph on a surface so that its edges do not intersect and dissecting along the edges gives a union of open cells.



The pairs were (i,i), (l,j), (j,l), (k,k).

The diagram has 3 cycles:

i=i

l=j

k=k

As a map it has 3 faces.

Calcuate the Euler characteristic of this map:

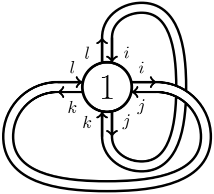
$$\chi = V - E + F = 2 - 2g$$

$$\chi = 1 - 2 + 3 = 2 - 2g$$

implies $\chi = 1$ and $g = 0$.

Considering diagrams as maps

Consider how to embed this graph on a surface so that its edges do not intersect and dissecting along the edges gives a union of open cells.



The pairs were (i,l), (k,j), (j,i), (l,k).
The diagram has 1 cycle:
i = l = k = j
As a map it has 1 face.

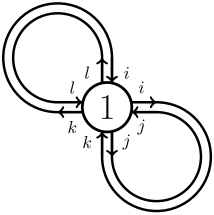
Calcuate the Euler characteristic of this map:

$$\chi = V - E + F = 2 - 2g$$
$$\chi = 1 - 2 + 1 = 2 - 2g$$

implies $\chi = 0$ and $g = 1$.

CONSIDERING DIAGRAMS AS MAPS

Consider how to embed this graph on a surface so that its edges do not intersect and dissecting along the edges gives a union of open cells.



The pairs were (i,k), (j,j), (k,i), (l,l).
The diagram has 3 cycles:
i=k
j=j
l=l
As a map it has 3 faces.

Calcuate the Euler characteristic of this map:

$$\chi = V - E + F = 2 - 2g$$
$$\chi = 1 - 2 + 3 = 2 - 2g$$

implies $\chi = 1$ and $g = 0$.

The count of 4-valent 1-vertex labeled maps

The method:

- We paired edges in all possible ways to make distinct diagrams.
- For each diagram, we found the number of cycles (= # faces),
- and calculated the Euler characteristic and genus of each diagram.

The count: there are exactly three 4-valent 1-vertex labeled maps.

- Two distinct labeled maps live on the sphere (genus = 0),
- only one labeled map lives on the torus (genus = 1).

A generating function for 1-vertex maps m_1:

$$m_1 = \sum_{g \geq 0} s_g z^g$$

where $s_0 = 2$, $s_1 = 1$, and $s_g = 0$ for $g \geq 2$.

Random matrices

Definition

An ensemble of random matrices is a family of square matrices whose entries are randomly determined by a given probability distribution.

Example: the Gaussian Unitary Ensemble is all $N \times N$ Hermitian matrices whose entries are independent Gaussian random variables.

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

Probability density functions for entries:

$$\begin{aligned} m_{11} : \quad & \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} m_{11}^2} dm_{11} \\ m_{12} : \quad & \frac{1}{\pi} e^{-\left((m_{12}^R)^2 + (m_{12}^I)^2\right)} dm_{12}^R dm_{12}^I \end{aligned}$$

Random matrices

We can calculate expectations of these random variables:

$$
\begin{aligned}
\langle m_{ij} m_{ji} \rangle &= \langle m_{ij} \overline{m_{ij}} \rangle \\
&= \left\langle |m_{ij}|^2 \right\rangle \\
&= \left\langle (m_{ij}^R)^2 + (m_{ij}^I)^2 \right\rangle \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((m_{ij}^R)^2 + (m_{ij}^I)^2) e^{-(m_{ij}^R)^2 - (m_{ij}^I)^2} dm_{ij}^R dm_{ij}^I \\
&= 1
\end{aligned}
$$

We learn the important fact:

$$\langle m_{ij} m_{kl} \rangle = \delta_{il} \delta_{jk}.$$

WICK'S FORMULA

THEOREM (WICK FORMULA)

For linear functions l_i on $\mathbb{R}^N$,

$$\langle l_1 l_2 \cdots l_{2k} \rangle = \sum \langle l_{r_1} l_{s_1} \rangle \langle l_{r_2} l_{s_2} \rangle \cdots \langle l_{r_k} l_{s_k} \rangle$$

where the sum is taken over all permutations $r_1 s_1 r_2 \cdots s_k$ of the set of indices $1, 2, 3, \ldots, 2k$ such that $r_1 < r_2 < \cdots < r_k$ and $r_i < s_i$ for each i.

Example: Wick Formula for $k = 2$

$$\langle l_1 l_2 l_3 l_4 \rangle = \langle l_1 l_2 \rangle \langle l_3 l_4 \rangle + \langle l_1 l_3 \rangle \langle l_2 l_4 \rangle + \langle l_1 l_4 \rangle \langle l_2 l_3 \rangle$$

since all the possible permutations of (1234) are (1234), (1324), and (1423).

Wick Formula applied

Our random variables are linear functions and our measure is the right kind of measure so we can use the Wick Formula to calculate some integrals in our random matrix setting.

First note: $\mathrm{Tr}(M^4) = \sum_{i,j,k,l=1}^{N} m_{ij}m_{jk}m_{kl}m_{li}$. Then

$$\begin{aligned}
\left\langle \mathrm{Tr}(M^4) \right\rangle &= \sum_{i,j,k,l=1}^{N} \langle m_{ij}m_{jk}m_{kl}m_{li} \rangle \\
&= \sum_{i,j,k,l=1}^{N} \left(\langle m_{ij}m_{jk} \rangle \langle m_{kl}m_{li} \rangle + \langle m_{ij}m_{kl} \rangle \langle m_{jk}m_{li} \rangle + \langle m_{ij}m_{li} \rangle \langle m_{jk}m_{kl} \rangle \right) \\
&= \sum_{i,j,k,l=1}^{N} \left(\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk} \right) \\
&= N^3 + N + N^3
\end{aligned}$$

Wick Formula applied

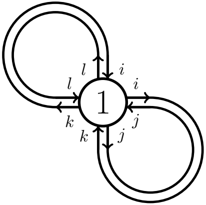
Notice the cycles in the indices given by these products of delta functions:

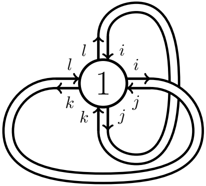
$\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll}$	$\delta_{il}\delta_{jk}\delta_{ji}\delta_{kl}$	$\delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}$
$i = k$	$i = l = k = j$	$i = i$
$j = j$		$j = l$
$l = l$		$k = k$

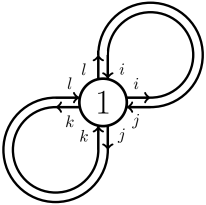
WICK FORMULA APPLIED

Notice the cycles in the indices given by these products of delta functions:

$\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll}$	$\delta_{il}\delta_{jk}\delta_{ji}\delta_{kl}$	$\delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk}$
$i = k$	$i = l = k = j$	$i = i$
$j = j$		$j = l$
$l = l$		$k = k$







THE CALCULATION OF $\left\langle (\mathrm{TR}(M^4))^n \right\rangle$ FOR $n = 1$

The method:

- Use the Wick calculus to break down $\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle$ into expectations of pairs.
- Reduce these pairs to delta functions and look for cycles in the indices.
- The number of cycles gives the power of N.

The count: we got 3 terms out of the integral.

- Two terms gave N^3,
- and only one term gave N^1.

Since we saw #(cycles of indices) = # (faces in diagram), let's compute the Euler characteristic:

$$\chi = n - 2n + F = 2 - 2g \quad \Rightarrow \quad F = 2 - 2g + n.$$

FINALLY, THE REAL GENERATING FUNCTION

A generating function for 4-valent 1-vertex diagrams:

$$\left\langle \mathrm{Tr}(M^4) \right\rangle = \sum_{g \geq 0} \# \left\{ \text{4-valent 1-vertex g-diagrams} \right\} N^{2-2g+n}.$$

Applying the same logic for $n \geq 1$, we have

$$\left\langle (\mathrm{Tr}(M^4))^n \right\rangle = \sum_{g \geq 0} \# \left\{ \text{4-valent n-vertex g-diagrams} \right\} N^{2-2g+n}.$$

The partition function of statistical mechanics has a formal series expansion

$$Z_N(t) \text{ “} = \text{”} \sum_{n \geq 0} \frac{1}{n!} \left(-\frac{t}{N} \right)^n \left\langle (\mathrm{Tr}(M^4))^n \right\rangle ,$$

so combining these two results gives a formal generating function for 4-valent g-diagrams.

Some References:

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- N. M. Ercolani and K. D. T-R McLaughlin, *Asymptotics of the Partition Function for Random Matrices via Riemann-Hilbert Techniques and Applications to Graphical Enumeration*, IMRN, Vol. 14 (2003), pp. 755-820.