Enumeration of maps on surfaces by the loop equations of random matrix theory

Tova Brown

Advisor: Nicholas Ercolani

Written Comprehensive Exam
Department of Mathematics
University of Arizona

April 21, 2014

Abstract

Rigorously derived as a consequence of the Schwinger-Dyson Equation in the setting of the Unitary Ensemble (Ercolani and McLaughlin in [EM07]), the loop equations give a recursive way of deriving coefficients in the loop expansion of the Cauchy transform of the one-point function. The loop expansion is related to another fundamental expansion called the genus expansion which enumerates labeled maps embedded on Riemann surfaces (conjectured in [BIZ80] and proven by [EM03]). Therefore, generating functions for certain families of labeled maps can be explicitly calculated by taking derivatives of the loop expansion coefficients. A derivation of the leading order in this recursion can be done by studying convergence of integrals with respect to eigenvalue densities [Joh98].

In this paper we derive the loop equations following the approach indicated above and then apply this method to analytically verify two generating function formulas appearing in the literature [Fra06].

Contents

1 Background and motivation

1.1 Introduction

A *graph* is a collection of vertices, edges, and incidence relations, with loops and multi-edges allowed, taken to be connected unless stated otherwise. The *degree* of a vertex is the number of edges incident to the vertex. A vertex of degree k or graph in which every vertex is of degree k is called *k-valent*.

A *map* is a graph which is topologically embedded into a Riemann surface so that the (images of the) edges do not intersect and dissecting the surface along the edges decomposes it into a union of open cells, called the *faces* of the map [EM03]. Equivalently, a map is the 1-skeleton in a 2-dimensional cell complex of a Riemann surface [Hat02]. By *Riemann surface* we mean a compact connected complex-analytic manifold of complex dimension one, but only up to orientation-preserving topological isomorphism (homeomorphism) [LZ04]. More simply put, a Riemann surface is a compact, connected, oriented topological 2-manifold as this is the only structure preserved by the isomorphism described above. Two maps are isomorphic if there exists an orientation-preserving homeomorphism from one surface to the other so that the restriction of the homeomorphism to the graph is a graph isomorphism.

Our interest is in enumerating maps. We do this by studying the coefficients in an asymptotic expansion of the logarithm of the partition function Z_N for certain families of random matrices,

$$\frac{1}{N^2}\log\left(\frac{Z_N(t)}{Z_N(0)}\right) = e_0(t) + \frac{1}{N^2}e_1(t) + \cdots + \frac{1}{N^{2g}}e_g(t) + \dots.$$

In particular, we are interested in calculating explicit formulae for these coefficients $e_g(t)$ and their derivatives, which can be interpreted as generating functions for certain families of labeled maps.

1.2 The Wick calculus

In this section we demonstrate how matrix integrals can be interpreted as enumerating labeled maps.

The Gaussian Unitary Ensemble (GUE) is the collection of Hermitian $N \times N$ matrices with measure

$$\frac{1}{2^{N/2}\pi^{N^2/2}}e^{-\frac{1}{2}\operatorname{tr}\left(M^2\right)}dM,$$

where dM is the product of the Lebesgue measures on the entries. This is a Gaussian measure ([EM03] p. 758) so the following formula holds for expectations with respect to it:

Lemma 1.1 (Wick Formula)**.** *For linear functions l_i on $\mathbb{R}^N$,*

$$\langle l_1 l_2 \cdots l_{2k}\rangle = \sum \langle l_{r_1} l_{s_1}\rangle \langle l_{r_2} l_{s_2}\rangle \cdots \langle l_{r_k} l_{s_k}\rangle$$

where $\langle\cdot\rangle$ denotes expectation with respect to a Gaussian measure and the sum is taken over all $(2k-1)!!$ Wick couplings of $1, 2, \dots, 2k$. A Wick coupling *is an ordering of $r_1, r_2, \dots, r_k, s_1, s_2, \dots, s_k$ so that $r_1 < r_2 < \dots < r_k$ and $s_i > r_i$.*

Each entry of a GUE matrix is a linear function and expectations of pairs of these random variables can be computed in the following way:

Lemma 1.2.

$$\langle M_{ij} M_{kl}\rangle = \delta_{il}\delta_{jk}.$$

Proof. If $i = l$ and $j = k$ then by the Hermitian structure of M and marginal density $\frac{1}{\pi}e^{-(x^2+y^2)}$ of an off-diagonal entry $x + iy$,

$$\begin{aligned}\langle M_{ij} M_{ji}\rangle &= \left\langle M_{ij}\overline{M_{ji}}\right\rangle \\ &= \left\langle |M_{ij}|^2\right\rangle \\ &= \left\langle (M_{ij}^R)^2 + (M_{ij}^I)^2\right\rangle\end{aligned}$$

$$= \frac{1}{2} + \frac{1}{2}$$
$$= 1,$$

but if $i \neq l$ or $j \neq k$ then M_{ij} and M_{kl} are independent random variables with mean 0 so

$$\langle M_{ij} M_{kl} \rangle = \langle M_{ij} \rangle \langle M_{kl} \rangle = 0.$$

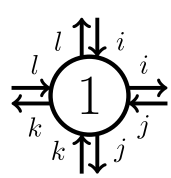
□

We first use the Wick calculus in a small, but important, example. The trace of M^4 is written in coordinates $\mathrm{tr}(M^4) = \sum_{i,j,k,l=1}^{N} M_{ij} M_{jk} M_{kl} M_{li}$ and the Wick formula applied to each term in the summand:

$$\begin{aligned}
\left\langle -\frac{t_4}{N} \mathrm{tr}(M^4) \right\rangle &= \left\langle -\frac{t_4}{N} \sum_{i,j,k,l=1}^{N} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle \\
&= -\frac{t_4}{N} \sum_{i,j,k,l=1}^{N} \left(\langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle + \langle M_{ij} M_{li} \rangle \langle M_{jk} M_{kl} \rangle \right) \\
&= -\frac{t_4}{N} \sum_{i,j,k,l=1}^{N} \left(\delta_{ik}\delta_{jj}\delta_{ki}\delta_{ll} + \delta_{il}\delta_{jk}\delta_{ji}\delta_{kl} + \delta_{ii}\delta_{jl}\delta_{jl}\delta_{kk} \right) \\
&= -\frac{t_4}{N} \left(N^3 + N + N^3 \right) \\
&= -t_4(2N^2 + 1)
\end{aligned}$$

The chains of equal indices given by the products of delta functions determine the power of N contributed by each term: $i = k$, $j = j$, and $l = l$ from the first term, $i = l = k = j$ from the second, and so on. Because each index appears exactly twice, each chain is a closed cycle. These cycles are fundamental to what follows, as are the pairs of matrices in the second line of the calculation. A term in the sum is a positive contribution if and only if each of the pairs of matrix entries is nonzero, if and only if the indices match as in Lemma 1.2.

The terms that give a nonzero contribution can be represented in the following graphical way: draw a vertex with four ribbon-darts coming out of it and label these ribbon-darts cyclically to correspond to the four matrix entries M_{ij}, M_{jk}, M_{kl}, and M_{li} on which every term depends:



The outgoing edge of each ribbon corresponds to the first subscript in the matrix entry and the incoming edge to the second (e.g. M_{ij} corresponds to the ribbon-dart on the right with outgoing i and incoming j). Now connect ribbon-darts with ribbons (no twisting of ribbons is allowed, but overlapping is): the three ways it can be done are

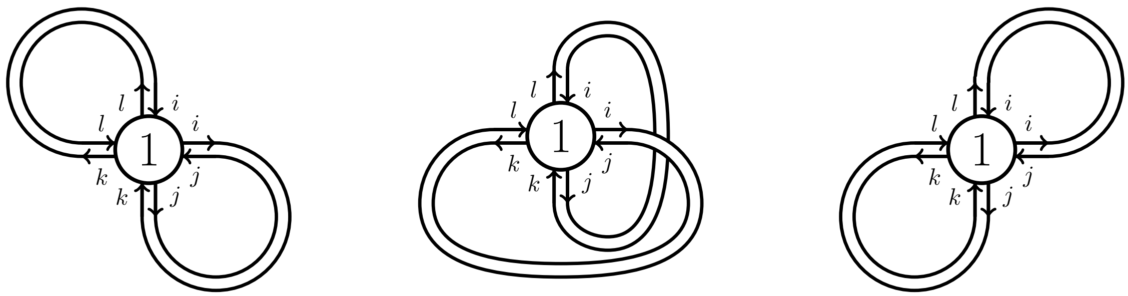


Figure 1: 4-valent 1-vertex ribbon diagrams

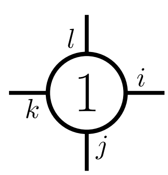
Such figures are called *ribbon diagrams*. The cycles of equal indices arising from the Wick calculations (e.g. $i = k$, $j = j$, and $l = l$ from the first term, $i = l = k = j$ from the second) appear in ribbon diagrams as sequences of edges following arrows and connecting into closed loops. Since each index appears exactly twice (as an outgoing and as an incoming edge) again the chains are seen to be closed.

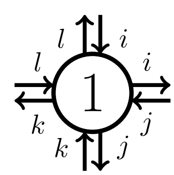
1.3 Ribbon diagrams, diagrams, and maps

A *diagram* on n vertices has the following parts:

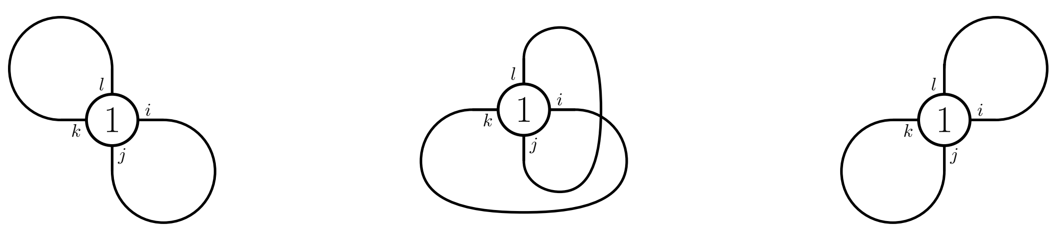
1. n vertices, labeled by numbers $1, 2, \ldots, n$, with vertex σ of prescribed valence k_σ;
2. a labeling of the edges incident to vertex σ by letters $i_1^{(\sigma)}, i_2^{(\sigma)}, \ldots, i_{k_\sigma}^{(\sigma)}$ (this order corresponds to the cyclic clockwise order of the edges around the vertex); and
3. a partitioning of the labeled edges into pairs.

Diagrams are in bijection with ribbon diagrams by relating each ribbon to an edge, with the edge labeled by the letter on the outgoing edge of the corresponding ribbon.





Here are the diagrams corresponding to the three ribbon diagrams of the previous section:



We claim that a diagram is a labeled map, i.e. that it can be embedded into a Riemann surface as a map (in the case of a disconnected diagram, it is embedded into a disjoint collection of Riemann surfaces, one for each connected component). In order to work with this structure, we declare an isomorphism of diagrams to be an isomorphism of maps that preserves the labelings.

Lemma 1.3. *A connected ribbon diagram has a natural embedding into a connected, compact, oriented surface in such a way that dissecting the diagram along the edges decomposes the surface into a union of open cells.*

Proof. View the diagram as a 2-dimensional cell complex: for 0-cells take the vertices and for 1-cells the edges, and attach them using the boundary relations of each edge connecting its two vertices (with multiplicity). For 2-cells take a collection of open disks, one for each cycle in the diagram. The cycles give the gluing maps for attaching each 2-cell to the complex: the boundary of a face is precisely the sequence of edges it gets glued to, which form a simple closed loop (cycle). When all the faces have been attached, the cell complex is now a closed surface (if it had a boundary, then there would be another cycle of edges, and therefore another 2-cell to attach). Compactness follows because finitely many cells were used. The surface is oriented because every ribbon diagram may be laid flat in the plane (with possibly some ribbons overlapping, e.g. Figure 1) and the faces oriented consistently by taking the induced orientation in the plane. □

Each connected component of a (ribbon) diagram is therefore a labeled map. If there is only one component call the diagram a *g-map* and if more than one a *g-diagram*. In the latter case, g is determined by the additivity of the Euler characteristic across disjoint unions.

1.4 The Unitary Ensemble

In order to precisely state the asymptotic expansion whose coefficients enumerate g-maps, we must briefly introduce certain aspects of the Unitary Ensemble, and we include others which will be used throughout the paper. The Unitary Ensemble with large parameter N consists of $n \times n$ Hermitian matrices with measure

$$\frac{1}{Z_{n,N}(\mathbf{t})} e^{-N \operatorname{tr}(V_{\mathbf{t}}(M))} dM$$

where dM is the product of the Lebesgue measures on the matrix entries, the potential is

$$V_{\mathbf{t}}(x) = V(x) = \frac{1}{2}x^2 + \sum_{j=1}^{\nu} t_j x^j, \tag{1}$$

for ν even and $\mathbf{t} = (t_1, t_2, \ldots, t_\nu) \in \mathbb{T}(T, \gamma) = \left\{ \mathbf{t} \in \mathbb{R}^\nu : |\mathbf{t}| \le T, t_\nu > \gamma \sum_{j=1}^{\nu-1} |t_j| \right\}$ and partition function

$$Z_{n,N}(\mathbf{t}) = Z_{n,N} = \int e^{-N \operatorname{tr}(V_{\mathbf{t}}(M))} dM. \tag{2}$$

For statistical questions of only the eigenvalues, the probability density on eigenvalues is useful:

$$P(\boldsymbol{\lambda})\, d^n\lambda = P(\lambda_1, \ldots, \lambda_n) d^n\lambda = \frac{1}{Z^e_{n,N}(\mathbf{t})} e^{-N \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 d^n\lambda \tag{3}$$

with partition function

$$Z^e_{n,N} = Z^e_{n,N}(\mathbf{t}) = \int e^{-N \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 d^n\lambda. \tag{4}$$

The *mean density of eigenvalues*, also called the *1-point function*,

$$\rho_{n,N}(\lambda) = \frac{1}{Z^e_{n,N}} \int e^{-N \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 d\lambda_2 \cdots d\lambda_n$$

is a special case of the *k-point function* for any $1 \le k < n$:

$$\rho_{n,N}(\lambda_1, \ldots, \lambda_k) = \frac{1}{Z^e_{n,N}} \int e^{-N \sum_{i=1}^n V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 d\lambda_{k+1} \cdots d\lambda_n. \tag{5}$$

The *equilibrium measure* (properly, its density) is the large n, N limit of the mean density of eigenvalues:

$$\psi(\lambda) = \lim_{n,N\to\infty, n/N\to 1} \rho_{n,N}(\lambda) \tag{6}$$

for a.e. $\lambda \in \operatorname{supp}(\psi)$. Under our conditions on the potential, called the *one-cut assumption*, $\operatorname{supp}(\psi) = [a, b]$. For purely even potentials, $a = -b$. The density $\psi(\lambda)$ is continuous on $\mathbb{R}$. The existence of the limit (6) and its properties are shown through analysis of a variational problem for the equilibrium measure [Dei00], [Joh98].

1.5 The genus expansion

That matrix integrals enumerate g-diagrams was seen by Bessis, Itzykson, and Zuber [BIZ80]. Extending the calculations to $\left\langle \left(-\frac{t_4}{N}\operatorname{tr}(M^4)\right)^n \right\rangle$ for arbitrary n, they saw that this enumerated the n-vertex diagrams. Expanding the exponential in the integrand of $\frac{Z_N(t_4)}{Z_N(0)} = \left\langle e^{-\frac{t_4}{N}\operatorname{tr}(M^4)} \right\rangle$ and integrating term-by-term, they calculated for the potential with $\mathbf{t} = (0,0,0,t_4,0,\ldots)$ that

$$\frac{Z_N(t_4)}{Z_N(0)} \text{ “ = ” } \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{t_4}{N}\right)^n \sum_g \#\left\{\text{4-valent n-vertex g-diagrams}\right\} N^{2-2g+n},$$

viewing the partition function as a formal generating function for these diagrams. The expansion is certainly formal; one must consider whether the term-by-term integration yields a convergent series on the right-hand side and if so, whether the result is equal to the well-defined left-hand side.

The exponential formula exchanges the generating function of possibly disconnected g-diagrams for its logarithm, a generating function of only connected g-diagrams (g-maps):

$$\log\left(\frac{Z_N(t_4)}{Z_N(0)}\right) \text{ “ = ” } \sum_{n\geq 0} \frac{1}{n!}(-t_4)^n \sum_g \#\left\{\text{4-valent n-vertex g-maps}\right\} N^{2-2g}.$$

Formally resumming to order the count by genus, the result is the genus expansion:

$$\frac{1}{N^2}\log\left(\frac{Z_N(t_4)}{Z_N(0)}\right) \text{ “ = ” } \sum_{g\geq 0} e_g(t_4) N^{-2g}, \tag{7}$$

where the interpretation of the coefficients is that $(-1)^n \frac{\partial^n}{\partial t_4^n} e_g(0) = \#\left\{\text{4-valent n-vertex g-maps}\right\}$.

The following two theorems are the precise statements by Ercolani and McLaughlin that make rigorous sense of these calculations:

Theorem 1.4 ([EM03] Theorem 1.1)**.** *There is $T > 0$, $\gamma > 0$, and $N_0 > 0$ so that for $\mathbf{t} \in \mathbb{T}(T,\gamma)$ and $N > N_0$, the $N \to \infty$ asymptotic expansion*

$$\log\left(\frac{Z_N(\mathbf{t})}{Z_N(0)}\right) = N^2 e_0(\mathbf{t}) + e_1(\mathbf{t}) + \frac{1}{N^2} e_2(\mathbf{t}) + \ldots$$

holds true. The meaning of this expansion is: if terms up to order N^{-2k} are kept, the error term is bounded by CN^{-2k-2}, where the constant C is independent of $\mathbf{t}$ for all $\mathbf{t} \in \mathbb{T}(T,\gamma)$. For each j, the function $e_j(\mathbf{t})$ is an analytic function of the (complex) vector $\mathbf{t}$ in a neighborhood of 0. Moreover, the asymptotic expansion of the derivatives of $\log(Z_N)$ may be calculated via term-by-term differentiation of the above series.

Theorem 1.5 ([EM03] Theorem 1.3)**.** *The coefficients in the asymptotic expansion of the preceeding theorem satisfy the following relations. Let g be a non-negative integer. Then*

$$e_g(t_1,\ldots,t_\nu) = \sum_{n_j \geq 1} \frac{1}{n_1! \cdots n_\nu!}(-t_1)^{n_1}\cdots(-t_\nu)^{n_\nu}\kappa_g(n_1,\ldots,n_\nu),$$

in which each of the coefficients $\kappa_g(n_1,\ldots,n_\nu)$ is the number of g-maps with n_j j-valent vertices for $j = 1,\ldots,\nu$.

For example, the generating function for 4-valent 0-maps with one 1-valent vertex is extracted by taking $\mathbf{t} = (t_1,0,0,t_4,0,\ldots)$ and calculating

$$e_0(t_1,t_4) = \lim_{N\to\infty} \frac{1}{N^2}\log\left(\frac{Z_N(t_1,t_4)}{Z_N(0)}\right) \quad \text{and} \quad -\frac{\partial}{\partial t_1} e_0(t_1,t_4)\bigg|_{t_1=0}.$$

We will develop a recursive scheme (the loop equations) for calculating such functions explicitly and use it to lay a foundation for enumeration of unlabeled planar maps and the study of geometric features like the geodesic distance between two marked points on a graph.

1.6 A combinatorial enumeration of unlabeled planar maps

Unlabeled objects are more fundamental than labeled objects, so it is desirable to be able to enumerate unlabeled maps. A situation where this is possible is described in this section. *Planar maps* are those embedded on the sphere. A *leg* is a 1-valent vertex. The *geodesic distance* of a 2-legged planar map is the minimal number of edges that must be crossed by any path (on the surface) joining the two legs. Define $\mathcal{M}$ to be the family of all 2-legged planar maps with fixed even valence and the two legs distinguished from one another, incoming and outgoing, with the incoming leg always represented in the external (unbounded) face.

These maps admit a traditional combinatorial enumeration because $\mathcal{M}$ is in bijection with a family of trees, which have a recursive structure that can be exploited to derive a generating function. Define generating functions R for the family $\mathcal{M}$ and R_n for the subset of that family which has geodesic distance at most n between the legs. Note that each of these must depend upon the valence of the maps to be considered.

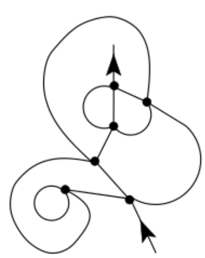
The bijection, due to Schaeffer in his PhD thesis [Sch98], is an algorithm which breaks apart a map in $\mathcal{M}$ into a *blossom tree*: a connected rooted tree whose leaves are decorated with a minimal labeling which records information about how extraneous edges (not required for maintaining connectedness) of the map were cut in the algorithm to obtain the tree. Here are the steps of the algorithm and its inverse [Fra06].

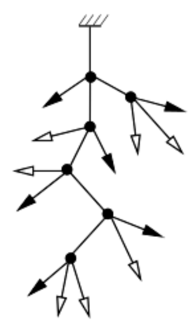
- Begin at the incoming leg and proceed around on the edges of the map counter-clockwise. At each edge, determine whether the edge is necessary for connectedness. If so, leave it and move on to the next edge. If not, cut it and label the half edge behind with a black leaf and the half edge in front with a white leaf.
- Terminate when there is only one face and the graph is still connected. This is now by definition a tree.
- Label the outgoing leg as the root and the incoming leg with a white leaf.

The inverse:

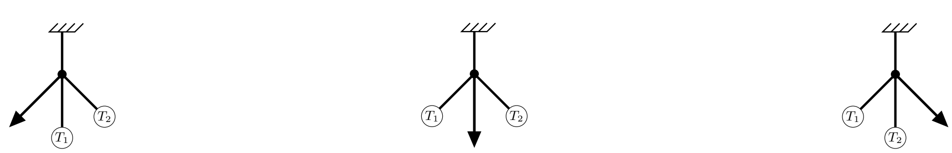
- Begin at any leaf on the tree and proceed on the edges counter-clockwise around the tree. If the present leaf is black and immediately followed in this order by a white leaf, then join the black leaf to the white leaf by an edge. Otherwise, do nothing and proceed to the next leaf.
- Repeat this procedure cyclically around the tree, joining pairs consisting of a black leaf immediately followed by a white leaf together into edges, until all black leaves are exhausted.
- One unmatched white leaf will remain at this point, and it will be in the external face. Replace it by an incoming leg. Replace the root by an outgoing leg.

For concreteness, we show a 4-valent map in this family and its associated blossom tree (images from [Fra06]).





As an example, we use this bijection to calculate R for 4-valent maps. The defining feature of 4-valent blossom trees is that at each vertex there is exactly one black leaf; the other two descending branches are subtrees (a white leaf is a subtree on zero vertices) which are also blossom trees. The environment of the root can then be decomposed in one of three ways:



Defining $r_k = \#\{\text{4-valent k-vertex blossom trees}\}$, this gives the recurrence relation

$$r_k = 3\sum_{i=0}^{k-1} r_i r_{k-1-i}.$$

We use this in the standard generating function calculation: let $R(s_4) = \sum_{k\geq 0} r_k s_4^k$. The parameter s_4 is labeled to reflect the valence of the maps and that it is the coefficient of x^4 in the related potential. The potential used by diFrancesco differs from the potential in this paper and the coefficients are related by $t_4 = -\frac{s_4}{4}$. Then

$$\begin{aligned}
\sum_{k\geq 1} r_k s_4^k &= \sum_{k\geq 1} 3\sum_{i=0}^{k-1} r_i r_{k-1-i} s_4^k \\
\sum_{k\geq 0} r_k s_4^k &= 1 + 3\sum_{k\geq 1}\sum_{i=0}^{k-1} r_i r_{k-1-i} s_4^k \\
&= 1 + 3s_4\sum_{k\geq 1}\sum_{i=0}^{k-1} r_i r_{k-1-i} s_4^{k-1} \\
&= 1 + 3s_4\sum_{n\geq 0}\sum_{i=0}^{n} r_i r_{n-i} s_4^n \\
&= 1 + 3s_4\left(\sum_{n\geq 0} r_n s_4^n\right)^2 \\
&= 1 + 3s_4 R(s_4)^2.
\end{aligned}$$

The conclusion is a functional relation for the generating function: that $R(s_4) = 1 + 3s_4 R(s_4)^2$.

A derivation in similar spirit gives closed formulas for the generating functions R_n [Fra06]. On page 72 is a formula for R_0, the generating function for 2-legged 4-valent planar maps with both legs in the same face:

$$R_0 = R - s_4 R^3 = \frac{R(4-R)}{3}. \tag{8}$$

1.7 Results and outline

The main results in this paper are the derivations of the loop equation hierarchy (Proposition 2.12) and an explicit formula for the leading coefficient $\widehat{P}_0$ (Proposition 3.2), a demonstration of the connection between the genus expansion and loop expansion, and the application of all this to calculate R and R_0 in the case of 4-valent maps.

Section 2 is the proof of our first main result, beginning with the derivation of an identity for the resolvent of a matrix. Applied in the setting of the Unitary Ensemble, the identity can be expressed in terms of the Cauchy transform of the mean density of eigenvalues $\int \frac{\rho_{n,N}(\lambda)}{z-\lambda} d\lambda$, which has a large N expansion called the loop expansion (Theorem 2.11). Viewing the resolvent identity as an equation to be solved for $\int \frac{\rho_{n,N}(\lambda)}{z-\lambda} d\lambda$ and inserting the expansion for this integral, we pull off coefficients at each order of N in the standard way, obtaining a hierarchy of equations to be solved recursively for the coefficients $\widehat{P}_g$ in the loop expansion.

Solving the leading order equation for $\widehat{P}_0$ is equivalent to the problem of finding the leading order of $\int \frac{\rho_{n,N}(\lambda)}{z-\lambda} d\lambda$. Doing this in section 3 is our second main result, culminating in an explicit formula for this function (Proposition 3.2). We prove this result using a change of variables trick and theorem about the leading order behavior of integrals with respect to the mean density of eigenvalues due to Johansson in [Joh98]. This gives a general formula for the desired function which we solve further under the one-cut assumption using some basic complex and harmonic analysis.

Our third and fourth main results are in section 4: connecting the loop expansion to the genus expansion and using this to calculate some key generating functions. The connection relies on the fact that both expansions are differentiable term-by-term (and more fundamentally, on the fact that both expansions arise as different modifications of the same expansion, see Appendix A); differentiating the genus expansion once eliminates a normalizing constant so that both expansions are of the same fundamental object, a logarithmic derivative of the partition function. Differentiating both expansions further give the derivatives of $\widehat{P}_g$ which correspond to the desired derivatives of e_g. By choosing specific potentials and derivatives, we use this method to find formulas for R and R_0 for comparison with the combinatorial calculations in section 1.6.

2 Derivation of the loop equations

In this section, we take the matrix size and large parameter to be the same: $n = N$.

2.1 Derivation of resolvent identity

2.1.1 Basic lemmas

The *resolvent* of a matrix M is defined by

$$G(z, M) = (zI - M)^{-1} \tag{9}$$

for $z \in \mathbb{C}$ such that the inverse matrix exists. The *Green's function* of M is the complex-valued function given by

$$g(z) = \mathrm{tr}(G(z, M)) \tag{10}$$

for $z \in \mathbb{C}$ away from the spectrum of M. In particular, for Hermitian M all the eigenvalues are real so we will take z to be bounded away from the real axis $\Im(z) \geq \epsilon > 0$.

Lemma 2.1.

$$g(z) = \sum_{i=1}^{N} \frac{1}{z - \lambda_i}$$

Proof. The proof is a simple calculation, relying on the diagonalization of M by unitary matrices: $M = U\Lambda U^\dagger$ for Λ the diagonal matrix of eigenvalues of M and U unitary with i^{th} column the eigenvector corresponding to λ_i.

$$\begin{aligned}
g(z) &= \mathrm{tr}((zI - M)^{-1}) \\
&= \mathrm{tr}((zI - U\Lambda U^\dagger)^{-1}) \\
&= \mathrm{tr}((U(zI - \Lambda)U^\dagger)^{-1}) \\
&= \mathrm{tr}(U(zI - \Lambda)^{-1}U^\dagger) \\
&= \mathrm{tr}((zI - \Lambda)^{-1}) \\
&= \sum_{i=1}^{N} \frac{1}{z - \lambda_i}.
\end{aligned}$$

□

The expectation of the Green's function is the Cauchy transform of the one-point function.

Lemma 2.2.

$$\langle g(z) \rangle = N \int_{-\infty}^{\infty} \frac{\rho_N(\lambda)}{z - \lambda} d\lambda$$

Proof. The density of the eigenvalue measure, $P(\boldsymbol{\lambda}) = \frac{1}{Z_N^e} e^{-N\sum_{i=0}^{N} V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2$, is symmetric in the eigenvalues, so that

$$\langle g(x) \rangle = \int \cdots \int \sum_{i=1}^{N} \frac{1}{z - \lambda_i} P(\boldsymbol{\lambda}) d\lambda_1 d\lambda_2 \cdots d\lambda_N$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int \cdots \int \frac{\delta(\lambda-\lambda_i)}{z-\lambda} P(\boldsymbol{\lambda}) d\lambda_1 d\lambda_2 \cdots d\lambda_N \\
&= N \int \cdots \int \frac{\delta(\lambda-\lambda_1)}{z-\lambda} P(\boldsymbol{\lambda}) d\lambda_1 d\lambda_2 \cdots d\lambda_N \\
&= N \int \frac{\delta(\lambda-\lambda_1)}{z-\lambda} \left(\int \cdots \int P(\boldsymbol{\lambda}) d\lambda_2 \cdots d\lambda_N \right) d\lambda_1 \\
&= N \int \frac{\delta(\lambda-\lambda_1)}{z-\lambda} \rho_N(\lambda_1) d\lambda_1 \\
&= N \int \frac{\rho_N(\lambda)}{z-\lambda} d\lambda.
\end{aligned}
$$

□

2.1.2 The identity

The first step towards the loop equations is a classical identity for the resolvent. We will develop the first statement of the identity in two propositions.

Proposition 2.3.

$$\langle G_{ki} G_{jl} \rangle = N \langle G_{kl} V'(M)_{ji} \rangle .$$

Proof. Differentiating $G(zI - M) = I$ with respect to the complex variable M_{ij} we have

$$G(-E_{ij}) + \frac{\partial G}{\partial M_{ij}} \cdot (zI - M) = 0$$

where E_{ij} is the elementary matrix with 1 in the ij^{th} entry and 0's elsewhere. Solve the last equality for the desired derivative:

$$\frac{\partial G}{\partial M_{ij}} = G E_{ij} (zI - M)^{-1} = G E_{ij} G,$$

the kl^{th} entry of which is $\frac{\partial G_{kl}}{\partial M_{ij}} = G_{ki} G_{jl}$. Averaging both sides with respect to the measure on the UE and integrating by parts we have the result:

$$
\begin{aligned}
\langle G_{ki} G_{jl} \rangle &= \frac{1}{Z_N} \int \frac{\partial G_{kl}}{\partial M_{ij}} e^{-N \operatorname{tr}(V(M))} dM \\
&= -\frac{1}{Z_N} \int G_{kl} \frac{\partial}{\partial M_{ij}} \left(e^{-N \operatorname{tr}(V(M))} \right) dM \\
&= -\frac{1}{Z_N} \int G_{kl} \left(-N \operatorname{tr}(V'(M) E_{ij}) \right) e^{-N \operatorname{tr}(V(M))} dM \\
&= \frac{N}{Z_N} \int G_{kl} \operatorname{tr}(V'(M) E_{ij}) e^{-N \operatorname{tr}(V(M))} dM \\
&= \frac{N}{Z_N} \int G_{kl} (V'(M))_{ji} e^{-N \operatorname{tr}(V(M))} dM \\
&= N \langle G_{kl} V'(M)_{ji} \rangle
\end{aligned}
$$

□

Proposition 2.4.

$$\left\langle (g(z))^2 \right\rangle - \langle g(z) \rangle^2 = N \langle \operatorname{tr}(G V'(M)) \rangle - \langle g(z) \rangle^2 .$$

Proof. This follows from the first proposition: set $i = k$ and $j = l$ and sum over the remaining indices. On the left-hand side, this sum clearly becomes a trace squared:

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \langle G_{kk} G_{ll} \rangle = \left\langle \sum_{k=1}^{N} G_{kk} \sum_{l=1}^{N} G_{ll} \right\rangle$$

$$= \left\langle (g(z))^2 \right\rangle .$$

On the right-hand side we have

$$\begin{aligned} N \sum_{k=1}^{N} \sum_{l=1}^{N} \langle G_{kl} V'(M)_{lk} \rangle &= N \left\langle \sum_{k=1}^{N} \sum_{l=1}^{N} G_{kl} V'(M)_{lk} \right\rangle \\ &= N \left\langle \sum_{k=1}^{N} \sum_{l=1}^{N} (G E_{ll} V'(M))_{kk} \right\rangle \\ &= N \left\langle \sum_{l=1}^{N} \operatorname{tr}(G E_{ll} V'(M)) \right\rangle \\ &= N \left\langle \operatorname{tr}(G V'(M)) \right\rangle . \end{aligned}$$

□

Proposition 2.4 is the first form of the identity. The more useful form is this identity recast at the eigenvalue level, in the spirit of Lemma 2.2, which takes care of the second and fourth terms. A similar calculation transforms the third term:

$$N \left\langle \operatorname{tr}(G V'(M)) \right\rangle = N^2 \int_{-\infty}^{\infty} \frac{V'(\lambda) \rho_N(\lambda)}{z - \lambda} d\lambda.$$

In the first term, the trace is squared before integration, so that the end result depends on two eigenvalues and what comes out of the calculation is the two-point function:

$$\begin{aligned} \left\langle (g(z))^2 \right\rangle &= \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} \left(\frac{1}{z - \lambda_i} \right) \left(\frac{1}{z - \lambda_j} \right) P(\boldsymbol{\lambda}) d^N \lambda \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left(\frac{1}{z - \lambda_i} \right)^2 P(\boldsymbol{\lambda}) d^N \lambda + \sum_{i \neq j} \int_{\mathbb{R}^N} \left(\frac{1}{z - \lambda_i} \right) \left(\frac{1}{z - \lambda_j} \right) P(\boldsymbol{\lambda}) d^N \lambda \\ &= N \int_{\mathbb{R}} \frac{\rho_N(\lambda)}{(z - \lambda)^2} d\lambda + N(N-1) \int_{\mathbb{R}^2} \frac{\rho_N(\lambda_1, \lambda_2)}{(z - \lambda_1)(z - \lambda_2)} d\lambda_1 d\lambda_2. \end{aligned}$$

Since the loop expansion is of the one-point function, this term must be worked at harder to get it into the correct form. This is the business of the vertex operator in the next section.

2.1.3 Vertex operator simplifications

The *vertex operator* is a formal sum of derivatives with respect to an infinite list of parameters t_j, the first ν of which we take to be the same parameters that the potential depends on:

$$\frac{d}{dV} = - \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{d}{dt_j}, \tag{11}$$

and we also define a finite truncation of this operator

$$\frac{d}{dV^{(m)}} = - \sum_{j=0}^{m-1} \frac{1}{z^{j+1}} \frac{d}{dt_j}. \tag{12}$$

Proposition 2.5. *For any $m \geq \nu$,*

$$\left\langle \frac{1}{N} g(z) \right\rangle = \int \frac{\rho_N(\lambda)}{z - \lambda} d\lambda$$

$$
\begin{aligned}
&= \sum_{j=0}^{m} \frac{1}{z^{j+1}} \int \lambda^j \rho_N(\lambda) d\lambda + O(z^{-(m+1)}) \\
&= \sum_{j=0}^{m} \frac{1}{z^{j+1}} \left\langle \frac{1}{N} \operatorname{tr}(M^j) \right\rangle + O(z^{-(m+1)}) \\
&= \frac{d}{dV^{(m)}} \left(\frac{1}{N^2} \log Z_N \right) + O(z^{-(m+1)}).
\end{aligned}
$$

The proof of this proposition will follow from the next three lemmas, working from the bottom equality up to the top.

Lemma 2.6.

$$
\frac{d}{dV} \left(\frac{1}{N^2} \log Z_N \right) = \sum_{j=1}^{\nu} \frac{1}{z^{j+1}} \left\langle \frac{1}{N} \operatorname{tr}(M^j) \right\rangle. \tag{13}
$$

Proof.

$$
\frac{d}{dt_j} \left(\frac{1}{N^2} \log Z_N \right) = \frac{1}{N^2} \frac{\frac{d}{dt_j} Z_N}{Z_N} = \left\langle -\frac{1}{N} \operatorname{tr}(M^j) \right\rangle
$$

since

$$
\frac{d}{dt_j} Z_N = \int \frac{d}{dt_j} e^{-N \operatorname{tr}(V_t(M))} dM = \int e^{-N \operatorname{tr}(V_t(M))} (-N \operatorname{tr}(M^j)) dM.
$$

Notice that the sum is now finite because Z_N depends only on t_j for $j \leq \nu$. In particular for all $m \geq \nu$ applying the truncated vertex operator yields exactly the same result. □

Lemma 2.7.

$$
\left\langle \frac{1}{N} \operatorname{tr}(M^j) \right\rangle = \int \lambda^j \rho_N(\lambda) d\lambda.
$$

Proof. By the Spectral Theorem representation $M = U \Lambda U^\dagger$ and conjugation invariance of the trace, $\operatorname{tr}(M^j) = \sum_{i=1}^{N} \lambda_i^j$. Then

$$
\begin{aligned}
\left\langle \frac{1}{N} \operatorname{tr}(M^j) \right\rangle &= \int \cdots \int \frac{1}{N} \operatorname{tr}(M^j) P(\boldsymbol{\lambda}) d\lambda_1 \cdots d\lambda_N \\
&= \frac{1}{N} \int \cdots \int \sum_{i=1}^{N} \lambda_i^j P(\boldsymbol{\lambda}) d\lambda_1 \cdots d\lambda_N \\
&= \frac{1}{N} \sum_{i=1}^{N} \int \cdots \int \lambda_i^j P(\boldsymbol{\lambda}) d\lambda_1 \cdots d\lambda_N \\
&= \int \cdots \int \lambda_1^j P(\boldsymbol{\lambda}) d\lambda_1 \cdots d\lambda_N \\
&= \int \lambda_1^j \left(\int \cdots \int P(\boldsymbol{\lambda}) d\lambda_2 \cdots d\lambda_N \right) d\lambda_1 \\
&= \int \lambda_1^j \rho_N(\lambda_1) d\lambda_1
\end{aligned}
$$

□

Lemma 2.8. *For large z with $\Im(z) \geq \epsilon > 0$ and any $m \geq \nu$,*

$$
\int \frac{\rho_N(\lambda)}{z - \lambda} d\lambda = \sum_{j=0}^{m} \frac{1}{z^{j+1}} \int \lambda^j \rho_N(\lambda) d\lambda + O(z^{-(m+1)}).
$$

Proof. By the geometric series expansion of $\frac{1}{1-\lambda/z}$ for $|\lambda| < |z|$, we have

$$\int \frac{\rho_N(\lambda)}{z-\lambda} d\lambda = \frac{1}{z} \int \left(\sum_{j=0}^{m} \frac{\lambda^j}{z^j} + \frac{\frac{\lambda^{m+1}}{z^{m+1}}}{1-\frac{\lambda}{z}} \right) \rho_N(\lambda) d\lambda$$
$$= \sum_{j=0}^{m} \frac{1}{z^{j+1}} \int \lambda^j \rho_N(\lambda) d\lambda + \frac{1}{z^{m+2}} \int \frac{\lambda^{m+1}\rho_N(\lambda)}{1-\frac{\lambda}{z}} d\lambda.$$

The error is order $z^{-(m+1)}$ by the following calculation:

$$\left| \frac{1}{z^{m+2}} \int \frac{\lambda^{m+1}\rho_N(\lambda)}{1-\frac{\lambda}{z}} d\lambda \right| = \left| \frac{1}{z^{m+1}} \int \frac{\lambda^{m+1}\rho_N(\lambda)}{z-\lambda} d\lambda \right|$$
$$\leq \left| \frac{1}{z^{m+1}} \right| \left| \frac{1}{\epsilon} \right| \left| \int \lambda^{m+1} \rho_N(\lambda) d\lambda \right|$$
$$\leq \frac{C}{\epsilon} \left| \frac{1}{z^{m+1}} \right|$$

since $|z-\lambda| \geq \Im(z) \geq \epsilon > 0$ for every $\lambda \in \mathbb{R}$ and where $C = \left| \int \lambda^{m+1} \rho_N(\lambda) d\lambda \right|$ is finite since moments of eigenvalues with respect to an exponential measure are bounded. □

This completes the proof of Proposition 2.4. Applying the vertex operator a second time connects this result to the resolvent identity by expressing the left-hand side as a derivative of the fundamental object $\int \frac{\rho_N(\lambda)}{z-\lambda} d\lambda$.

Proposition 2.9.

$$\left\langle g(z)^2 \right\rangle - \langle g(z) \rangle^2 = \frac{d}{dV^{(m)}} \int \frac{\rho_N(\lambda)}{z-\lambda} d\lambda + O(z^{-(m+2)}). \tag{14}$$

Proof. By a geometric series expansion of $(z-M)^{-1}$ we have

$$\left\langle g(z)^2 \right\rangle = \frac{1}{Z_N} \int \left(\operatorname{tr}((z-M)^{-1}) \right)^2 e^{-N \operatorname{tr}(V_t(M))} dM$$
$$= \frac{1}{Z_N} \int \operatorname{tr} \left(\sum_{j \geq 0} \frac{1}{z} \left(\frac{M}{z} \right)^j \right) \operatorname{tr} \left(\sum_{k \geq 0} \frac{1}{z} \left(\frac{M}{z} \right)^k \right) e^{-N \operatorname{tr}(V_t(M))} dM + O(z^{-(m+2)})$$
$$= \frac{1}{Z_N} \sum_{0 \leq j+k \leq m-1} \frac{1}{z^{j+k+2}} \int \operatorname{tr}(M^j) \operatorname{tr}(M^k) e^{-N \operatorname{tr}(V_t(M))} dM + O(z^{-(m+2)})$$
$$= \sum_{0 \leq j+k \leq m-1} \frac{1}{z^{j+k+2}} \left\langle \operatorname{tr}(M^j) \operatorname{tr}(M^k) \right\rangle + O(z^{-(m+2)}).$$

Squaring the right representation of $\langle g(z) \rangle$ from Proposition 2.8 and subtracting we have:

$$\left\langle g(z)^2 \right\rangle - \langle g(z) \rangle^2 = \sum_{0 \leq j+k \leq m-1} \frac{1}{z^{j+k+2}} \left(\left\langle \operatorname{tr}(M^j) \operatorname{tr}(M^k) \right\rangle - \left\langle \operatorname{tr}(M^j) \right\rangle \left\langle \operatorname{tr}(M^k) \right\rangle \right) + O(z^{-(m+2)}).$$

Calculating now from the right-hand side of (14), the result is proved.

$$\frac{d}{dV^{(m)}} \int \frac{\rho_N(\lambda)}{z-\lambda} d\lambda = \frac{d}{dV^{(m)}} \left(\sum_{j=0}^{m} \frac{1}{z^{j+1}} \left\langle \frac{1}{N} \operatorname{tr}(M^j) \right\rangle + O(z^{-(m+1)}) \right)$$
$$= \sum_{0 \leq j+k \leq m-1} \frac{1}{z^{j+k+2}} \left(\left\langle \operatorname{tr}(M^j) \operatorname{tr}(M^k) \right\rangle - \left\langle \operatorname{tr}(M^j) \right\rangle \left\langle \operatorname{tr}(M^k) \right\rangle \right) + O(z^{-(m+2)})$$

since

$$\begin{aligned}
\frac{d}{dt_k}\left\langle \frac{1}{N}\operatorname{tr}(M^j)\right\rangle &= \frac{d}{dt_k}\left(\frac{\int \frac{1}{N}\operatorname{tr}(M^j)e^{-N\operatorname{tr}(V_t(M))}dM}{\int e^{-N\operatorname{tr}(V_t(M))}dM}\right)\\
&= \frac{\int e^{-N\operatorname{tr}(V_t(M))}dM \int \frac{1}{N}\operatorname{tr}(M^j)\left(-N\operatorname{tr}(M^k)\right)e^{-N\operatorname{tr}(V_t(M))}dM}{\left(\int e^{-N\operatorname{tr}(V_t(M))}dM\right)^2}\\
&\quad - \frac{\int \frac{1}{N}\operatorname{tr}(M^j)e^{-N\operatorname{tr}(V_t(M))}dM \int \left(-N\operatorname{tr}(M^k)\right)e^{-N\operatorname{tr}(V_t(M))}dM}{\left(\int e^{-N\operatorname{tr}(V_t(M))}dM\right)^2}\\
&= -\left\langle \operatorname{tr}(M^j)\operatorname{tr}(M^k)\right\rangle + \left\langle \operatorname{tr}(M^j)\right\rangle\left\langle \operatorname{tr}(M^k)\right\rangle.
\end{aligned}$$

□

The conclusion of this section is the resolvent identity in its nearly-final form:

$$\frac{d}{dV}\int_{-\infty}^{\infty}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda = N^2\int_{-\infty}^{\infty}\frac{V'(\lambda)\rho_N(\lambda)}{z-\lambda}d\lambda - N^2\left(\int_{-\infty}^{\infty}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda\right)^2. \tag{15}$$

where this equality is understood to mean that applying the truncated vertex operator at level m yields the errors explicitly calculated above.

2.2 Large N asymptotics and the loop equations

2.2.1 Large N asymptotic results: the equilibrium measure

Theorem 2.10 ([EM03]).

$$\int_{-\infty}^{\infty}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda = \int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda + O(e^{-cN}) \quad as \quad N\to\infty,$$

where $[a,b]$ is the support of the equilibrium measure, $\delta>0$, and c is a positive constant depending on δ.

The essentially compact support of these integrals allows the second term in the resolvent identity, $\int_{-\infty}^{\infty}\frac{V'(\lambda)\rho_N(\lambda)}{z-\lambda}d\lambda$, to be rewritten with the functions in the integrand decoupled from one another by applying Cauchy's Integral Formula to re-write $V'(\lambda)$ as an integral over a contour enclosing the interval and excluding the point z.

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{V'(\lambda)\rho_N(\lambda)}{z-\lambda}d\lambda &= \int_{a-\delta}^{b+\delta}\frac{V'(\lambda)\rho_N(\lambda)}{z-\lambda}d\lambda + O(e^{-cN})\\
&= \int_{\alpha-\delta}^{\beta+\delta}\left(\frac{1}{2\pi i}\oint\frac{V'(w)}{w-\lambda}dw\right)\frac{\rho_N(\lambda)}{z-\lambda}d\lambda + O(e^{-cN})\\
&= \frac{1}{2\pi i}\oint\int_{a-\delta}^{b+\delta}\left(\frac{V'(w)}{w-\lambda}\right)\frac{\rho_N(\lambda)}{z-\lambda}d\lambda dw + O(e^{-cN})\\
&= \frac{1}{2\pi i}\oint\int_{a-\delta}^{b+\delta}V'(w)\rho_N(\lambda)\left(\frac{\frac{1}{z-w}}{w-\lambda} - \frac{\frac{1}{z-w}}{z-\lambda}\right)d\lambda dw + O(e^{-cN})\\
&= \frac{1}{2\pi i}\oint\int_{a-\delta}^{b+\delta}V'(w)\rho_N(\lambda)\left(\frac{1}{z-w}\right)\left(\frac{1}{w-\lambda}\right)d\lambda dw\\
&\quad - \frac{1}{2\pi i}\left(\oint\frac{V'(w)}{z-w}dw\right)\left(\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda\right) + O(e^{-cN})\\
&= \frac{1}{2\pi i}\oint\frac{V'(w)}{z-w}\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{w-\lambda}d\lambda dw + O(e^{-cN})
\end{aligned}$$

The final form of the resolvent identity is then

$$\frac{1}{2\pi i}\oint \frac{V'(w)}{z-w}\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{w-\lambda}d\lambda\, dw = \frac{1}{N^2}\frac{d}{dV}\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda + \left(\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda\right)^2 + O(e^{-cN}). \tag{16}$$

2.2.2 Large N asymptotic results: the loop expansion

The loop expansion follows from a result in Ercolani and McLaughlin [EM03].

Theorem 2.11. *For each $\delta > 0$ there exists $T > 0$ and $\gamma > 0$ so that for $\mathbf{t} \in \mathbb{T}(T,\gamma)$, the expansion*

$$\int_{a-\delta}^{b+\delta}\frac{\rho_N(\lambda)}{z-\lambda}d\lambda \sim -i\pi\sum_{g=0}^{\infty}\frac{1}{N^{2g}}\widehat{P}_g(z) \tag{17}$$

holds for $z \in \mathbb{C}\setminus[a-\delta, b+\delta]$. The coefficients depend analytically on z and $\boldsymbol{t} \in \mathbb{T}(T,\gamma)$, and have convergent Laurent expansions for $z \to \infty$. The asymptotic expansion may be differentiated term by term.

2.2.3 The loop equations hierarchy

Proposition 2.12. *The leading order loop equation is*

$$\frac{1}{2\pi i}\oint \frac{-i\pi V_t'(w) + \pi^2\widehat{P}_0(w)}{z-w}\widehat{P}_0(w)dw = 0$$

where the integral is taken over a simple closed counter-clockwise contour enclosing $[a,b]$ and excluding z, and for $g \geq 1$,

$$\frac{1}{2\pi i}\oint \frac{-i\pi V_t'(w) + 2\pi^2\widehat{P}_0(w)}{z-w}\widehat{P}_g(w)dw = -i\pi\frac{d}{dV}\widehat{P}_{g-1}(z) - \pi^2\sum_{g'=1}^{g-1}\widehat{P}_{g'}(z)\widehat{P}_{g-g'}(z). \tag{18}$$

These equations come from inserting the loop expansion into the resolvent identity (16) and pulling off the coefficients of each order of N. For $n \geq 1$ the N^{-2g} order terms yield

$$\frac{1}{2\pi i}\oint \frac{-i\pi V_t'(w)}{z-w}\widehat{P}_g(w)dw = -i\pi\frac{d}{dV}\widehat{P}_{g-1}(z) - \pi^2\sum_{g'=0}^{g}\widehat{P}_{g'}(z)\widehat{P}_{g-g'}(z)$$

and subtracting $2\widehat{P}_0(z)\widehat{P}_g(z)$ from both sides gives the final form in the proposition.

3 The leading order loop equation

3.1 A change of variables

Johansson's derivation of the leading order $-i\pi\widehat{P}_0(z)$ of $\int\frac{\rho_N(\lambda)}{z-\lambda}d\lambda$ is applicable in a more general setting than the rest of the orders can be known. Define the probability measure

$$Z_{n,N}^{e,\beta} = \int_{\mathbb{R}^n} e^{\frac{\beta}{2}\left(\sum_{i\neq j}\log|x_i-x_j| - N\sum_{i=0}^{n}V_t(x_i)\right)}d^n x. \tag{19}$$

Taking $\beta = 2$ reduces to the measure introduced in section 1.4 and gives the Unitary Ensemble, but the measure is interesting to study for any $\beta > 0$, giving the eigenvalue distributions of the β-ensembles [DE02].

The first step is a trick common in quantum field theory: reparametrization invariance. The idea is to make a change of variables depending on a small parameter, differentiate with respect to the parameter, and set it equal to zero, yielding an identity called the variational formula. Make the change of variables

$x_j = y_j + \gamma\phi(y_j)$, where $\phi \in C^1(\mathbb{R})$ and ϕ' bounded from below, and $\gamma \geq 0$ is small enough that $\gamma\phi'(y) > -1$ for all $y \in \mathbb{R}$. Then

$$Z_{n,N}^{e,\beta} = \int_{\mathbb{R}^n} e^{\frac{\beta}{2}\left(\sum_{i\neq j}\log|y_i+\gamma\phi(y_i)-y_j-\gamma\phi(y_j)|-N\sum_{i=0}^n V(y_i+\gamma\phi(y_i))\right)}\prod_{i=1}^n(1+\gamma\phi'(y_i))d^n y.$$

For relative brevity of notation, let

$$w(y) = w(y_1,\ldots,y_n) = e^{\frac{\beta}{2}\left(\sum_{i\neq j}\log|y_i+\gamma\phi(y_i)-y_j-\gamma\phi(y_j)|-N\sum_{i=0}^n V(y_i+\gamma\phi(y_i))\right)}.$$

Then using this trick:

$$\begin{aligned}
\frac{d}{d\gamma}\log Z_{n,N}^{e,\beta}\Big|_{\gamma=0^+} &= \frac{1}{Z_{n,N}^{e,\beta}}\Bigg[\int_{\mathbb{R}^n} w(y)\frac{\beta}{2}\sum_{i\neq j}\frac{|\phi(y_i)-\phi(y_j)|}{|y_i-y_j+\gamma\phi(y_i)-\gamma\phi(y_j)|}\prod_{i=1}^n(1+\gamma\phi'(y_i))d^n y \\
&\quad + \int_{\mathbb{R}^n} w(y)\left(-\frac{\beta}{2}N\sum_{i=0}^n V'(y_i+\gamma\phi(y_i))\phi(y_i)\right)\prod_{i=1}^n(1+\gamma\phi'(y_i))d^n y \\
&\quad + \int_{\mathbb{R}^n} w(y)\left(\sum_{i=1}^n\phi'(y_i)\prod_{j\neq i}(1+\gamma\phi'(y_j))\right)d^n y\Bigg]\Bigg|_{\gamma=0^+} \\
&= \frac{1}{Z_{n,N}^{e,\beta}}\Bigg[\int_{\mathbb{R}^n} w(y)\frac{\beta}{2}\sum_{i\neq j}\frac{|\phi(y_i)-\phi(y_j)|}{|y_i-y_j|}d^n y \\
&\quad + \int_{\mathbb{R}^n} w(y)\left(-\frac{\beta}{2}N\sum_{i=0}^n V'(y_i)\phi(y_i)\right)d^n y \\
&\quad + \int_{\mathbb{R}^n} w(y)\sum_{i=1}^n\phi'(y_i)d^n y\Bigg] \\
&= \frac{\beta}{2}n(n-1)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\phi(y_1)-\phi(y_2)|}{|y_1-y_1|}\rho_{n,N}(y_1,y_2)dy_1dy_2 \\
&\quad - \frac{\beta}{2}nN\int_{\mathbb{R}} V'(y_1)\phi(y_1)\rho_{n,N}(y_1)dy_1 + n\int_{\mathbb{R}}\phi'(y_1)\rho_{n,N}(y_1)dy_1
\end{aligned}$$

we obtain the variational formula:

$$\frac{\beta}{2}n(n-1)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\phi(t)-\phi(s)}{t-s}\rho_{n,N}(t,s)dtds - \frac{\beta}{2}nN\int_{\mathbb{R}} V'(t)\phi(t)\rho_{n,N}(t)dt + n\int_{\mathbb{R}}\phi'(t)\rho_{n,N}(t)dt = 0. \quad (20)$$

Choose $\phi(t) = \frac{1}{z-t}$.

$$\begin{aligned}
0 &= \frac{\beta}{2}n(n-1)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\frac{1}{z-t}-\frac{1}{z-s}}{t-s}\rho_{n,N}(t,s)dtds - \frac{\beta}{2}nN\int_{\mathbb{R}}\frac{V'(t)}{z-t}\rho_{n,N}(t)dt + n\int_{\mathbb{R}}\frac{-1}{(z-t)^2}\rho_{n,N}(t)dt \\
&= \frac{\beta}{2}(n-1)\int\int\frac{\rho_{n,N}(t,s)}{(z-t)(z-s)}dtds - \frac{\beta}{2}N\int\frac{V'(t)}{z-t}\rho_{n,N}(t)dt - \int\frac{\rho_{n,N}(t)}{(z-t)^2}dt \\
&= \frac{\beta}{2}(n-1)\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds + \frac{\beta}{2}N\int\frac{V'(t)}{t-z}\rho_{n,N}(t)dt - \int\frac{\rho_{n,N}(t)}{(t-z)^2}dt
\end{aligned}$$

Dividing both sides by $\frac{\beta}{2}(i\pi)^2$ we obtain

$$0 = \frac{n-1}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds + \frac{N}{(i\pi)^2}\int\frac{V'(t)}{t-z}\rho_{n,N}(t)dt - \frac{2}{\beta}\frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt. \quad (21)$$

Define the following functions:

$$U_{n,N}(z) = \frac{1}{i\pi}\int \frac{\rho_{n,N}(t)}{t-z}dt, \tag{22}$$

$$T_{n,N}(z) = \frac{1}{i\pi}\int \frac{V'(z)-V'(t)}{t-z}\rho_{n,N}(t)dt, \tag{23}$$

and

$$K_{n,N}(z) = n\frac{1}{i\pi}\int\frac{1}{i\pi}\int\left(\frac{1}{(t-z)(s-z)} - \frac{1}{2(t-z)^2} - \frac{1}{2(s-z)^2}\right)k_{n,N}(t,s)dtds \tag{24}$$

where $k_{n,N}(t,s) = n\rho_{n,N}(t)\rho_{n,N}(s) - (n-1)\rho_{n,N}(t,s)$. Expand and simplify $K_{n,N}(z)$:

$$\begin{aligned}
\frac{1}{n}K_{n,N}(z) &= \frac{n}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t)\rho_{n,N}(s)}{(t-z)(s-z)}dtds - \frac{n-1}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds \\
&\quad - \frac{n}{2(i\pi)^2}\int\int\left(\frac{1}{(t-z)^2}+\frac{1}{(s-z)^2}\right)\rho_{n,N}(t)\rho_{n,N}(s)dtds \\
&\quad + \frac{n-1}{2(i\pi)^2}\int\int\left(\frac{1}{(t-z)^2}+\frac{1}{(s-z)^2}\right)\rho_{n,N}(t,s)dtds \\
&= \frac{n}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{t-z}dt\int\frac{\rho_{n,N}(s)}{s-z}ds - \frac{n-1}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds \\
&\quad - \frac{n}{2(i\pi)^2}\left(\int\frac{\rho_{n,N}(t)}{(t-z)^2}\int\rho_{n,N}(s)dsdt + \int\frac{\rho_{n,N}(s)}{(s-z)^2}\int\rho_{n,N}(t)dtds\right) \\
&\quad + \frac{n-1}{2(i\pi)^2}\left(\int\frac{1}{(t-z)^2}\int\rho_{n,N}(t,s)dsdt + \int\frac{1}{(s-z)^2}\int\rho_{n,N}(t,s)dtds\right) \\
&= nU_{n,N}(z)^2 - \frac{n-1}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds \\
&\quad - \frac{n}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt + \frac{n-1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt \\
&= nU_{n,N}(z)^2 - \frac{n-1}{(i\pi)^2}\int\int\frac{\rho_{n,N}(t,s)}{(t-z)(s-z)}dtds - \frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt.
\end{aligned}$$

Solving the resulting equation for the second term in this last line and substituting this in for the first term in equation (21) we have

$$\begin{aligned}
0 &= -\frac{1}{n}K_{n,N}(z) + nU_{n,N}(z)^2 - \frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt \\
&\quad + \frac{N}{(i\pi)^2}\int\frac{V'(t)}{t-z}\rho_{n,N}(t)dt - \frac{2}{\beta}\frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt \\
&= -\frac{1}{n}K_{n,N}(z) + nU_{n,N}(z)^2 - \left(1+\frac{2}{\beta}\right)\frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt \\
&\quad + \frac{N}{(i\pi)^2}\int\frac{V'(t)}{t-z}\rho_{n,N}(t)dt - \frac{N}{(i\pi)^2}V'(z)\int\frac{\rho_{n,N}(t)}{t-z}dt + \frac{N}{(i\pi)^2}V'(z)\int\frac{\rho_{n,N}(t)}{t-z}dt \\
&= -\frac{1}{n}K_{n,N}(z) + nU_{n,N}(z)^2 - \left(1+\frac{2}{\beta}\right)\frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt \\
&\quad - \frac{N}{i\pi}T_{n,N}(z) + \frac{N}{i\pi}V'(z)U_{n,N}(z)
\end{aligned}$$

Finally, divide through by n:

$$0 = -\frac{1}{n^2}K_{n,N}(z) + U_{n,N}(z)^2 - \frac{1}{n}\left(1+\frac{2}{\beta}\right)\frac{1}{(i\pi)^2}\int\frac{\rho_{n,N}(t)}{(t-z)^2}dt - \frac{N}{ni\pi}T_{n,N}(z) + \frac{N}{ni\pi}V'(z)U_{n,N}(z). \tag{25}$$

3.2 Leading order integral asymptotics

We will now study the asymptotics of the Borel transform of the one-point function through the following theorem, a piece of one of the main results (Theorem 2.1, p. 156) in [Joh98].

Theorem 3.1. *If the equilibrium measure $\psi(y)dy$ exists, then for any continuous bounded function f on $\mathbb{R}^k$ we have*

$$\lim_{n,N\to\infty,n/N\to 1}\int_{\mathbb{R}^k} f(y_1,\dots,y_k)\rho_{n,N}(y_1,\dots,y_k)d^k y = \int_{\mathbb{R}^k} f(y_1,\dots,y_k)\psi(y_1)\cdots\psi(y_k)d^k y.$$

Taking the double limit as $n, N\to\infty$ and $\frac{n}{N}\to 1$ in (25) and applying a truncation argument to the polynomial in the integrand of $T_{n,N}$ (details in Appendix B), we obtain

$$U(z)^2 + \frac{1}{i\pi}V'(z)U(z) - \frac{1}{i\pi}T(z) = 0, \tag{26}$$

where $U(z)$ and $T(z)$ are defined to be the limits of $U_{n,N}(z)$ and $T_{n,N}(z)$. The most important function to come out of this limit is

$$U(z) = \lim_{n,N\to\infty,n/N\to 1} U_{n,N}(z) = \frac{1}{i\pi}\int_{\mathbb{R}}\frac{\psi(t)}{t-z}dt. \tag{27}$$

It is completely characterized by three things: it is analytic in $\mathbb{C}\setminus[a,b]$, it behaves like $-\frac{1}{i\pi z}$ as $z\to\infty$ in the first quadrant away from $\mathbb{R}$, and it solves the quadratic equation (26).

3.3 Explicit formula for $P_0(z) = U(z)$

The first step towards an explicit formula for $U(z)$ is simply solving the quadratic equation:

$$U(z) = \frac{iV'(z) - \sqrt{-V'(z)^2 - 4\pi i T(z)}}{2\pi}. \tag{28}$$

Since $U(z)$ is analytic away from the support $[a,b]$ of the equilibrium measure, the square root on the right-hand side must have its branch cut along $[a,b]$. Therefore $\sqrt{-V'(z)^2 - 4\pi i T(z)}$ can be written as the product of an analytic function and $\sqrt{z-a}\sqrt{z-b}$ with square roots defined by the principal branch $\sqrt{w} = \sqrt{|w|}e^{\frac{i}{2}\mathrm{Arg}(w)}$ for $\mathrm{Arg}(w)\in(-\pi,\pi]$. Write

$$U(z) = \frac{iV'(z) - iq(z)\sqrt{z-a}\sqrt{z-b}}{2\pi}$$

for some function $q(z)$:

$$q(z) = \frac{V'(z)}{\sqrt{z-a}\sqrt{z-b}} + \frac{2\pi i U(z)}{\sqrt{z-a}\sqrt{z-b}}. \tag{29}$$

Proposition 3.2. *If $[a,b] = supp(\psi)$, then*

$$U(z) = \frac{iV'(z) - iq(z)\sqrt{z-a}\sqrt{z-b}}{2\pi}$$

with $q(z)$ a polynomial of degree $\nu - 2$ given by

$$q(z) = \frac{1}{2\pi i}\oint \frac{V'(w)}{\sqrt{w-a}\sqrt{w-b}}\frac{dw}{w-z}$$

for a simple closed contour traversed counter-clockwise and encircling $[a,b]$ and z.

Corollary 3.3. *If $[a,b] = supp(\psi)$, the equilibrium measure is given by*

$$\psi(x) = \frac{1}{2\pi i}q(x)\sqrt{x-a}\sqrt{x-b}\;\chi_{[a,b]}(x).$$

We will prove the proposition, but first require a result from harmonic analysis that gives the boundary values of $U(z)$ as z approaches the real axis perpendicularly. These are defined precisely for any complex-analytic function f and $x \in \mathbb{R}$ for which the limit exists as

$$f_{\pm}(x) = \lim_{\epsilon \to 0^+} f(x \pm i\epsilon).$$

We will also need the Hilbert transform of the equilibrium measure:

$$H\psi(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\psi(y)}{x-y} dy. \tag{30}$$

The boundary values of $U(z)$ exist and are given by the following (see [Joh98] and [Dei00]).

Lemma 3.4. *For all real x,*

$$U_{\pm}(x) = iH\psi(x) \pm \psi(x).$$

Furthermore, for $x \in supp(\psi)$ we have

$$U_{\pm}(x) = i\frac{V'(x)}{2\pi} \pm \psi(x).$$

Proof of Proposition 3.2. We will show first that $q(z)$ is an entire function, then that it is a polynomial, and finally calculate its explicit formula. To show $q(z)$ is entire, we apply the Schwarz Reflection Principle on the domain $\mathbb{C}$ with line of symmetry the real axis. To show that $q(z)$ is continuous across the real axis, we will calculate the difference of boundary values for $x \in [a, b]$:

$$\begin{aligned}
q_+(x) - q_-(x) &= \frac{V'(x) + 2\pi i U_+(x)}{(\sqrt{x-a}\sqrt{x-b})_+} - \frac{V'(x) + 2\pi i U_-(x)}{(\sqrt{x-a}\sqrt{x-b})_-} \\
&= \frac{V'(x) + (-V'(x) + 2\pi i\psi(x))}{(\sqrt{x-a}\sqrt{x-b})_+} + \frac{V'(x) + (-V'(x) - 2\pi i\psi(x))}{(\sqrt{x-a}\sqrt{x-b})_+} \\
&= \frac{2\pi i\psi(x) - 2\pi i\psi(x)}{(\sqrt{x-a}\sqrt{x-b})_+} \\
&= 0
\end{aligned}$$

so q is continuous across the entire real line. Therefore q is entire.

To see that q is a polynomial, we first consider its large z behavior in the first quadrant away from the real line. The asymptotics are that $U(z) \sim -\frac{1}{i\pi z}$ and first term on the right-hand side of $q(z)$ goes like $\frac{z^{\nu-1}}{z} \sim z^{\nu-2}$, so we must have $q(z) \sim z^{\nu-2}$. Therefore, there exists a constant M so that $|q(z)| \leq M|z^{\nu-2}|$ for z large enough. By analyticity, a Taylor series

$$q(z) = \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!} z^n.$$

for $q(z)$ is valid in the whole plane. By Cauchy's Integral Formula,

$$q^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{q(w)}{(w-z)^{n+1}} dw$$

for $|z| < R$ where the contour is a circle of radius $R > 0$ centered at the origin. Then for R large enough

$$|q^{(n)}(0)| \leq \frac{n!}{2\pi} \frac{MR^{\nu-2}}{R^{n+1}} 2\pi R = \frac{n! M R^{\nu-2}}{R^n},$$

which goes to 0 as $R \to \infty$ if $n > \nu - 2$. Therefore $|q^{(n)}(0)| = 0$ for $n > \nu - 2$, so $q(z)$ is a polynomial:

$$q(z) = \sum_{n=0}^{\nu-2} \frac{q^{(n)}(0)}{n!} z^n.$$

Finally we calculate the explicit formula for this polynomial. By the balancing of asymptotics, $q(z)$ must be equal to the polynomial part of the first term in (29). Define:

$$H(z) := \frac{V'(z)}{\sqrt{z-a}\sqrt{z-b}} = H_P(z) + H_L(z) \tag{31}$$

where $H_P(z)$ is the polynomial part of $H(z)$ and $H_L(z)$ the Laurent part. Then $q(z) = H_P(z) = \sum_{j=0}^{\nu-2} h_j z^j$, where the coefficients h_j are given by Cauchy's formula

$$h_j = \frac{1}{2\pi i} \oint \frac{V'(w)}{\sqrt{w-a}\sqrt{w-b}} \frac{dw}{w^{j+1}}$$

with the contour a simple closed loop encircling $[a, b]$ and z, so

$$\begin{aligned} q(z) &= \sum_{j=0}^{\nu-2} \frac{1}{2\pi i} \oint \frac{V'(w)}{\sqrt{w-a}\sqrt{w-b}} \frac{dw}{w^{j+1}} z^j \\ &= \frac{1}{2\pi i} \oint \frac{V'(w)}{\sqrt{w-a}\sqrt{w-b}} \frac{dw}{w-z}. \end{aligned}$$

□

4 Explicit formulas for map generating functions

4.1 The genus expansion and loop expansion connected

Recall the genus expansion and the loop expansion, equations (7) and (17), respectively:

$$\frac{1}{N^2} \log\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right) \sim \sum_{g=0}^{\infty} \frac{1}{N^{2g}} e_g(\mathbf{t})$$

and

$$\frac{d}{dV} \frac{1}{N^2} \log Z_N(\mathbf{t}) \sim -i\pi \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \widehat{P}_g(z; \mathbf{t}).$$

Since the vertex operator $\frac{d}{dV} = -\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{d}{dt_j}$ is composed of derivatives with respect to the t_j parameters, then

$$\frac{d}{dV}\left(\frac{1}{N^2} \log\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right)\right) = \frac{1}{N^2} \frac{\frac{d}{dV}\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right)}{\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}} = \frac{1}{N^2} \frac{\frac{d}{dV}\left(Z_N(\mathbf{t})\right)}{Z_N(\mathbf{t})} = \frac{d}{dV}\left(\frac{1}{N^2} \log\left(Z_N(\mathbf{t})\right)\right),$$

which implies that

$$\frac{d}{dV} \sum_{g=0}^{\infty} \frac{1}{N^{2g}} e_g(\mathbf{t}) \sim -i\pi \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \widehat{P}_g(z; \mathbf{t}). \tag{32}$$

The meaning of this expansion is that if the vertex operator is truncated at level m, then the error in z is $O(z^{-(m+1)})$, and if the sum is truncated at level k, then the error in N is $O(N^{-2(k+1)})$.

4.2 Example: 2-legged 4-valent maps on the sphere

With potential $V_t(z) = \frac{1}{2}z^2 + t_1 z + t_4 z^4$, we use the correspondence to calculate the generating function for 2-legged 4-valent 0-maps and compare it to $R(s_4)$ from section 1.6. It will be given by

$$\left.\frac{d^2}{dt_1^2} e_0(t_1, t_4)\right|_{t_1=0}.$$

Taking the leading order in N of (32) and truncating the vertex operator at level 2 to include derivatives with respect to t_1 at most, we have:

$$\frac{1}{z^2}\frac{1}{z^2}\frac{d^2}{dt_1^2}e_0(t_1,t_4) = -\frac{1}{z^2}\frac{d}{dt_1}(-i\pi)\widehat{P}_0(z;t_1,t_4),$$

and therefore

$$\frac{d^2}{dt_1^2}e_0(t_1,t_4) = i\pi\frac{d}{dt_1}[z^{-2}]\widehat{P}_0(z;t_1,t_4),$$

where $[z^k]$ denotes the coefficient of z^k in a series expansion of the function in question. We use the formula in Proposition 3.2 for $\widehat{P}_0(z) = U(z)$ to find $[z^{-2}]\widehat{P}_0(z)$.

$$\begin{aligned}
U(z) &= \frac{iV'(z) - iq(z)\sqrt{z-a}\sqrt{z-b}}{2\pi} \\
&= \frac{i}{2\pi}\left(V'(z) - zq(z)\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}\right) \\
&= \frac{i}{2\pi}\left(\frac{V'(z)}{\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}} - zq(z)\right)\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}
\end{aligned}$$

Since $U(z) \sim \frac{1}{i\pi z}$ and $zq(z) \sim z^{\nu-2}$, this polynomial must cancel the polynomial part of the fractional expression. Writing the Laurent part of this fractional expression as $\sum_{j=1}^{\infty} h_{-j}z^{-j}$ we have

$$\begin{aligned}
[z^{-2}]U(z) &= \frac{i}{2\pi}[z^{-2}]\left(\left[\frac{V'(z)}{\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}}\right]_L\right)\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}} \\
&= \frac{i}{2\pi}[z^{-2}]\left(\frac{h_{-1}}{z} + \frac{h_{-2}}{z^2}\right)\left(1 - \frac{a+b}{2z}\right) \\
&= \frac{i}{2\pi}\left(h_{-2} - \frac{a+b}{2}h_{-1}\right).
\end{aligned}$$

When setting $t_1 = 0$ the potential becomes even, so that $a = -b$. Taking the t_1 derivative of the second term will vanish when this happens, so we include only the first term in the next calculation:

$$\begin{aligned}
i\pi\frac{d}{dt_1}[z^{-2}]U(z)\bigg|_{t_1=0} &= i\pi\left(\frac{i}{2\pi}\right)\frac{d}{dt_1}h_{-2}\bigg|_{t_1=0} \\
&= -\frac{1}{2}[z^{-2}]\frac{d}{dt_1}\left(\frac{V'(z)}{\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}}\right)\Bigg|_{t_1=0} \\
&= -\frac{1}{2}[z^{-2}]\left(\frac{1}{\sqrt{1-\frac{-b}{z}}\sqrt{1-\frac{b}{z}}} - \frac{V'(z)\,\frac{d}{dt_1}\left(-\frac{a+b}{2z}+\frac{ab}{z^2}\right)\Big|_{t_1=0}}{((1-\frac{-b}{z})(1-\frac{b}{z}))^{3/2}}\right) \\
&= -\frac{1}{2}[z^{-2}]\frac{1}{\sqrt{1-\frac{b^2}{z^2}}} \\
&= \frac{b^2}{4}.
\end{aligned}$$

The conclusion of this calculation is therefore that

$$\frac{d^2}{dt_1^2}e_0(t_1,t_4)\bigg|_{t_1=0} = i\pi[z^{-2}]\frac{d}{dt_1}\widehat{P}_0(z;t_1,t_4)\bigg|_{t_1=0} = \frac{b^2}{4}.$$

The comparison to the combinatorial result $R(s_4) = 1 + 3s_4 R(s_4)^2$ needs two pieces: first, the parameters s_4 and t_4 are related by $t_4 = -\frac{s_4}{4}$ and second, this generating function $\frac{b^2}{4}$, referred to as z_0 in the paper [EMP08], is shown independently to satisfy $1 = z_0 + 12tz_0^2$. Then $1 = \frac{b^2}{4} + 12\left(\frac{-s_4}{4}\right)\left(\frac{b^2}{4}\right)^2$ reduces to $\frac{b^2}{4} = 1 + 3s_4\left(\frac{b^2}{4}\right)^2$, the defining equation for $R(s_4)$.

4.3 Example: all such maps with geodesic distance 0

We next find the generating function for 2-legged 4-valent 0-maps with both legs in the same face, and compare this to R_0 in equation (8). Such a map can be viewed as coming from one with a 2-valent vertex which was split apart into two 1-valent vertices, the generating function for which which can be calculated by taking a t_2 derivative of e_0, which introduces a negative sign (see 1.5). If breaking the 2-valent vertex apart disconnects the graph into two pieces, then they will each be 1-leg maps. Since we are interested only in connected maps, this contribution must be subtracted off. Taking the potential $V(z) = \frac{1}{2}z^2 + t_1 z + t_2 z^2 + t_4 z^4$ we will therefore calculate

$$-\frac{d}{dt_2}e_0(t_2,t_4)\bigg|_{t_2=0} - \left(-\frac{d}{dt_1}e_0(t_1,t_4)\bigg|_{t_1=0}\right)^2$$

one piece at a time.

The potential for the first piece is even so $a = -b$. Calculating Taylor expansions in SAGE we have

$$\begin{aligned}
\frac{d}{dt_2}e_0 &= i\pi\frac{i}{2\pi}[z^{-3}]\left(\frac{V'(z)}{\sqrt{1-\frac{b^2}{z^2}}}\right)_L\sqrt{1-\frac{b^2}{z^2}}\\
&= -\frac{1}{2}[z^{-3}]\left(\frac{h_{-1}}{z}+\frac{h_{-2}}{z^2}+\frac{h_{-3}}{z^3}\right)\left(1-\frac{a+b}{2z}-\frac{(a-b)^2}{8z^2}\right)\\
&= -\frac{1}{2}\left(h_{-3}-h_{-1}\frac{b^2}{2}\right)\\
&= -\frac{1}{2}\left(\frac{10b^6t_4+6b^4t_2+3b^4}{8}-\frac{3b^4t_4+2b^2t_2+b^2}{2}\cdot\frac{b^2}{2}\right)\\
&= -\frac{1}{16}\left(4b^6t_4+b^4\right).
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt_1}e_0(\mathbf{t}) &= i\pi\frac{i}{2\pi}[z^{-2}]\left(\frac{V'(z)}{\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}}\right)_L\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}}\\
&= -\frac{1}{2}[z^{-2}]\left(\frac{h_{-1}}{z}+\frac{h_{-2}}{z^2}\right)\left(1-\frac{a+b}{2z}\right)\\
&= -\frac{1}{2}\left(h_{-2}-h_{-1}\frac{a+b}{2}\right)\\
&= -\frac{1}{2}\left(h_{-2}\right)\\
&= -\frac{1}{2}\left(\frac{b^2t_1}{2}\right)\\
&= 0.
\end{aligned}$$

Since the second term is 0 then the desired generating function is given by

$$-\frac{d}{dt_2}e_0(t_2,t_4)\bigg|_{t_2=0} = \frac{1}{16}\left(4b^6t_4+b^4\right)$$

and the comparison to $R_0(s_4)$ comes by relating $t_4 = -\frac{s_4}{4}$, $R(s_4) = \frac{b^2}{4}$ from the previous section, and $R(s_4) = 1 + 3s_4 R(s_4)^2$ implying $s_4 R^2 = \frac{R-1}{3}$. We have:

$$\begin{aligned}\frac{1}{16}\left(4b^6 t_4 + b^4\right) &= \frac{1}{16}\left(-b^6 s_4 + b^4\right) \\ &= -4\left(\frac{b^2}{4}\right)\left(\frac{b^2}{4}\right)^2 s_4 + \left(\frac{b^2}{4}\right)^2 \\ &= -4R\left(\frac{R-1}{3}\right) + R^2 \\ &= \frac{R(4-R)}{3} \\ &= R_0.\end{aligned}$$

Appendix A The theorem behind the genus & loop expansions

Both the genus expansion and the loop expansion come from the foundational theorem proved in [EM03] (Theorem 1.4 p. 767):

Theorem A.1. *There exists $T > 0$ and $\gamma > 0$ such that for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$ the expansion*

$$\int f(\lambda)\rho_N(\lambda)d\lambda = f_0 + \frac{1}{N^2}f_1 + \frac{1}{N^4}f_2 + \dots$$

holds, provided that $f(\lambda)$ is C^∞ smooth and grows no faster than a polynomial for $\lambda \to \infty$. The coefficients f_j depend analytically on $\mathbf{t}$ for $\mathbf{t} \in \mathbb{T}(T, \gamma)$, and the asymptotic expansion may be differentiated term-by-term with respect to $\mathbf{t}$.

The genus expansion was derived in [EM03] as a consequence of this theorem. An outline of the proof is found in §1.6 of [EM03], and we record the basic steps here as well. First, the same calculations as in Lemmas 2.6 and 2.7 (with a single t_j derivative instead of the whole vertex operator) give

$$\frac{\partial}{\partial t_j}\log(Z_N) = -N^2 \int \lambda^j \rho_N(\lambda)d\lambda.$$

The uniform asymptotics of the right-hand side are established by the above theorem, and integrating up by the Fundamental Theorem of Calculus one has

$$Z_N(\mathbf{t}) = Z_N(\mathbf{0})\exp\left\{-N^2 \int_0^{\mathbf{t}} \int_{\mathbb{R}} \rho_N(\lambda)\vec{\nabla}_t V d\lambda \cdot \vec{dj}\right\},$$

from which the asymptotics of $\log\left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})}\right)$ can be discovered.

The loop expansion was derived in [EM07] as a consequence of the same theorem. It had to be extended to include a parameter $z \in \mathbb{C} \setminus [a-\delta, b+\delta]$, by replacing $f(\lambda)$ with $\frac{w(\lambda)}{\lambda - z}$, with slightly stronger conditions on $w(\lambda)$ (analytic in a neighborhood of $\mathbb{R}$ and growing no faster than a polynomial as $\lambda \to \infty$). For z near $\mathbb{R}$ and bounded away from $[a, b]$, this is done by deforming the contour of integration so that λ is bounded uniformly away from z and using Riemann-Hilbert analysis of the asymptotics of the associated orthogonal polynomials to show that the contribution from $\rho_N(\lambda)$ on this deformation is uniformly exponentially small.

Appendix B Double-limit convergence argument for $T_{n,N}$

The substance of this appendix is a proof that for

$$T_{n,N}(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{V'(z) - V'(t)}{t - z}\rho_{n,N}(t)dt \quad \text{and} \quad T(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{V'(z) - V'(t)}{t - z}\psi(t)dt,$$

we have

$$\lim_{n,N\to\infty,\frac{n}{N}\to 1} T_{n,N}(z) = T(z), \tag{33}$$

a necessary result in section 3.2. This limit does not follow directly from Theorem 3.1 as do the other limits in the section, since $f(t,z) = \frac{V'(z)-V'(t)}{t-z}$ is a polynomial and therefore unbounded on $\mathbb{R}$. Our strategy will be to decompose $\mathbb{R}$ into a compact interval $[-M,M]$ and its complement for M large enough so that $\operatorname{supp}(\psi) \subset [-M,M]$. Theorem 3.1 applies on this compact set:

$$\lim_{n,N\to\infty,\frac{n}{N}\to 1} \frac{1}{i\pi}\int_{-M}^{M} \frac{V'(z)-V'(t)}{t-z}\rho_{n,N}(t)dt = \frac{1}{i\pi}\int_{-M}^{M} \frac{V'(z)-V'(t)}{t-z}\psi(t)dt = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{V'(z)-V'(t)}{t-z}\psi(t)dt.$$

We will show that the contribution to the integral from the complement goes to 0 in the limit $n,N\to\infty$.

First: since $f(t,z)$ is a polynomial of degree $\nu-2$ in both t and z, for t large enough we have

$$|f(t,z)| \le c_1 |p(z)||t|^{\nu-2}$$

where $p(z) = \frac{f(z,t)}{t^{\nu-2}}$ is bounded in t for each z. Similarly,

$$|\rho_{n,N}(t)| \le c_2 e^{-NV(t)}|t^N|,$$

so that

$$\begin{aligned}
|f(t,z)\rho_{n,N}(t)| &\le c_1c_2e^{-NV(t)}e^{N\log|t|+(\nu-2)\log|t|}\\
&\le c_1c_2e^{-NV(t)}e^{N(\log|t|+(\nu-2)\log|t|)}\\
&= c_1c_2e^{-N(V(t)-(\nu-1)\log|t|)}.
\end{aligned}$$

Since $\deg(V)=2\nu$ and $\frac{-(\nu-1)\log|t|}{c_3t^{2\nu}} \to 0$ as $t\to\infty$, then

$$V(t)-(\nu-1)\log|t| = c_3t^{2\nu}(1+o(1))$$

and so finally by an easy application of the Dominated Convergence Theorem,

$$\lim_{n,N\to\infty,\frac{n}{N}\to 1}\left|\frac{1}{i\pi}\int_{[-M,M]^C}\frac{V'(z)-V'(t)}{t-z}\rho_{n,N}(t)dt\right| \le \lim_{n,N\to\infty,\frac{n}{N}\to 1}\frac{1}{\pi}\int_{[-M,M]^C} c_1c_2e^{-N(c_3t^{2\nu}+o(1))}dt = 0.$$

Since the tail contribution vanishes, the desired result (33) holds. This argument is a standard one in questions such at this one which are related to the law of large numbers (here the number of eigenvalues is growing and their distribution is settling down to the equilibrium measure ψ).

References

[BIZ80] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum Field Theory Techniques in Graphical Enumeration. *Advances in Applied Mathematics*, 1:109–157, 1980.

[DE02] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[Dei00] Percy Deift. *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, volume 3 of *Courant Lecture Notes*. American Mathematical Society, 2000.

[EM03] N. M. Ercolani and K. D. T.-T. McLaughlin. Asymptotics of the Partition Function for Random Matrices via Riemann-Hilbert Techniques and Applications to Graphical Enumeration. *International Mathematics Research Notes*, 14:755–820, 2003.

[EM07] N. M. Ercolani and K. D. T-R McLaughlin. A Quick Derivation of the Loop Equations for Random Matrices. In *Probability, Geometry and Integrable Systems*, volume 55, pages 149–162. MSRI Publications, 2007.

[EMP08] N. M. Ercolani, K. D. T.-T. McLaughlin, and V. U. Pierce. Random Matrices, Graphical Enumeration and the Continuum Limit of Toda Lattices. *Communications in Mathematical Physics*, 278:31–81, 2008.

[Fra06] P. Di Francesco. 2D Quantum Gravity, Matrix Models and Graph Combinatorics. In E. Brezin et al, editor, *Applications of Random Matrices in Physics*, pages 33–88. Springer, 2006.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[Joh98] Kurt Johansson. On Fluctuations of Eigenvalues of Random Hermitian Matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.

[LZ04] Sergei Lando and Alexander Zvonkin. *Graphs on Surfaces and Their Applications*, volume 141 of *Encyclopedia of Mathematical Sciences*. Springer, 2004.

[Sch98] G. Schaeffer. *Conjugaison d'arbreset cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux, 1998.

[Zub12] Jean-Bernard Zuber. Introduction to Random Matrices. 2012.

[Zvo97] A. Zvonkin. Matrix Integrals and Map Enumeration: An Accessible Introduction. *Mathematical and Computer Modelling*, 26(8), 1997.