Trigonometric Substitution

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November 3, 2015

1.1 Trig Identities

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\cos^2(\theta) = \dfrac{1}{2} + \dfrac{1}{2}\cos(2\theta)$
- $\sin^2(\theta) = \dfrac{1}{2} - \dfrac{1}{2}\cos(2\theta)$

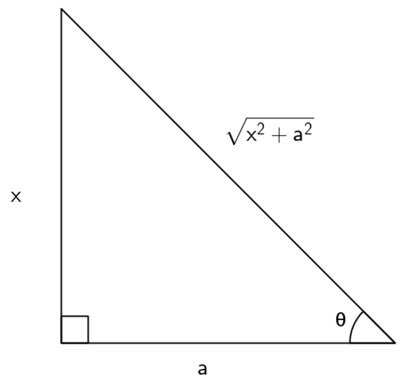
1.2 Trig Integrals

- $\displaystyle\int \sin(\theta)\,d\theta = -\cos(\theta) + C$
- $\displaystyle\int \sec^2(\theta)\,d\theta = \tan(\theta) + C$
- $\displaystyle\int \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta) + C$
- $\displaystyle\int \sec(\theta)\,d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$
- $\displaystyle\int \tan(\theta)\,d\theta = -\ln|\cos(\theta)| + C$
- $\displaystyle\int \cos(\theta)\,d\theta = \sin(\theta) + C$
- $\displaystyle\int \csc^2(\theta)\,dx\theta = -\cot(\theta) + C$
- $\displaystyle\int \csc(\theta)\cot(\theta)\,d\theta = -\csc(\theta) + C$
- $\displaystyle\int \csc(\theta)\,d\theta = \ln|\csc(\theta) - \cot(\theta)| + C$
- $\displaystyle\int \cot(\theta)\,d\theta = \ln|\sin(\theta)| + C$

1.3 Reference Triangles

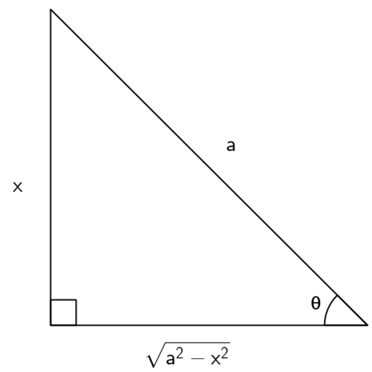
The following triangles are helpful for determining where to place the square root and determine what the trig functions are.

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$



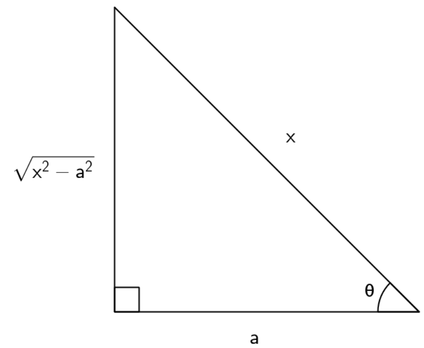
- $a^2 + x^2 = \left(\sqrt{x^2 + a^2}\right)^2$
- x-stuff and number stuff are positive
  $\Rightarrow$ Hypotenuse

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx$$



- $\left(\sqrt{a^2 - x^2}\right)^2 + x^2 = a^2$
- x-stuff is negative, number stuff is positive
  $\Rightarrow$ Adjacent

$$\int \frac{dx}{\sqrt{x^2 - a^2}}$$



- $a^2 + \left(\sqrt{x^2 - a^2}\right)^2 = x^2$
- x-stuff is positive, number stuff is negative
  $\Rightarrow$ Opposite

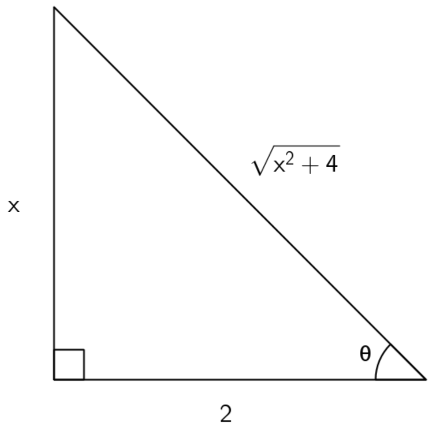
1.3.1 Strategies

- We always have to satisfy the Pythagorean Theorem ($a^2 + b^2 = c^2$) when deciding what each side should equal
- We always start by looking at the inside of the root, specifically at the x-stuff to figure out where to place it. Generally we think of positive as up and negative as down. If the x-stuff is negative, we know immediately that the root goes in the "down" position, which is along the bottom of the triangle. If the x-stuff is positive, the root goes in one of the two "upper" positions. To decide which, we now look at the number stuff. If the number stuff is also positive, we put the root on the hypotenuse. If the number stuff is negative, it goes on the side opposite the angle θ.
- Once we have the triangle, it can be helpful to just write out all of the trig functions that correspond to the triangle. That way we can see all of the available options for solving for what we need.

1.4 Examples

1. Compute $\int \frac{1}{\sqrt{4+x^2}}\,dx$.

   Solution: We notice both the x term and the number are positive, so we are using the first reference triangle with $a=2$. So, using the reference triangle we obtain the following trig substitutions :



$$\sin(\theta) = \frac{x}{\sqrt{4+x^2}} \qquad \csc(\theta) = \frac{\sqrt{4+x^2}}{x}$$
$$\cos(\theta) = \frac{2}{\sqrt{4+x^2}} \qquad \sec(\theta) = \frac{\sqrt{4+x^2}}{2}$$
$$\tan(\theta) = \frac{x}{2} \qquad \cot(\theta) = \frac{2}{x}$$

The only two x-things in our integral are $\sqrt{4+x^2}$ and dx. Thus, we need to replace those. Notice that $\sec(\theta)$ has the root on top, so it is easy to solve for. The reason we want $\sec(\theta)$ instead of $\csc(\theta)$ is because the $\csc(\theta)$ has extra x's in the denominator that we don't need. The constants we can always deal with, but we don't want to introduce more x's than we already have. So, solving for the root, we get $2\sec(\theta) = \sqrt{4+x^2}$.
We also need to get rid of the dx and turn it into $d\theta$ stuff. The best way to do that is to find the substitution that only involves x and a number. In this case, we'll choose $\tan(\theta)$ because again the x is already on top and ready to be solved for. Hence, we get $2\tan(\theta) = x$. Next, to get the dx that we want to get rid of, we take derivatives of both sides. Thus, we get $2\sec^2(\theta)d\theta = dx$.
Now that we have all of our substitutions ready we can plug them all in at once. So, we get

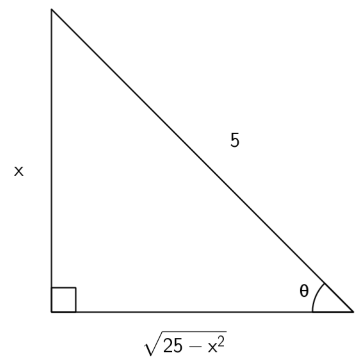
$$\int \frac{1}{\sqrt{4+x^2}}\,dx = \int \frac{1}{2\sec(\theta)} \cdot 2\sec^2(\theta)\,d\theta.$$

From here all we have to do is simplify and integrate using the integrals from section 1.2. Once we have the antiderivative, we just plug back in for all the theta stuff to go back to x's and finish the problem. Therefore,

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{2\sec(\theta)} \cdot 2\sec^2(\theta)\,d\theta \\
&= \int \sec(\theta)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C
\end{aligned}$$

2. Compute $\int \frac{\sqrt{25-x^2}}{x^2}\,dx$

Solution: There are three x terms in this problem. We have the numerator, $\sqrt{25-x^2}$, the denominator, x^2, and the dx. The first thing we need is the reference triangle. Notice that in the root, the number stuff is positive and the x stuff is negative. That means we get the following triangle:



This time we won't list all of the trig substitutions, we'll only list the ones we want as we need them. To start with the root on top, we need a trig sub that has the root on top and number stuff in the denominator. That means we're looking for adjacent over hypotenuse, which is $\cos(\theta)$. So, $\cos(\theta) = \frac{\sqrt{25-x^2}}{5} \Rightarrow 5\cos(\theta) = \sqrt{25-x^2}$.

Now we need the x^2. We aren't given anything with x^2 in the numerator, but we can just solve for x and then square it. So, we want a trig function that has x by itself with just numbers elsewhere. We notice that choosing $\sin(\theta) = \frac{x}{5} \Rightarrow 5\sin(\theta) = x$. This serves two purposes: we can square it to get the x^2 that we want, but also to solve for dx by taking a derivative. So, we get $x^2 = 25\sin^2(\theta)$ and $dx = 5\cos(\theta)d\theta$.

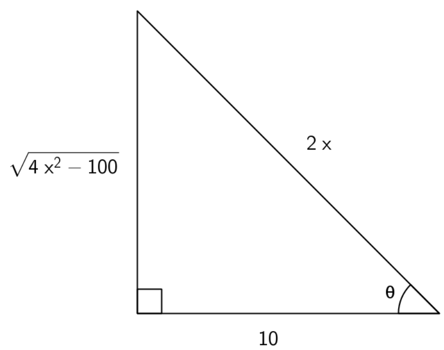
Now that we have everything taken care of, we can plug it all in to get

$$\int \frac{\sqrt{25-x^2}}{x^2}\,dx = \int \frac{5\cos(\theta)}{25\sin^2(\theta)} \cdot 5\cos(\theta)\,d\theta.$$

Simplify, we get it down to $\int \cot^2(\theta)\,d\theta$. We don't have a rule for this integral, so we have to transform it into something we do know how to integrate. In this case we are going to use one of the trig identities in section 1.1 to make it into something familiar. Thus, we need to compute $\int \cot^2(\theta)\,d\theta = \int \csc^2(\theta) - 1\,d\theta$. These we do know how to do, so integrating leaves us with $-\cot(\theta) - \theta + C$. The first term we can easily solve from our triangle and plug back in, but what about θ? Whenever we want θ, its always stuck inside of trig functions. To get it back out, all we need to do is take the inverse. Since $\sin(\theta) = \frac{x}{5}$, we know that $\theta = \sin^{-1}\left(\frac{x}{5}\right)$.

Now we can plug everything back in and get our final answer of $-\frac{\sqrt{25-x^2}}{x} - \sin^{-1}\left(\frac{x}{5}\right) + C$.

3. Compute $\displaystyle\int \frac{1}{\sqrt{4x^2-100}}\,dx$.
Solution: Looking at the root, we see that the x stuff is positive and the number stuff is negative. That means we get the reference triangle



Just like last time, we will solve for the trig subs that we need rather than listing all of them. We notice that there are two pieces to the integral, the root on the bottom and the dx. To find the root, we are looking for a trig sub that has the root on top and number stuff in the bottom. It looks like $\tan(\theta)$ will fit the bill, so we find that $\tan(\theta) = \dfrac{\sqrt{4x^2-100}}{10} \Rightarrow 10\tan(\theta) = \sqrt{4x^2-100}$. We also need the dx, so first we need just x. Again we look for the x on top, and numbers in the denominator. We notice that $\sec(\theta) = \dfrac{2x}{10} \Rightarrow 5\sec(\theta) = x$. Thus, taking a derivative on both sides gives us $5\sec(\theta)\tan(\theta)d\theta = dx$.
Replacing all this is the integral gives us

$$\int \frac{1}{\sqrt{4x^2-100}}\,dx = \int \frac{1}{10\tan(\theta)} \cdot 5\sec(\theta)\tan(\theta)\,d\theta.$$

From here we just simplify, then integrate and plug back in for the trig functions that we need. Therefore,

$$\begin{aligned}
\int \frac{1}{10\tan(\theta)} \cdot 5\sec(\theta)\tan(\theta)\,d\theta &= \frac{1}{2}\int \sec(\theta)\,d\theta \\
&= \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C \\
&= \frac{1}{2}\ln\left|\frac{x}{5} + \frac{\sqrt{4x^2-100}}{10}\right| + C
\end{aligned}$$