RIEMANN SUM EXAMPLE

We want to compute the area under the curve $f(x) = -x^2 + 3$ on the interval $[1, 3]$. The easy way is to compute the integral using the Fundamental Theorem of Calculus $\left(\text{i.e. } \int_1^3 -x^2 + 3\, dx\right)$. But if we want to do it the "proper" way using the sums, we should do the following steps.

Step 1: The General Formula

The general formula for the area under the curve $f(x)$ (for any f) on the interval $[a, b]$ using the right hand Riemann Approximation is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right).$$

This means that we are splitting the interval up into n rectangles. The terms in the sum are the height of the rectangles multiplied by the width respectively. Thus, the sum is adding up the areas of the rectangles which gives us a rough approximation. To make the approximation better and better, we take the limit as n goes to infinity to pack more and more rectangles into the interval. In the limit the approximation becomes exact.

Step 2: Our case

In our case, the interval is $[1, 3]$. So $a = 1$ and $b = 3$. Thus, the height of each rectangle is

$$f\left(a + k\frac{(b-a)}{n}\right) = -\left(1 + k\frac{(3-1)}{n}\right)^2 + 3 = -\left(1 + \frac{2k}{n}\right)^2 + 3$$

Date: April 21, 2015.

and the width is

$$\frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}.$$

Step 3: Compute the Approximation Here we begin by just computing the approximation, and in the next step we will take the limit as $n \to \infty$. So, we plug and chug:

$$\begin{aligned}
\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) &= \sum_{k=1}^{n}\left(-\left(1+\frac{2k}{n}\right)^2 + 3\right)\left(\frac{2}{n}\right) \\
&= \sum_{k=1}^{n}\left(-\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right) + 3\right)\left(\frac{2}{n}\right) \\
&= \sum_{k=1}^{n}\left(-1-\frac{4k}{n}-\frac{4k^2}{n^2} + 3\right)\left(\frac{2}{n}\right) \\
&= \sum_{k=1}^{n}\left(\frac{4}{n}-\frac{8k}{n^2}-\frac{8k^2}{n^3}\right) \\
&= \sum_{k=1}^{n}\frac{4}{n} - \sum_{k=1}^{n}\frac{8k}{n^2} - \sum_{k=1}^{n}\frac{8k^2}{n^3} \\
&= \frac{4}{n}\sum_{k=1}^{n}1 - \frac{8}{n^2}\sum_{k=1}^{n}k - \frac{8}{n^3}\sum_{k=1}^{n}k^2 \\
&= \frac{4}{n}\cdot n - \frac{8}{n^2}\cdot\frac{n(n+1)}{2} - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= 4 - \frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2}
\end{aligned}$$

Step 4: Make the Approximation Exact Now that we have the approximation for the area, we can make it exact by using infinitely many rectangles. So, we just compute the following limit:

$$\begin{aligned}
\lim_{n\to\infty} 4 - \frac{4(n+1)}{n} - \frac{4(n+1)(2n+1)}{3n^2} &= \lim_{n\to\infty} 4 - \frac{4(n+1)}{n}\cdot\frac{1/n}{1/n} - \frac{4(n+1)(2n+1)}{3n^2}\cdot\frac{1/n^2}{1/n^2} \\
&= \lim_{n\to\infty} 4 - \frac{4(1+1/n)}{1} - \frac{4(1+1/n)(2+1/n)}{3} \\
&= 4 - \frac{4(1+0)}{1} - \frac{4(1+0)(2+0)}{3} \\
&= 4 - 4 - \frac{8}{3} \\
&= -\frac{8}{3}
\end{aligned}$$

Step 5:Check Using our massive shortcut (FTC part 2), we can check this easily:

$$\int_1^3 -x^2+3\,dx = -\frac{1}{3}x^3+3x\Big|_1^3 = \left(-\frac{1}{3}(3)^3+3(3)\right)-\left(-\frac{1}{3}(1)^3+3(1)\right) = 0-\left(3-\frac{1}{3}\right) = -\frac{8}{3}.$$

So what did we just compute? We just found how much signed area there is between $-x^2+3$ and the x-axis.