PARTIAL FRACTIONS EXAMPLE

Question: Suppose you want to make a batch of muffins, and the recipe calls for $2\,\frac{1}{6}$ cups of flour. However, being a normal household, you don't have any 1/6-cups. What you do have are 1/2 and 1/3 cups. What combination of the cups you do have will result in the correct amount of flour?

Answer: We want to choose some amount A of 1/2 cups and some amount B of 1/3 cups so that

$$2\,\frac{1}{6} = \frac{13}{6} = \frac{A}{2} + \frac{B}{3}.$$

We can see that choosing $A = 3$ and $B = 2$ will get us the result we want.

Partial fractions is the same way: We turn a rational function into sums of pieces of that function that are easier to deal with. To start with, let's compute two integrals.

- $\displaystyle\int \frac{8}{x+1}\,dx = 8\ln|x+1| + C$
- $\displaystyle\int \frac{5}{x-4}\,dx = 5\ln|x-4| + C$

Now the integral I want to compute is the following $\displaystyle\int \frac{3x-37}{x^2-3x-4}\,dx$. What's the connection? Notice the following:

$$\frac{3x-37}{x^2-3x-4} = \frac{8}{x+1} - \frac{5}{x-4}$$

Ok, what the heck? How am I supposed to know that off the top of my head? Or at least how do we check this? We make a common denominator

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on the right hand side and simplify the top and see if they actually are the same.

Check: $\frac{8}{x+1} \cdot \frac{x-4}{x-4} - \frac{5}{x-4} \cdot \frac{x+1}{x+1} = \frac{8(x-4) - 5(x+1)}{(x+1)(x-4)} = \frac{3x-37}{x^2-3x-4}$

But if we didn't already know that we need 8 pieces of $x+1$ and -5 pieces of $x-4$, how would we get those numbers? Guess and check is not a very good option, so we do the following process.

- First we factor the denominator into the pieces that we want to split the rational function into. So,
$$\frac{3x-37}{x^2-3x-4} = \frac{3x-37}{(x+1)(x-4)}.$$
- Then we want to split this up into the factors in the denominator, but we don't know how much of each we want. So we just make placeholder values A and B. So we set
$$\frac{3x-37}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}.$$
- Now we have a rational function (think a fraction) equal to the sum of two rational functions. As before with the check, we find a common denominator on the right side and see if we can set the two equal to each other. The common denominator is the product of the two denominators on the right, so we get
$$\frac{3x-37}{(x+1)(x-4)} = \frac{A(x-4) + B(x+1)}{(x+1)(x-4)}.$$
- Now we have two rational functions equal to each other. The question becomes: how do you know when two rational functions are

equal to each other? Let's ask an easier question first. How do you know when do fractions are equal to each other? Take the example of $\frac{1}{2} = \frac{2}{4}$. How do you know these are equal as fractions? One way is to cross multiply and see that the two numbers you get are the same. The other is to reduce the right side and get $\frac{1}{2} = \frac{1}{2}$. But how do you know that 1/2 is equal to 1/2? It sounds like a stupid question, but its actually kind of deep. But a stupid question deserves a stupid response. The stupid answer: the tops and bottoms are the same thing. However, even though this sounds like a smartass response, it's still correct and actually solves our problem. We are dealing with rational functions, but they are just fractions with polynomials in the top and bottom rather than just numbers. But the answer still applies. Two rational functions are equal if the numerators are equal *and* the denominators are equal. Notice in our problem so far, we already have that the bottoms are equal. So that means the tops have to be equal to.

- If these two rational functions are going to be equal, then the tops must be equal. So, let's expand out the right hand side to make a polynomial. Then we get

$$3x - 37 = Ax - 4A + Bx + B = (A + B)x - 4A + B.$$

Next we need to know what it means for these two polynomials to be equal to each other. Essentially we have 3 x's on the left and $A + B$ x's on the right. But if these are going to be equal as polynomials, then that should be the same amount. So the rule is: set the coefficients of each power of x equal to each other.

- Doing this gives us a system of equations. When we set each coefficient on the left equal to each corresponding coefficient on the right, we get the two following equations:

$$3 = A + B \qquad \text{(from the } x \text{ terms)} \tag{1}$$

$$-37 = -4A + B \qquad \text{(from the constant terms)} \tag{2}$$

- Now we solve the system of equations. Starting with equation (1), we solve for B and get $B = 3 - A$. Substituting this into (2) we get $-37 = -4A + B = -4A + (3 - A) = 3 - 5A$. So, solving for A, we get $A = 8$. Plugging back in, we get $B = 3 - A = 3 - 8 = -5$.
- Now that we have A and B, we can go back up and plug them in. so we get

$$\frac{3x - 37}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4} = \frac{8}{x+1} + \frac{-5}{x-4}.$$

- Lastly, we plug everything back in back at the beginning to actually compute the antiderivative.

$$\int \frac{3x - 37}{x^2 - 3x - 4}\, dx = \int \frac{8}{x+1} + \frac{-5}{x-4}\, dx = 8 \ln|x+1| - 5 \ln|x-4| + C$$

Method:

(1) Check degree of top and bottom. If top is bigger than the bottom, use polynomial long division
(2) Factor the denominator
(3) Split up as sum of each factor with $A, B, C, \ldots$ on top
(4) Make a common denominator
(5) Set the tops equal to each other and distribute
(6) Gather like terms and set coefficients equal to each other
(7) Solve the system of equations
(8) Plug back in for each $A, B, C, \ldots$
(9) Set integrals equal to each other, and integrate each piece