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1.1 Exponentials

- $x^a = \overbrace{x \cdot x \cdot x \cdot \cdots \cdot x}^{a\text{-times}}$
- $x^a \cdot x^b = x^{a+b}$
- $(x^a)^b = x^{a \cdot b}$
- $x^{-a} = \dfrac{1}{x^a}$
- $\dfrac{x^a}{x^b} = x^{a-b}$
- $x^{\frac{1}{b}} = \sqrt[b]{x}$
- $x^{\frac{a}{b}} = \sqrt[b]{x^a}$
- $x^a \cdot y^a = (xy)^a$

1.2 Logarithms

- $\log_b(x) = y$ if and only if $b^y = x$.
- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^a) = a\log_b(x)$
- $\log_b(b^x) = x = b^{\log_b(x)}$
- $\log_b(b) = 1$
- $\log_b(1) = 0$
- When we do not specify base, it can mean $\log_{10}$ or ln.
- The natural logarithm is log with base e, and is denoted ln. So, $\log_e(x) = \ln(x)$.
- **Change of Base:** When we get a logarithm that we don't like or understand how to do, sometimes its easier to change the base so that we can simplify. To do so, we take the quotient $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

1.3 Useful Facts

- $\ln(1) = 0 \iff e^0 = 1$
- $\ln(e) = 1 \iff e^1 = e$
- $\ln(\frac{1}{e}) = -1 \iff e^{-1} = \frac{1}{e}$

1.4 Examples

1. Calculate 2^x for x ranging from 0 to 10.

   Solution:

$$\begin{aligned} 2^0 &= 1 \\ 2^1 &= 2 \\ 2^2 &= 4 \\ 2^3 &= 8 \\ 2^4 &= 16 \\ 2^5 &= 32 \\ 2^6 &= 64 \\ 2^7 &= 128 \\ 2^8 &= 256 \\ 2^9 &= 512 \\ 2^{10} &= 1024 \end{aligned}$$

   This is a helpful benchmark to memorize: $2^{10} \approx 1000$.

2. Find $\log_3(81)$.

   Solution: Instead of trying to compute this directly (and without a calculator), we turn it into exponentials where just need to guess the power. So, we recall that if $\log_3(81) = y$, then that is the same as saying $3^y = 81$. Now we just need to find what power to raise 3 to in order to get 81. We can list the powers of 3, or just recall that $9 \cdot 9 = 81$, and that $9 = 3^2$. So, $9 \cdot 9 = 3^2 \cdot 3^2 = 3^{2+2} = 3^4 = 81$. So, $\log_3(81) = 4$.

3. Simplify $\ln(x) + \dfrac{\ln(y)}{2}$.

   Solution: Using our log rules, we see that

$$\ln(x) + \frac{\ln(y)}{2} = \ln(x) + \frac{1}{2} \cdot \ln(y) = \ln(x) + \ln(y^{1/2}) = \ln(xy^{1/2}) = \ln(x\sqrt{y}).$$

4. Simplify $\ln(x(x+1)^2)$.

   Solution: First we notice that only the $x+1$ is begin squared, so we can't bring the 2 in front yet. Looking at just the inside of the log, the outer most operation happening is multiplication. So the first thing we can do is use our log rule for the product on the inside. Then we can use the power rule on the second piece. So,

$$\ln(x(x+1)^2) = \ln(x) + \ln((x+1)^2) = \ln(x) + 2\ln(x+1).$$

   Notice that $\ln(x+1) \neq \ln(x) + \ln(1)$. We can only split logs when the inside is a product or quotient.

5. Simplify $((8^{-1})(8^{2/3}))^3$.

   Solution: There are two ways to solve this problem that use the same rules, but in different orders. The first is to distribute the outside 3 across both terms on the inside. So, $((8^{-1})(8^{2/3}))^3 = (8^{-1\cdot 3})(8^{2/3\cdot 3}) = (8^{-3})(8^2)$. Then since we have the same base we can add the exponents to get $(8^{-3})(8^2) = 8^{-3+2} = 8^{-1}$.
   The second method is to add the exponents first, then multiply by the 3. Since we have the same base on the inside, we can start by adding the exponents. So, $((8^{-1})(8^{2/3}))^3 = (8^{-1+2/3})^3 = (8^{-1/3})^3$. Now we can multiply the exponents to get $(8^{-1/3})^3 = 8^{-1/3\cdot 3} = 8^{-1}$.

6. Simplify $5^{2-3\log_5(3)}$.

   Solution: In order to use the inverse properties, we first need to get the 5 raised to the $\log_5$ next to each other. First, we can use exponent rules to turn the difference in the exponent into division. So, $5^{2-3\log_5(3)} = \dfrac{5^2}{5^{3\log_5(3)}}$. We still don't have what we want in the denominator, but our log rules can save us. The 3 in front of the log can be brought up as an exponent to give us $\dfrac{5^2}{5^{3\log_5(3)}} = \dfrac{5^2}{5^{\log_5(3^3)}} = \dfrac{5^2}{5^{\log_5(27)}}$. Now we can simplify like we wanted to finally obtain $\dfrac{5^2}{5^{\log_5(27)}} = \dfrac{5^2}{27} = \dfrac{25}{27}$.