LOGARITHMIC DIFFERENTIATION

As we learn to differentiate all the old families of functions that we knew from algebra, trigonometry and precalculus, we run into two basic rules. The first is for polynomials. When taking the derivative of a polynomial, we use the power rule (both basic and with chain rule):

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(f(x))^n = n((f(x))^{n-1} \cdot f'(x).$$

This rule is used when we run into a function of x being raised to a power than is a constant. Examples of this are

$$\frac{\mathrm{d}}{\mathrm{d}x}4x^3 + 2x + 1 = 12x^2 + 2 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x))^6 = 6(\sin(x))^5 \cdot \cos(x)$$

We also have a rule for exponential functions (both basic and with the chain rule):

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x \cdot \ln(a) \qquad \frac{\mathrm{d}}{\mathrm{d}x}a^{f(x)} = a^{f(x)} \cdot \ln(a) \cdot f'(x).$$

This rule is used when we have a constant being raised to a function of x. Examples of this are

$$\frac{\mathrm{d}}{\mathrm{d}x}\pi^x = \pi^x \cdot \ln(\pi) \qquad \frac{\mathrm{d}}{\mathrm{d}x}6^{\sin(x)} = 6^{\sin(x)}\ln(6) \cdot \cos(x)$$

But of course this doesn't cover all the types of functions that we might see.

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Question: What happens when we encounter a function of the form $y = \sin(x)^{\cos(x)}$ and ask for its derivative?

We can't just use the power rule because that only works when the exponent is a number, not a function. But we also can't use the exponential rule because it requires that the base be a number, not a function. So what can we do? Is there a way to combine the two rules into one?

The answer (unfortunately) is no. There is no "nice" way to combine the rules into one little compact new one. So does that mean there is no way to take the derivative of this function? Not so fast. Just because there is no obvious method doesn't mean we can give up. We do have one type of function that has some very nice properties. Recall that logarithms have the property

$$\ln(x^a) = a\ln(x).$$

The key thing to know about this property is that it works for *any* x and *any* a (as long as $x > 0$). Why shouldn't it work with functions too? Let's see how to use this to solve our derivative problem.

Problem 1: Compute the derivative of $y = \sin(x)^{\cos(x)}$.

Solution: We notice that there are functions of x in both the base and the exponent. That means we can't use our normal rules. The only thing we know that pulls things out of the exponent is a logarithm, so let's take the natural log of both sides of the equation. We get $\ln(y) = \ln\left(\sin(x)^{\cos(x)}\right)$. But now we can use the properties of the log to bring down the exponent to the front to get $\ln(y) = \cos(x) \cdot \ln(\sin(x))$. From here we do know how to take derivatives. On the left side we have to use implicit differentiation (aka chain rule), and on the right we see that we will have a product rule to compute. So, taking the derivative with respect to x on both sides gives us the following:

$$\begin{aligned} & \ln(y) && = \cos(x) \cdot \ln(\sin(x)) \\ \Rightarrow\quad & \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(y)\right] && = \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x) \cdot \ln(\sin(x))\right] \\ \Rightarrow\quad & \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} && = -\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x) \end{aligned}$$

Ok, we've taken a derivative, but let's remind ourselves what we are looking for. We were asked for the derivative of the function y. Notice that on the left side we have $\frac{\mathrm{d}y}{\mathrm{d}x}$. That's exactly what we want! We can treat it just like any other algebraic object and solve for it. So, we just multiply across by y and we've found the derivative. So,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \left(-\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x)\right).$$

But before we can claim victory and go on with our lives, there's a problem. On the left side we have the derivative of a function y, and on the right side there is a y. We usually just want x-stuff on the right side, otherwise we just have a recursive definition and we never actually found the derivative of y. However, there is a way to fix this. We already know what y is because it was given to us at the beginning of the problem. So all we have to do is plug back in for y and we really will be done. Finally we get the derivative of y is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\sin(x)^{\cos(x)}\right) \cdot \left(-\sin(x) \cdot \ln(\sin(x)) + \cos(x) \cdot \frac{1}{\sin x} \cdot \cos(x)\right).$$

Problem 2: Find the derivative of $y = (x^5 + e^x - 2x^2)^{x^3-16x}$.

Solution:

$$\begin{aligned}
y &= (x^5 + e^x - 2x^2)^{x^3-16x} \\
\Rightarrow \ln(y) &= \ln\left((x^5 + e^x - 2x^2)^{x^3-16x}\right) \\
&= (x^3 - 16x) \cdot \ln(x^5 + e^x - 2x^2) \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(x^3 - 16x) \cdot \ln(x^5 + e^x - 2x^2)\right] \\
\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= (3x^2 - 16) \cdot \ln(x^5 + e^x - 2x^2) + (x^3 - 16x) \cdot \frac{1}{x^5 + e^x - 2x^2} \cdot (5x^4 + e^x - 4x) \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= y \cdot \left((3x^2 - 16) \cdot \ln(x^5 + e^x - 2x^2) + (x^3 - 16x) \cdot \frac{1}{x^5 + e^x - 2x^2} \cdot (5x^4 + e^x - 4x)\right)
\end{aligned}$$

Finally, replacing y in the last equation we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x^5+e^x-2x^2)^{x^3-16x} \cdot \left((3x^2 - 16) \cdot \ln(x^5 + e^x - 2x^2) + (x^3 - 16x) \cdot \frac{5x^4 + e^x - 4x}{x^5 + e^x - 2x^2}\right).$$

Method:

(1) Check that you have a function of the form $y = f(x)^{g(x)}$

(2) Take the natural log of both sides of the equation

(3) Use log properties to bring down $g(x)$

(4) Take a derivative on both sides. The left will always result in $\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$ and the right side will always be a product rule.

(5) Multiply across by y to solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

(6) Substitute back in the original equation $f(x)^{g(x)}$ for y