

## 1.1 Basic Integrals

Let  $C$  be an arbitrary constant in the following

- $\int a \, dx = ax + C$
- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$
- $\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C$
- $\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C$
- $\int \sec(ax) \tan(ax) \, dx = \frac{1}{a} \sec(ax) + C$
- $\int \csc(ax) \cot(ax) \, dx = -\frac{1}{a} \csc(ax) + C$
- $\int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax) + C$
- $\int \csc^2(ax) \, dx = -\frac{1}{a} \cot(ax) + C$
- $\int \tan(ax) \, dx = -\frac{1}{a} \ln |\cos(ax)| + C$
- $\int \cot(ax) \, dx = \frac{1}{a} \ln |\sin(ax)| + C$
- $\int \sec(ax) \, dx = \frac{1}{a} \ln |\sec(ax) + \tan(ax)| + C$
- $\int \csc ax \, dx = -\frac{1}{a} \ln |\csc(ax) + \cot(ax)| + C$
- $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$
- $\int \frac{1}{x} \, dx = \ln |x| + C$
- $\int a^x \, dx = \frac{a^x}{\ln a} + C \quad a \neq 1, a > 0$
- $\int \frac{1}{1+x^2} \, dx = \arctan(x) + C$

## 1.2 The Fundamental Theorem of Calculus

- *Part 1:* If  $f$  is continuous on  $[a, b]$  then  $F(x) = \int_a^x f(t) \, dt$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and its derivative is  $f(x)$ . So,

$$F'(x) = \frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

- *Part 2:* If  $f$  is continuous at every point in  $[a, b]$ , and  $F$  is an antiderivative of  $f$  on  $[a, b]$ , then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

## 1.3 Techniques of Integration

- **U-Substitution:** A U-sub is used when you see that we can't integrate, but a substitution of a function of  $x$  can be used to turn it into an integrable function. As an example, consider the integral  $\int \sqrt{2x+1} \, dx$ . Notice that we only know how to easily integrate  $\sqrt{x}$ . However, if we make a substitution,  $u(x) = 2x+1$ , then we can transform the integral into something that is easy to integrate. However, we need to make sure that we integrate with respect to

the right variable. So far we have  $\int \sqrt{u} dx$ , but we need to change the  $dx$  into a  $du$ . So we take our substitution,  $u = 2x + 1$ , and take the derivative. Thus  $du = 2 dx$ . But we don't have a  $2 dx$  in the integral, just a  $dx$ . So we divide by 2, and obtain  $\frac{du}{2} = dx$ . Now we can make the substitution complete by replacing the  $dx$ , and we get

$$\int \sqrt{2x+1} dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{2} \frac{(u)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3} u^{3/2} + C.$$

Now we replace  $u$  with  $2x + 1$ , and we get our final answer,  $\int \sqrt{2x+1} dx = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C$ .

- **Integration by Parts:**  $\int u dv = uv - \int v du$ . When a U-sub won't work, we might have to use integration by parts. To see if we can use integration by parts, we look for a function that we know how to integrate, and a function that we know how to derive that are being multiplied together. For an example, consider  $\int x e^x dx$ . Notice that each function is simple; we know how to derive and anti-derive each one. We choose our functions as follows

$$\int \underbrace{x}_u \underbrace{e^x dx}_{dv}.$$

Then we have

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= \int e^x dx = e^x \end{aligned}$$

Plugging these into our formula, we have

$$x e^x - \int e^x dx = x e^x - e^x + C = e^x(x - 1) + C$$

\* A technique called *Tabular Integration* is a fast version of integration by parts, but because it lacks sufficient notation, it can easily be done wrong. This worksheet will not cover tabular integration.

- **Trigonometric Substitution:** This technique will not be covered in this worksheet.

## 1.4 Facts About Integrals

- A definite integral  $\int_a^b f(x) dx$  is a *number*. An indefinite integral  $\int f(x) dx$  is a *function* plus and arbitrary constant.
- The integral of a function measures the area under the curve.
- We need to take care of which variable with which we are integrating. Notice that  $\int 5y^2 dx = 5y^2x + C$ , since we are taking the integral with respect to  $x$ . Since no  $x$ 's show up in the equation,  $5y^2$  is treated as a constant. Similarly,  $\int e^x + z dz = e^x z + \frac{z^2}{2} + C$ .
- The substitution  $u = x$  is completely useless. All it will do is change all  $x$ 's to  $u$ 's and  $dx$  to  $du$ . However it doesn't simplify anything, just changes the names of the variables.

## 1.5 Examples

1. Compute  $\int e^{3x} dx$  using a U-substitution.

*Solution:* This problem would be easier if that 3 wasn't there and it was just  $e^x$ . So the 3 is kind of annoying. When we see things that could be easier but are a bit annoying you should think of a u-sub. We usually want to sub out the annoying part. So, let's start with

$$\begin{aligned}u &= 3x \\ du &= 3dx\end{aligned}$$

But we don't have a  $3dx$  sitting in the integral, we only have a  $dx$ . So let's solve the second equation for  $dx$ . Dividing both sides by 3, we get  $\frac{1}{3}du = dx$ . Recall that equals signs mean replacement. Wherever we see a  $3x$  we can replace it with  $u$ , and we can replace the  $dx$  with  $\frac{1}{3}du$ . Doing so, we get an integral we do know how to calculate. So,

$$\int e^{3x} dx = \int e^u \left(\frac{1}{3}du\right) = \frac{1}{3} \int e^u du = \frac{1}{3}e^u + C.$$

But this isn't a very nice answer. Its like asking a friend for change for a \$10 bill so you can buy a candy bar from a vending machine, but your friend gives you two \$5 bills instead of any single dollars. The 5's still don't do you much good for a \$1 candy bar. Here we asked a question about an anti derivative of  $e^{3x}$ , a function of  $x$ . But the answer we've given is  $e^u$ , a function of  $u$ . It still doesn't solve the problem we had originally. Fortunately we know exactly what  $u$  is equal to in  $x$ -terms. All we have to do is re-substitute  $3x$  for all the  $u$ 's and we'll be finished. In the end we get  $\int e^{3x} dx = \frac{1}{3}e^{3x} + C$ .

2. Compute  $\int 2x \cos(x) dx$ .

*Solution:* We notice that this is a product of two functions, and that we know how to derive and antiderive each one. This should make us think of integration by parts. So, let's choose the  $u$  function to be one that gets easier when we take its derivative and the  $dv$  function be one that at least doesn't get any harder when we take its antiderivative. Using the formula, we get the following:

$$\int 2x \cos(x) dx = 2x \sin(x) - \int 2 \sin(x) dx = 2x \sin(x) + 2 \cos(x) + C$$

$$\begin{aligned}u &= 2x & dv &= \cos(x) dx \\ du &= 2dx & v &= \sin(x)\end{aligned}$$

3. Compute  $\int \ln(x) dx$ .

*Solution:* This one is a little weird because it doesn't look like it can be a u-sub or integration by parts. We can use one of the oldest (and stupidest) tricks in the book and write  $\ln(x)$  as

$\ln(x) \cdot 1$ . By doing this, we can see that we now have a product of functions, so we should think about integration by parts. This time, however, it matters which function we choose as  $u$  and  $dv$ . If we choose  $dv = \ln(x)$ , then to get to  $v$  we have to take an antiderivative. This means we'd need to find  $\int \ln(x) dx$ . But that's exactly the problem we are trying to solve! So let's not do that since we don't know the answer to that problem. But we do know how to take derivatives of  $\ln(x)$ , so we'll choose that as our  $u$ . Doing so we get the following:

$$\int \ln(x) dx = x \ln(x) - \int x \cdot \frac{1}{x} dx = x \ln(x) - \int 1 dx = x \ln(x) - x + C.$$

$$\begin{aligned} u &= \ln(x) & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

4. Compute the following integral:  $\int x^3 e^{x^2} dx$

*Solution:* We see a product of functions, so the first thought is to use integration by parts. But that would mean taking derivatives or antiderivatives of  $e^{x^2}$  depending on if we choose it for  $u$  or  $dv$ . Derivatives get crazy because we'll have to use chain rule, and antiderivatives we don't know how to do (in fact there is **no** antiderivative for this function!). So there's something weird here. One thing we notice is that the  $x^2$  in the exponent is kind of annoying; if it was just an  $x$  we could do this. So the first thing we can do is a u-sub. We'll use  $w$  instead of  $u$  (the reason why will be revealed soon). By letting  $w = x^2$ , we get  $dw = 2x dx$ . Now hidden inside that  $x^3$  is an  $x$ . Technically we know that  $x^3 = x \cdot x^2$ . We don't have the  $2x dx$  we want, but there is the  $x dx$ . So,  $\frac{1}{2} dw = x dx$ . Subbing all this in, we get

$$\int x^3 e^{x^2} dx = \int x \cdot x^2 e^{x^2} dx = \int \frac{1}{2} x^2 e^w dw.$$

But this is almost worse because now we have  $x$ 's and  $w$ 's mixed in the same integral! However, one small observation makes this doable again. Notice that we chose  $w = x^2$ , which means *wherever* we see an  $x^2$  we can replace it with  $w$ . So we get  $\frac{1}{2} \int w e^w dw$ . This is a classic integration by parts problem now. Here we will use  $u = w$  and  $dv = e^w dw$  (this is why we used  $w$ , it'd be confusing if we had two different  $u$ 's). So,

$$\frac{1}{2} \int w e^w dw = \frac{1}{2} \left[ w e^w - \int e^w dw \right] = \frac{1}{2} [w e^w - e^w] + C = \frac{1}{2} [x^2 e^{x^2} - e^{x^2}] + C.$$

$$\begin{aligned} u &= w & dv &= e^w dw \\ du &= dw & v &= e^w \end{aligned}$$

The last step comes from re-substituting  $x^2$  in for  $w$ . This problem illustrates that we can combine more than one technique of integration into one problem.