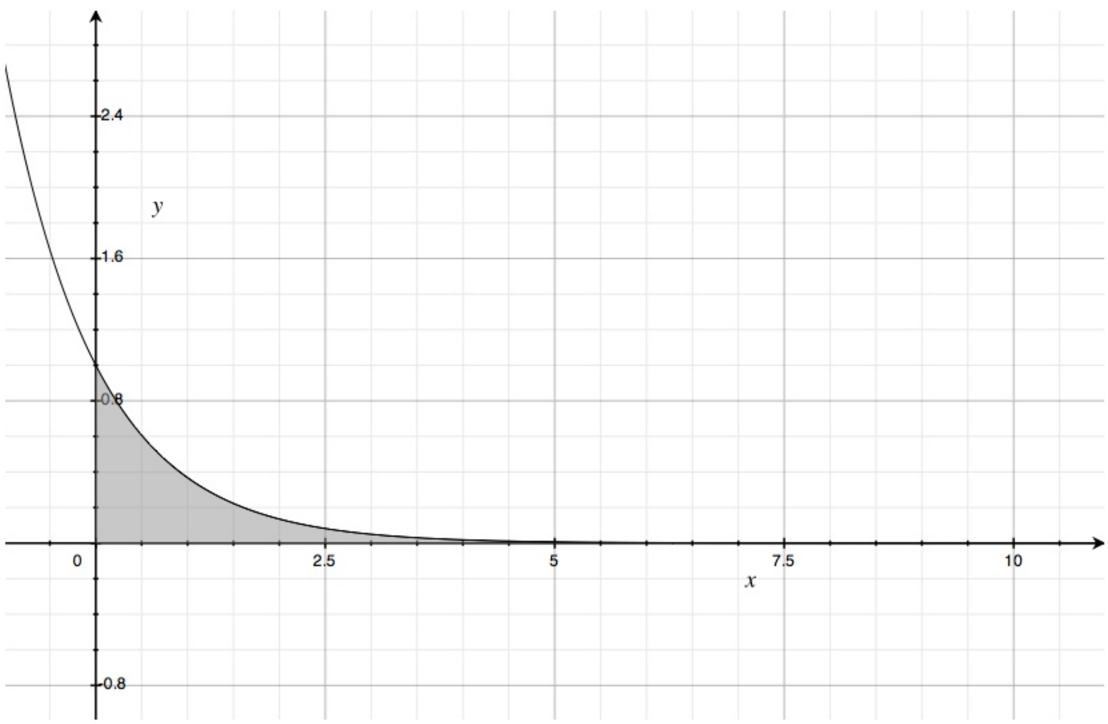
IMPROPER INTEGRATION AT INFINITY

Question: Compute the area under the curve $f(x) = e^{-x}$ that is above the x-axis and to the right of the y-axis.

Answer: The first thing we can do to try to answer this question is to graph the function and figure out what the region looks like. The graph of e^{-x} is shown below, with the region that we are trying to find the area of filled in with grey:



The first thing we notice is that there is no right hand bound, which means the area that is contained under the curve goes off infinitely far in the positive direction. That's a little weird...

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Now before we even try computing the area, let's talk about something a little bit easier to recognize: the perimeter. What is the perimeter of this object? Well the left hand side is a straight line of length 1, and the bottom is a straight line (the x-axis) that is infinitely long. We also have the sloping top which is also infinitely long. That means is looks like the perimeter is $P \approx 1 + \infty + \infty$, which is definitely infinity. Why should the area of an object that has infinite perimeter have anything but infinite area? That wouldn't make much sense, so we'll go with the guess that this problem is just asking for a long way of computing infinity.

Now that we understand a little about this region, let's write down how to describe it in calculus terms. We want the area under a curve, which means a definite integral. We also know that we start at the y-axis, which means $x = 0$ and stop at ... well, we don't stop. Which means we go all the way out to infinity. Putting that stuff together, we think that we should compute

$$\int_0^\infty e^{-x}\, dx.$$

The first question I need to ask myself is, "Why isn't this a normal calculus problem? What is stopping me from just doing things like normal?" The answer is the ∞ that is sitting there as a limit of integration. We've never dealt with that before. If you recall, the Fundamental Theorem of Calculus states that the curve that forms the top of your area must be continuous on the closed interval $[a, b]$. We know that e^{-x} is continuous, but the interval we are integrating over is $[0, \infty)$, which is not of the form that the FTC requires. This is why we can't just compute like normal.

In order to be able to use the FTC, we need a way to talk about infinity without actually talking about infinity. We want to talk bout the behavior of the function rather than actually plugging in infinity because infinity is not a number and it doesn't act like one either. Our solution is going to be the following: Why don't we just integrate from 0 to, say, 100, and then 1,000 and then 10,000 . . . and see what happens? But I don't want to calculate five or ten or fifty integrals to see if the value gets close to something, so we'll do it in one step by using a limit. This is exactly the kind of case that limits are useful for. Limits tell us about the behavior of a function or object as you get close to a value rather than the specific value itself. So, we will integrate from 0 to b, and then take the limit as b tends to ∞ and see what happens.

The reason we can now use the fundamental theorem is that the interval $[0, b]$ is closed and of the form that we want. Then after taking the integral we will take the limit. Since we will have already taken the integral, there is no problem taking the limit since we've already applied the FTC. Thus, computing the integral gives us

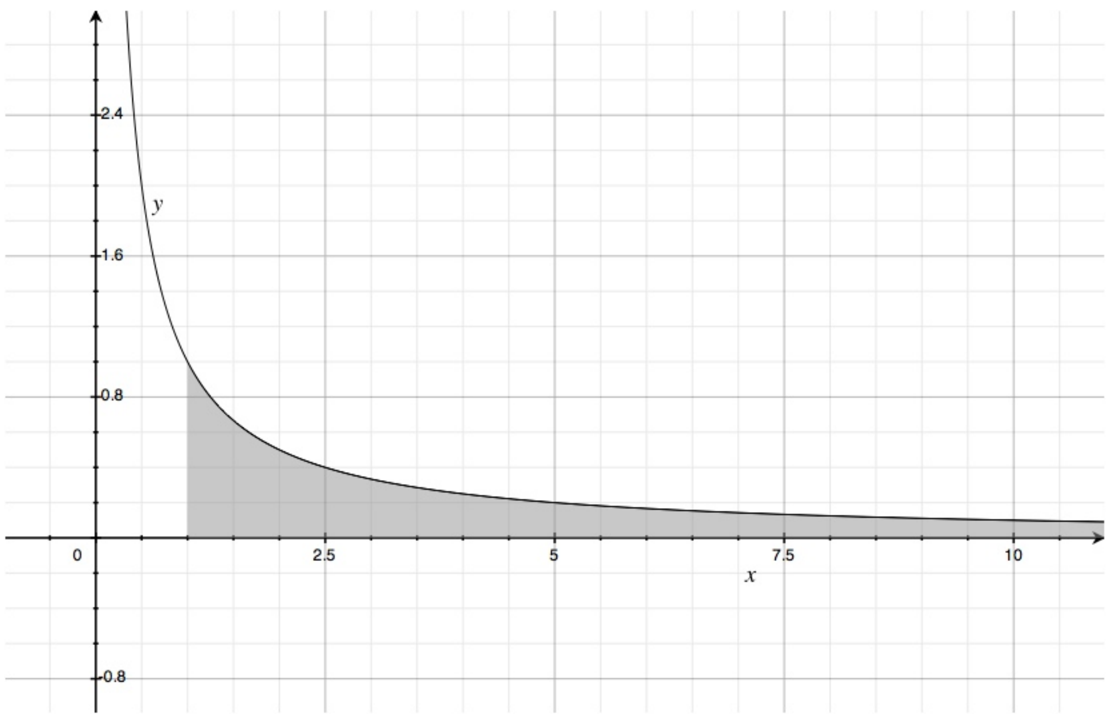
$$\begin{aligned}
\int_0^\infty e^{-x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-x}\,dx \\
&= \lim_{b\to\infty} -e^{-x}\Big|_0^b \\
&= \lim_{b\to\infty}\left(-e^{-b}\right) - \left(-e^{-0}\right) \\
&= \lim_{b\to\infty}\frac{-1}{e^b} - \frac{-1}{e^0} \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

Wait . . . what just happened here? We set it up correctly, and found a way to compute it, but the answer wasn't infinity. In fact, it wasn't just finite, but it was a *small* finite number. Apparently the area under e^{-x} to the right of the y-axis is just 1 unit^2. That's pretty crazy! So if this region was my plot of land, and I wanted to fence it in I would never be able to because the perimeter is infinite. But if I wanted to plant it with grass, I'd only need enough to cover 1 square acre.

Let's try another example.

Question: Compute the area under the graph of $f(x) = \dfrac{1}{x}$ to the right of $x = 1$.

Answer: First we can graph the function to see what the region looks like:



It looks very similar to the last problem that we did. So maybe this one will have the same kind of properties. To start, we notice that this region also has a left side length of 1, a bottom side length of infinity and a top sloping side that is infinitely long. So again the perimeter is $P \approx 1 + \infty + \infty = \infty$. To compute the actual area, we set it up just like last time and use the same method as before. So, computing the area we get

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b \\ &= \lim_{b\to\infty} \ln|b| - \ln|1| \\ &= \infty\end{aligned}$$

So this time the area is infinite. That means this one diverges, which is what our intuition about infinite perimeter objects tells us should happen.

What makes these regions act differently? These regions look similar and have almost the same properties. We have a side length of 1, a flat infinitely long bottom, a sloping infinitely long top, and the method of integration was the same. Why is one of them infinite and the other a small number?

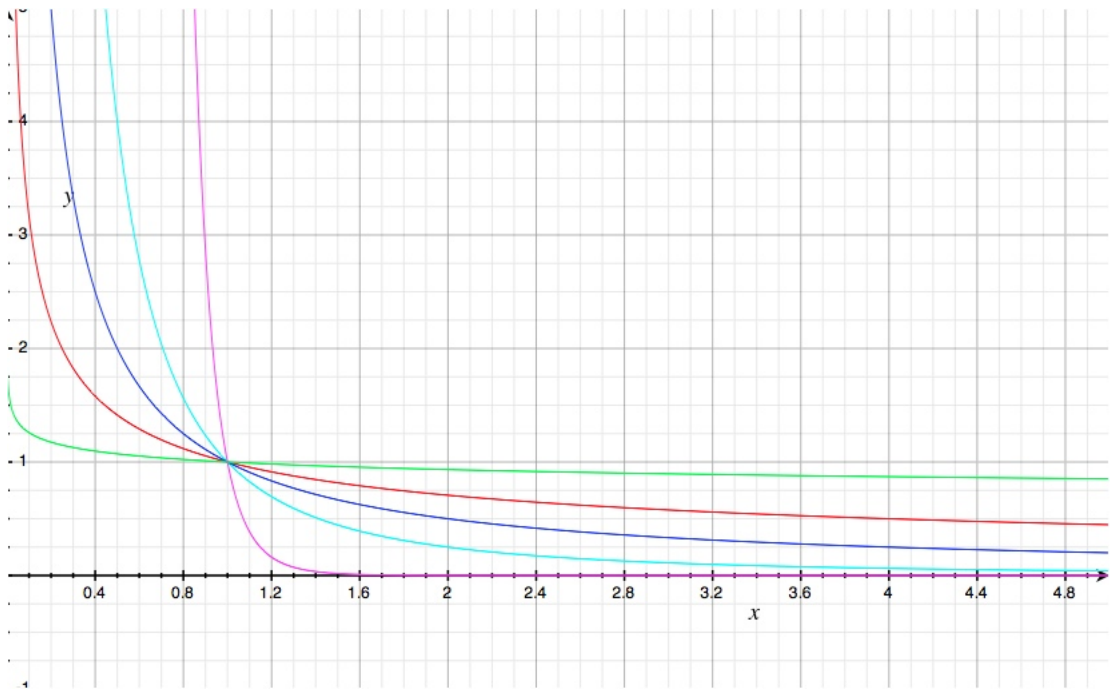
The answer comes down to speed. Both these functions approach zero, but one of them approaches very quickly and the other takes its time. We can see that e^{-x} gets close to zero very, very quickly, where as $\frac{1}{x}$ doesn't move so fast. It turns out that the faster you get close to zero, the better chance you will have of converging to a number.

In fact, we can determine exactly how fast a certain family of functions needs to be in order to converge.

Problem: Determine for which values of p the following integral will converge or diverge.

$$\int_1^\infty \frac{1}{x^p}\,dx$$

Solution: Here p is some real number that is greater than zero. These graphs have the following general shape:



As we can see each one approaches zero as x goes off to infinity, but some do so faster than others. We want to determine just how fast they converge to zero and when that speed is enough for the area underneath to converge as well. The difference with this problem is that we are going to leave p alone and try to solve for specific cases of it. We start like any other improper integral by using a limit,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \frac{1}{-p+1}x^{-p+1}\bigg|_1^b \qquad (\star)$$

Here we have a small problem. What we have is x raised to a power, but we don't know exactly what the power is. The power depends on the p value. Recall that if x has a negative power then you can drop it to the bottom, and if x has a positive power then it stays in the numerator. Since b is going off to infinity this completely determines if the integral will converge (because $1/\infty \approx 0$ and $\infty^n \approx \infty$). So we have the following cases.

Case 1: If the exponent of the x is positive, then $-p+1>0$. Thus, $p<1$. Then

$$\begin{aligned}
(\star) &= \lim_{b\to\infty} \frac{1}{-p+1} x^{-p+1}\Big|_1^b \\
&= \lim_{b\to\infty} \frac{1}{-p+1} b^{-p+1} - \frac{1}{-p+1} 1^{-p+1} \\
&= \lim_{b\to\infty} \frac{1}{-p+1} b^{-p+1} - \frac{1}{-p+1} \\
&= \infty - \frac{1}{-p+1} \\
&= \infty
\end{aligned}$$

Hence the integral would diverge if $p<1$.

Case 2: If the exponent of the x is negative, then $-p+1<0$. Thus, $p>1$. Then

$$\begin{aligned}
(\star) &= \lim_{b\to\infty} \frac{1}{-p+1} \frac{1}{x^{p-1}}\Big|_1^b \\
&= \lim_{b\to\infty} \frac{1}{-p+1} \frac{1}{b^{p-1}} - \frac{1}{-p+1} \frac{1}{1^{p-1}} \\
&= \lim_{b\to\infty} \frac{1}{-p+1} \frac{1}{b^{p-1}} - \frac{1}{-p+1} \\
&= 0 - \frac{1}{-p+1} \\
&= \frac{1}{p-1}
\end{aligned}$$

Hence the integral converges to $\frac{1}{p-1}$.

Case 3: Notice that by doing the anti-power rule in the first step of computing the integral, we divide by $-p+1$. We can only do this if $p \neq 1$ because otherwise we'd be dividing by zero (which creates black holes and breaks math). So we save one extra case when $p=1$. But if $p=1$, we can plug that in at the beginning and easily deduce the following:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln(b) - \ln(1) = \infty.$$

Therefore, we have classified all the positive p values that will make the integral converge or diverge. To write it succinctly,

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

Method:

(1) Rewrite as a limit with a variable replacing the infinity
(2) Use the FTC to compute the antiderivative like normal
(3) Take the limit
(4) If the answer comes out as $\pm\infty$ then it diverges. If the answer is a finite number then it converges