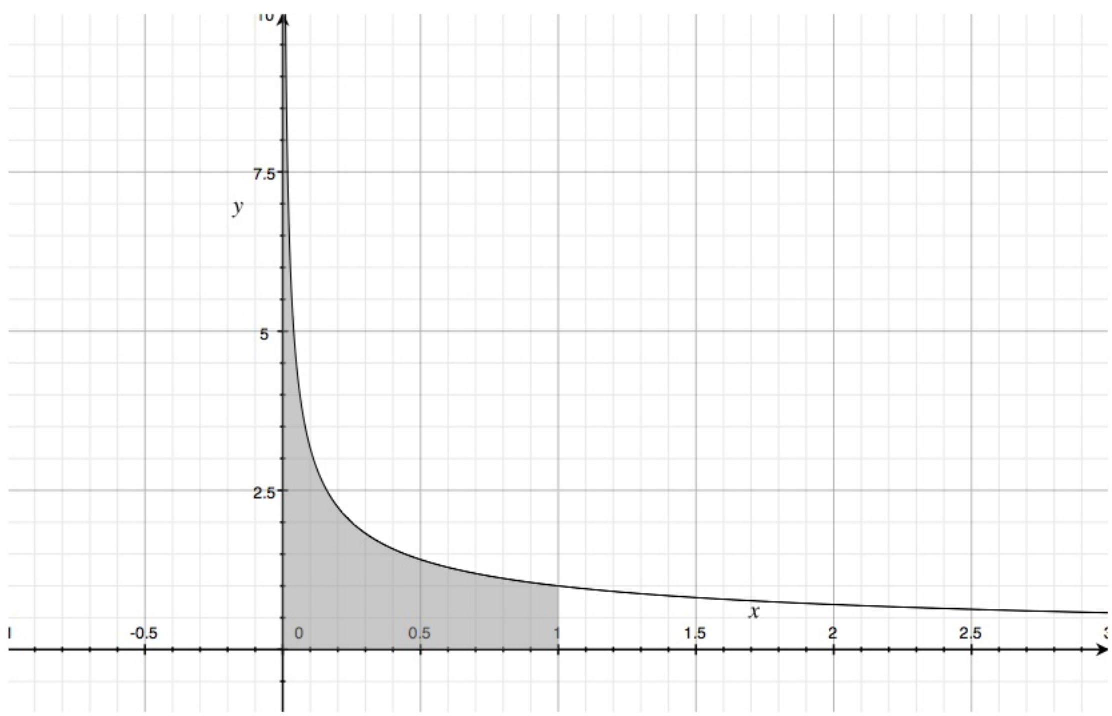
IMPROPER INTEGRATION AT DISCONTINUITIES

In the last worksheet we looked at the area under a curve on an unbounded region. Counter to how our intuition worked, sometimes the area inside a shape of infinite perimeter did *not* have infinite area. In this worksheet we will work with the same techniques, but instead of going off to infinity in an x direction, we will see what happens if we go off to infinity in the y direction.

Question: Compute the area under the curve $f(x) = \dfrac{1}{\sqrt{x}}$ on $[0, 1]$.

Answer: As usual, the first thing to do is to graph the function and get a handle on what the region looks like. The graph of $\dfrac{1}{\sqrt{x}}$ is shown below, with the region that we are trying to calculate shaded in grey:



Date: January 1, 2016 By Tynan Lazarus.

The first thing we notice is that this function escapes off to infinity near zero. This behavior comes from the fact that $\frac{1}{\sqrt{x}}$ has an asymptote as $x \to 0$. In particular, the function itself is discontinuous at $x = 0$. These kinds of discontinuities arise when you try to break math and divide by zero.

Now before we even try computing the area, let's talk about something a little bit easier to recognize: the perimeter. What is the perimeter of this object? Well the left bottom is a straight line of length 1, the right side is a vertical line of length 1 as well, but then we have the left side (the y-axis) which is infinitely long, and the sloping top that is also infinitely long. We also have the sloping top which is also infinitely long. That means is looks like the perimeter is $P \approx 1 + 1 + \infty + \infty = \infty$. Why should the area of an object that has infinite perimeter have anything but infinite area? (Sound familiar?) As with last time, we can write down in calculus terms what the area should be

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

The first question I need to ask myself is, "Why isn't this a normal calculus problem? What is stopping me from just doing things like normal?" Again the answer has to do with the Fundamental Theorem of Calculus. If you recall, the Fundamental Theorem of Calculus states that the curve that forms the top of your area must be continuous on the closed interval $[a, b]$. We are operating on $[0, 1]$, so that's not a problem. There is a theorem that says a continuous function on a closed set is bounded, so the FTC requires a bounded function. Ours is definitely not bounded because it grows infinitely tall. The problem is that this function is not continuous on the *entire* interval. There's a problem at the left end point. This is why we can't just compute like normal.

In order to be able to use the FTC, we need a way to talk about infinity without actually talking about infinity. We want to talk bout the behavior of the function rather than actually plugging in zero because we aren't allowed to divide by zero. Again we'll use our limit trick. Limits tell us about the behavior of a function or object as you get close to a value rather than the specific value itself. So, we will integrate from a to 1, and then take the limit as a tends to 0 and see what happens.

The reason we can now use the fundamental theorem is that the interval $[a, 1]$ is closed and $\frac{1}{\sqrt{x}}$ is continuous on this interval (since $a \neq 0$). Then after taking the integral we will take the limit. Since we will have already taken the integral, there is no problem taking the limit since we've already applied the FTC. Thus, computing the integral gives us

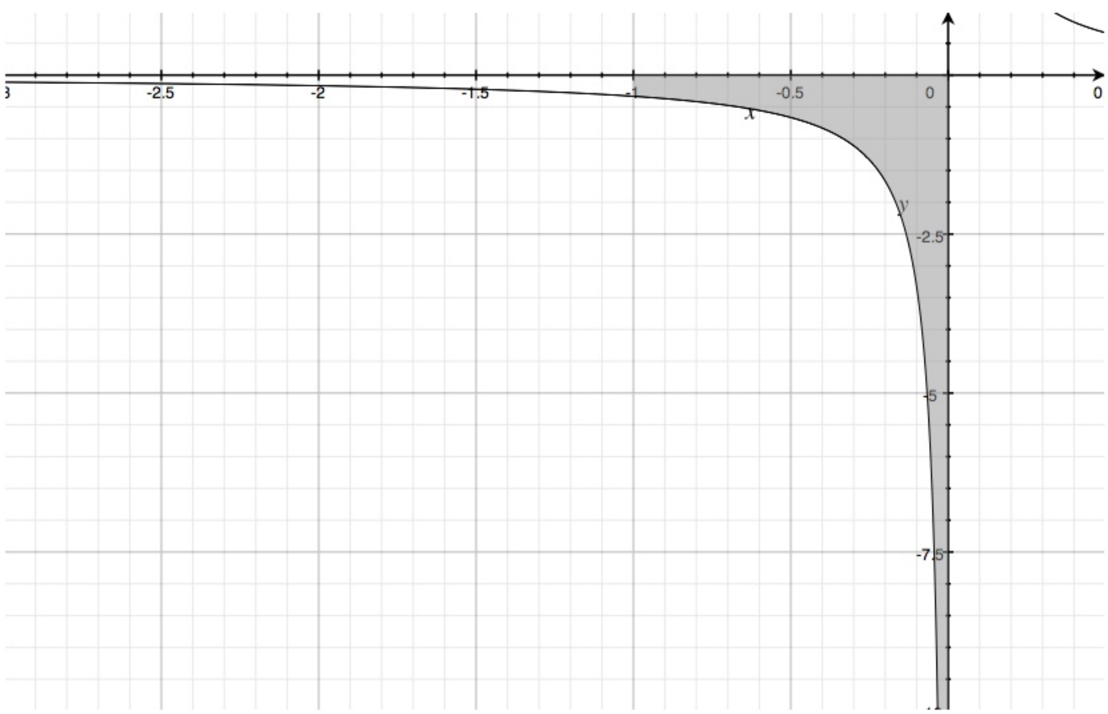
$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0}\int_a^1 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{a\to 0}\int_a^1 x^{-1/2}\,dx \\
&= \lim_{a\to 0} 2x^{1/2}\Big|_a^1 \\
&= \lim_{a\to 0} 2\sqrt{1} - 2\sqrt{a} \\
&= 2
\end{aligned}$$

Even though the region has infinite perimeter, it has a finite amount of area. So again our intuition was wrong.

Let's try another example.

Question: Compute the area under the graph of $f(x) = \dfrac{1}{x^3}$ between $x = -1$ and $x = 0$.

Answer: First we can graph the function to see what the region looks like:



It looks very similar to the last problem that we did, just turned upside down. So maybe this one will have the same kind of properties. To start, we notice that this region also has a top side length of 1, a left side length of infinity, a right side length of infinity, and a bottom sloping side that is infinitely long. So again the perimeter is $P \approx 1 + 1 + \infty + \infty = \infty$. To compute the actual area, we set it up just like last time and use the same method as before. So, computing the area we get

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{x^3}\,dx &= \lim_{b\to 0} \int_{-1}^{b} \frac{1}{x^3}\,dx \\
&= \lim_{b\to 0} \frac{-1}{2x^2}\bigg|_{-1}^{b} \\
&= \lim_{b\to 0} \frac{-1}{2b^2} - \frac{-1}{2(-1)^2} \\
&= \lim_{b\to 0} \frac{-1}{2b^2} + \frac{1}{2} \\
&= -\infty
\end{aligned}$$

So this time the area is infinite (note that the negative sign just tells us that the area is under the x-axis, and the amount of it is infinite). That means this one diverges, which is what our intuition about infinite perimeter objects tells us should happen.

What makes these regions act differently? These regions look similar and have almost the same properties. We have a side length of 1, a flat, infinitely long side, a sloping infinitely long piece, and the method of integration was the same. Why is one of them infinite and the other a small number?

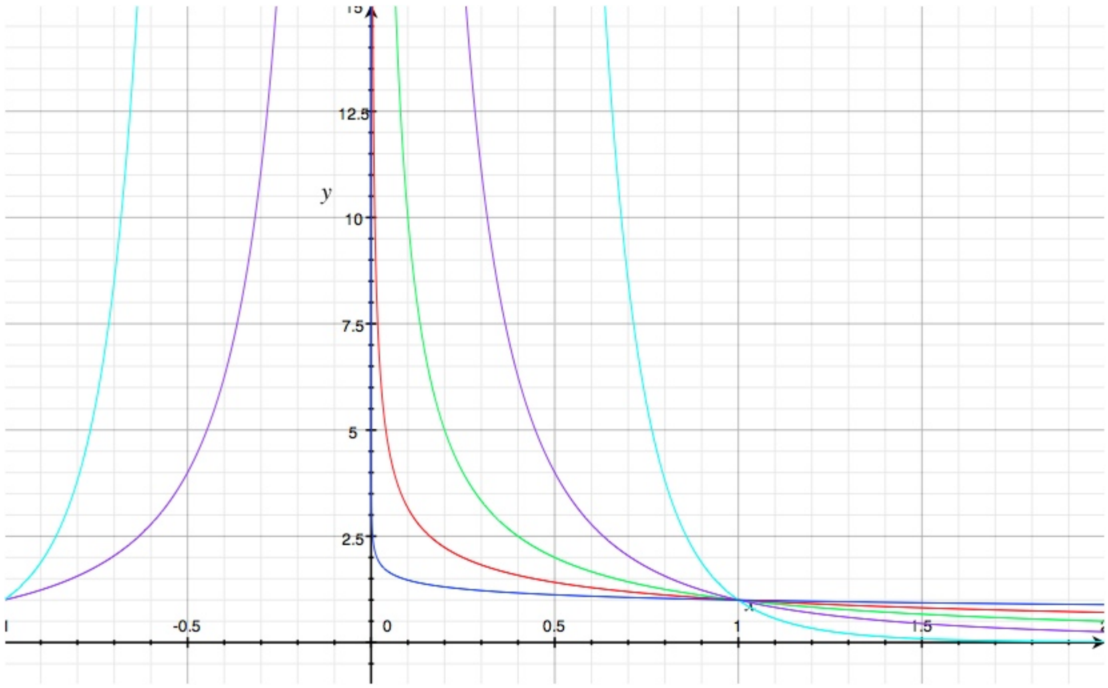
The answer comes down (again) to speed. Both these functions approach their respective asymptotes, but one of them approaches very quickly and the other is slower. It turns out that the faster you get close to the asymptote, the better chance you will have of converging to a number.

In fact, we can determine exactly how fast a certain family of functions needs to be in order to converge.

Problem: Determine for which values of p the following integral will converge or diverge.

$$\int_{0}^{1} \frac{1}{x^p}\,dx$$

Solution: Here p is some real number that is greater than zero. These graphs have the following general shape:



We can see that each graph approaches infinity as x goes gets close to zero, but some do so faster than others. We want to determine just how fast they converge to the asymptote and when that speed is enough for the area underneath to converge as well. We start like any other improper integral by using a limit,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \frac{1}{-p+1}x^{-p+1}\bigg|_a^1 \qquad (\star)$$

We run into the same cases as last time. What we have is x raised to a power, but we don't know exactly what the power is (because it depends on p and we haven't chosen any specific p values yet). Recall that if x has a negative power then you can drop it to the bottom, and if x has a positive power then it stays in the numerator. Since a is going to zero, this will completely determine if the integral will converge. So we have the following cases.

Case 1: If the exponent of the x is positive, then $-p+1>0$. Thus, $p<1$. Then

$$\begin{aligned}
(\star) &= \lim_{a\to 0^+} \frac{1}{-p+1} x^{-p+1}\Big|_a^1 \\
&= \lim_{a\to 0^+} \frac{1}{-p+1}(1)^{-p+1} - \frac{1}{-p+1}(a)^{-p+1} \\
&= \lim_{a\to 0^+} \frac{1}{-p+1} - \frac{a^{-p+1}}{-p+1} \\
&= \frac{1}{-p+1}
\end{aligned}$$

Thus, the integral converges to $\frac{1}{1-p}$.

Case 2: If the exponent of the x is negative, then $-p+1<0$. Thus, $p>1$. Then

$$\begin{aligned}
(\star) &= \lim_{a\to 0^+} \frac{1}{-p+1} x^{-p+1}\Big|_a^1 \\
&= \lim_{a\to 0^+} \frac{1}{-p+1}\frac{1}{x^{p-1}}\Big|_a^1 \\
&= \lim_{a\to 0^+} \frac{1}{-p+1}\frac{1}{(1)^{p-1}} - \frac{1}{-p+1}\frac{1}{(a)^{p-1}} \\
&= \lim_{a\to 0^+} \frac{1}{-p+1} - \frac{1}{-p+1}\frac{1}{a^{p-1}} \\
&= \frac{1}{-p+1} + \infty \\
&= \infty
\end{aligned}$$

Hence the integral diverges for $p>1$.

Case 3: Notice that by doing the anti-power rule in the first step of computing the integral, we divide by $-p+1$. We can only do this if $p \neq 1$

because otherwise we'd be dividing by zero (which creates black holes and breaks math and stuff). So we save one extra case when $p = 1$. But if $p = 1$, we can plug that in at the beginning and easily deduce the following:

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln|x|\Big|_a^1 = \lim_{a\to 0^+} \ln(1) - \ln(a) = \infty.$$

Therefore, we have classified all the positive p values that will make the integral converge or diverge. To write it succinctly,

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases}$$

Method:

(1) Determine what x values make the function discontinuous
(2) If a point of discontinuity lies in the interval we are integrating over, split the integral so that the discontinuity lies at one of the end points
(3) Rewrite as a limit with a variable replacing the discontinuity point
(4) Use the FTC to compute the antiderivative like normal
(5) Take the limit
(6) If the answer comes out as $\pm\infty$ then it diverges. If the answer is a finite number then it converges