

1.1 Basic Derivatives

- Definitions of the Derivative: $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$
- $\frac{d}{dx}(x) = 1$ • $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(cx) = c$ • $\frac{d}{dx}(cu) = c \cdot u'$
- $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule) • $\frac{d}{dx}(u^n) = nu^{n-1} \cdot u'$ (Power Rule)
- $\frac{d}{dx}(\sin x) = \cos(x)$ • $\frac{d}{dx}(\sin u) = \cos(u) \cdot u'$
- $\frac{d}{dx}(\cos x) = -\sin(x)$ • $\frac{d}{dx}(\cos u) = -\sin(u) \cdot u'$
- $\frac{d}{dx}(\sec x) = \sec(x) \cdot \tan(x)$ • $\frac{d}{dx}(\sec u) = \sec(u) \cdot \tan(u) \cdot u'$
- $\frac{d}{dx}(\csc x) = -\csc(x) \cdot \cot(x)$ • $\frac{d}{dx}(\csc u) = -\csc(u) \cdot \cot(u) \cdot u'$
- $\frac{d}{dx}(\tan x) = \sec^2(x)$ • $\frac{d}{dx}(\tan u) = \sec^2(u) \cdot u'$
- $\frac{d}{dx}(\cot x) = -\csc^2(x)$ • $\frac{d}{dx}(\cot u) = -\csc^2(u) \cdot u'$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$ • $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot u'$
- $\frac{d}{dx}(e^x) = e^x$ • $\frac{d}{dx}(e^u) = e^u \cdot u'$
- $\frac{d}{dx}(a^u) = a^u(\ln a) \cdot u' \quad a > 0$ • $\frac{d}{dx}(\log_a u) = \frac{1}{u \cdot \ln a} \cdot u' \quad a > 0$
- $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}} \cdot u'$ • $\frac{d}{dx}(\arccos u) = \frac{-1}{\sqrt{1-u^2}} \cdot u'$
- $\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot u'$ • $\frac{d}{dx}(\operatorname{arccsc} u) = \frac{-1}{|u|\sqrt{u^2-1}} \cdot u'$
- $\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2} \cdot u'$ • $\frac{d}{dx}(\operatorname{arccot} u) = \frac{-1}{1+u^2} \cdot u'$
- $\frac{d}{dx}(u \pm v) = u' \pm v'$ • $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$ (Quotient Rule)
- $\frac{d}{dx}(uv) = u'v + uv'$ (Product Rule) • $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ (Chain Rule)

1.2 Facts About Derivatives

- Derivatives have many names. Problems may ask you to find
 - the derivative at a point
 - the slope of the tangent line at a point
 - the rate of change at a point
 - the velocity at a point

Each of the above ask for the exact same thing: to compute the derivative.

- When using the Chain Rule, its advisable to work from the outside function inwards. This will be described in the Examples section.
- We need to take care of which variable with which we are taking a derivative. Notice that $\frac{d}{dx}5y^2 = 0$, since we are taking the derivative with respect to x . Since no x 's show up in the equation, $5y^2$ is treated as a constant. Similarly, $\frac{d}{dz}e^x + z = 0 + 1 = 1$.
- Position, velocity, acceleration and jerk are all linked through derivatives. If we denote position as a function of time, $s(t)$, velocity as $v(t)$, acceleration as $a(t)$, and jerk as $j(t)$, then we have the following relations:

$$\begin{aligned}v(t) &= \frac{ds}{dt} \\a(t) &= \frac{dv}{dt} = \frac{d^2s}{dt^2} \\j(t) &= \frac{da}{dt} = \frac{d^3s}{dt^3}\end{aligned}$$

1.3 Examples

1. Find the derivative of $\ln(\cos(4x + 2x^2))$.

Solution: Using the Chain Rule, we start from the outside function, the natural log, and work our way inwards to the cosine, then the polynomial. Thus,

$$\begin{aligned}\frac{d}{dx} \ln(\cos(4x + 2x^2)) &= \frac{1}{\cos(4x + 2x^2)} \cdot \frac{d}{dx}(\cos(4x + 2x^2)) \\&= \frac{1}{\cos(4x + 2x^2)} \cdot -\sin(4x + 2x^2) \cdot \frac{d}{dx}4x + 2x^2 \\&= \frac{1}{\cos(4x + 2x^2)} \cdot -\sin(4x + 2x^2) \cdot (4 + 4x) \\&= -\frac{\sin(4x + 2x^2) \cdot (4 + 4x)}{\cos(4x + 2x^2)} \\&= -4 \tan(4x + 2x^2)(1 + x).\end{aligned}$$

2. Find $\frac{d}{dy} \sec(\operatorname{arccot}(6y^5))$.

Solution: Since we see that we need the Chain Rule, we start from the outside function, secant, and work our way in through arc-cotangent to the polynomial. Thus,

$$\begin{aligned} \frac{d}{dy} \sec(\operatorname{arccot}(6y^5)) &= \sec(\operatorname{arccot}(6y^5)) \cdot \tan(\operatorname{arccot}(6y^5)) \cdot \frac{d}{dy} \operatorname{arccot}(6y^5) \\ &= \sec(\operatorname{arccot}(6y^5)) \cdot \tan(\operatorname{arccot}(6y^5)) \cdot \frac{-1}{1+(6y^5)^2} \cdot \frac{d}{dy} 6y^5 \\ &= \sec(\operatorname{arccot}(6y^5)) \cdot \tan(\operatorname{arccot}(6y^5)) \cdot \frac{-1}{1+(6y^5)^2} \cdot 30y^4 \end{aligned}$$

3. Find the slope of the tangent line of $2^{e^{13x}}$ at $x = 0$.

Solution: First we need the general form of the derivative of our function. Notice that again we need the Chain Rule. Thus,

$$\begin{aligned} \frac{d}{dx} 2^{e^{13x}} &= 2^{e^{13x}} \cdot \ln(2) \cdot \frac{d}{dx} e^{13x} \\ &= 2^{e^{13x}} \cdot \ln(2) \cdot e^{13x} \frac{d}{dx} 13x \\ &= 2^{e^{13x}} \cdot \ln(2) \cdot e^{13x} \cdot 13 \end{aligned}$$

Now that we have the derivative, we can plug in $x = 0$ to find the derivative at our point. Hence,

$$2^{e^{13(0)}} \cdot \ln(2) \cdot e^{13(0)} \cdot 13 = 2^{e^0} \cdot \ln(2) \cdot e^0 \cdot 13 = 2 \cdot \ln(2) \cdot 13 = 26 \cdot \ln(2).$$

4. Let $v = x^2 + y^2$, where $x = \cos(t) + \sin(t)$ and $y = \cos(t) - \sin(t)$. Find $\frac{dv}{dt}$.

Solution: There are two ways to do this problem. One requires the Chain Rule, the other just requires some algebra. We will first show the algebra method.

Way 1: Instead of just trying to brute force the problem, we try to simplify our work for us. If we make the substitutions for x and y , we see that

$$v = (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2.$$

FOILing out each piece, we see that

$$v = [\cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)] + [\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)].$$

Notice that we can cancel the middle terms and combine the squared terms, and we are left with

$$v = 2\cos^2(t) + 2\sin^2(t) = 2(\cos^2(t) + \sin^2(t)) = 2.$$

So, taking the derivative is now very simple, since the derivative of a constant is 0. Thus, $\frac{dv}{dt} = 0$.

Way 2: Making the substitution, we see that

$$v = (\cos(t) + \sin(t))^2 + (\cos(t) - \sin(t))^2.$$

Taking the derivative with respect to t , we use the Chain Rule on each set of parenthesis and get that

$$\frac{dv}{dt} = [2(\cos(t) + \sin(t)) \cdot (-\sin(t) + \cos(t))] + [2(\cos(t) - \sin(t)) \cdot (-\sin(t) - \cos(t))].$$

FOILing the terms, we get that

$$\frac{dv}{dt} = 2(\cos^2(t) - \sin^2(t)) + 2(-\cos^2(t) + \sin^2(t)) = 2\cos^2(t) - 2\sin^2(t) - 2\cos^2(t) + 2\sin^2(t) = 0.$$

As we can see, both methods arrive at the same conclusion, that the derivative is equal to 0. Its up to personal preference which way to choose. Usually it is more advisable to simplify first and then take derivates, so Way 1 is more common. However, both methods are equally valid.

5. Compute $\frac{d}{dx} \ln(\sin(x)) \cdot \arcsin(e^{2x})$.

Solution: Sometimes it is advisable to break up our functions into pieces to better keep track of what we need to do. First, we notice that this is going to be a problem involving the Product Rule. So, lets identify our f and g functions. Set $f(x) = \ln(\sin(x))$ and $g(x) = \arcsin(e^{2x})$. Since this is a product rule, we need $f'(x)$ and $g'(x)$. In both cases, we have to use the Chain Rule. So, we have

$$\begin{aligned} f(x) &= \ln(\sin(x)) & f'(x) &= \frac{1}{\sin(x)} \cdot \cos(x) \\ g(x) &= \arcsin(e^{2x}) & g'(x) &= \frac{1}{\sqrt{1 - (e^{2x})^2}} \cdot e^{2x} \cdot 2 \end{aligned}$$

Recall that the Product Rule says $\frac{d}{dx}(fg) = f'g + fg'$. Thus, all we need to do is insert everything into the right places. So we get

$$\begin{aligned} \frac{d}{dx} \ln(\sin(x)) \cdot \arcsin(e^{2x}) &= \left[\frac{1}{\sin(x)} \cdot \cos(x) \right] \cdot \arcsin(e^{2x}) + \ln(\sin(x)) \cdot \left[\frac{1}{\sqrt{1 - (e^{2x})^2}} \cdot e^{2x} \cdot 2 \right] \\ &= \cot(x) \cdot \arcsin(e^{2x}) + \ln(\sin(x)) \cdot \left(\frac{2e^{2x}}{\sqrt{1 - e^{4x}}} \right) \end{aligned}$$