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1.1 The Fundamental Theorem of Calculus

- *Part 1:* If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$. So,

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,dt = f(x).$$

- *Part 2:* If f is continuous at every point in $[a, b]$, and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

- Recall that $\displaystyle\int f(x)\,dx$ is a family of *functions*, whereas $\displaystyle\int_a^b f(x)\,dx$ is a *number*.

1.2 Techniques of Integration

- U**-Substitution:** Suppose we want to compute the following integral:

$$\int_0^{\pi/2} \sin(x)\cos(x)\,dx.$$

We notice first that it is a definite integral, so we are looking for a number as our answer. The second thing we notice is that this problem will require a u-substitution. In this case we have two nice choices for u. This time we will choose $u = \sin(x)$. That means $du = \cos(x)dx$. But we can't just plug in the u and du and forget about our limits. The limits of integration in the original integral are for x, but by doing a u-sub we are no longer dealing with x's. So why should we deal with x limits any more if there won't be any x's? That means if we want to keep things equal, we have to change the x limits to u limits. We do this using our relationship that we established for x and u, namely the substitution $u = \sin(x)$. When $x = 0$, that means $u = \sin(0) = 0$. So our new bottom limit is going to be 0. When $x = \pi/2$, we get $u = \sin(\pi/2) = 1$. So our new top limit will be 1. Now that we've changed the limits, we can put everything together to get

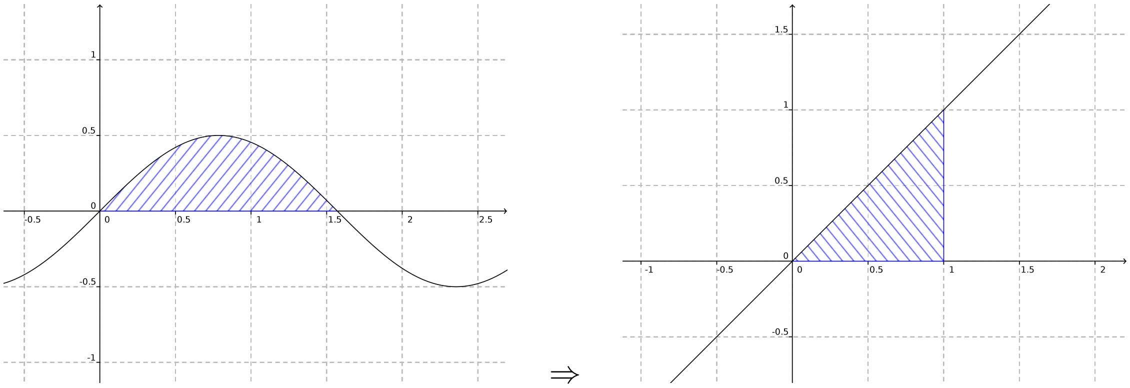
$$\int_0^{\pi/2} \sin(x)\cos(x)\,dx = \int_0^1 u\,du.$$

Notice that when we plug in the u-sub, we also plug in the u limits. we have to keep all of our information for each variable together. From here we can just use the fundamental theorem and get

$$\int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.$$

What does this all mean? When we do the u-sub we are changing the geometry of the space. Essentially we turn the left hand area (pictured below) into the right by redefining what x

and y are along the axis. The actual amount of the area doesn't change, just how we shape it and organize it.



Here the left is the function $f(x) = \sin(x)\cos(x)$ with the area between 0 and $\pi/2$ highlighted. On the right is the new function, $f(u) = u$ with the area between 0 and 1 highlighted.

Example: Compute $\displaystyle\int_0^4 \frac{1}{\sqrt{1+2x}}\,dx$.

Solution: Start with the u-sub $u = 1+2x$. Then $du = 2dx \Rightarrow \frac{1}{2}du = dx$. Changing the limits, we get $u = 1+2(0) = 1$ and $u = 1+2(4) = 9$. Plugging everything in, we see that

$$\int_0^4 \frac{1}{\sqrt{1+2x}}\,dx = \frac{1}{2}\int_1^9 \frac{1}{\sqrt{u}}\,du = \sqrt{u}\Big|_1^9 = \sqrt{9} - \sqrt{1} = 3 - 1 = 2.$$

- **Integration by Parts:** Recall our formula $\displaystyle\int u\,dv = uv - \int v\,du$. When we do a definite integral we just evaluate each piece, so the formula becomes

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example: Compute $\displaystyle\int_0^{\pi/2} x\cos(x)\,dx$.

Solution: To start we choose

$$\begin{aligned} u &= x \qquad & dv &= \cos(x)\,dx \\ du &= dx \qquad & v &= \sin(x) \end{aligned}$$

Plugging these into our formula, we have

$$\begin{aligned} x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)dx &= x\sin(x)\Big|_0^{\pi/2} + \cos(x)\Big|_0^{\pi/2} \\ &= \left(\frac{\pi}{2}\sin(\pi/2) - 0\right) + (\cos(\pi/2) - \cos(0)) \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

- **Partial Fractions:** Partial fractions works nicely with definite integrals because all we have to do is apply the fundamental theorem to each piece. The only thing we have to be very careful of is making sure we don't integrate through a discontinuity. If one of the discontinuities of the function is in the interval that we are integrating over, then we have an improper integral. This worksheet does not cover improper integration.

  Example: Compute $\int_0^2 \frac{16}{(x-3)^2(x+1)}\,dx$.
  Solution: We start by running partial fraction decomposition on the integrand. So,

$$\begin{aligned}
\frac{16}{(x-3)^2(x+1)} &= \frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}+\frac{D}{x+1} \\
&= \frac{A(x-3)^2(x+1)+B(x-3)(x+1)+C(x+1)+D(x-3)^3}{(x-3)^3(x+1)} \\
&= \frac{A(x^3-5x^2+3x+9)+B(x^2-2x-3)+C(x+1)+D(x^3-9x^2+27x-27)}{(x-3)^3(x+1)}
\end{aligned}$$

  This gives us the following system of equations:

$$\begin{aligned}
0 &= A+D \\
0 &= -5A+B-9D \\
0 &= 3A-2B+C+27D \\
16 &= 9A-3B+c-27D
\end{aligned}$$

  The top equation gives us $A=-D$. Plugging that into the second equation, we get $4D=B$. Plugging both of those into the third gives $C=-16D$. Finally plugging A,B,C into the last equation gives us that $D=-\frac{1}{4}$. Back substituting for each of the variables gives us $C=4, B=-1, A=\frac{1}{4}$. So, we get

$$\begin{aligned}
\int_0^2 \frac{16}{(x-3)^2(x+1)}\,dx &= \int_0^2 \frac{1/4}{x-3}+\frac{-1}{(x-3)^2}+\frac{4}{(x-3)^3}+\frac{-1/4}{x+1}\,dx \\
&= \frac{1}{4}\ln|x-3|+\frac{1}{x-3}-2\frac{1}{(x-3)^2}-\frac{1}{4}\ln|x+1|\Big|_0^2 \\
&= \left(0+\frac{1}{-1}-2\cdot\frac{1}{(-1)^2}-\frac{1}{4}\ln(3)\right)-\left(\frac{1}{4}\ln(3)-\frac{1}{3}-2\cdot\frac{1}{9}-0\right) \\
&= -\frac{22}{9}-\frac{1}{2}\ln(3)
\end{aligned}$$