Continuity

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Continuity is one of the fundamental properties that is studied in precalculus and calculus classes. It becomes one of the basic requirements for a function to be reasonably well behaved. Our intuitive definition is that if you can draw the graph of a function without picking up your pencil, then the function is continuous. This makes sense because you can continually draw a continuous function. Most of the shapes from geometry (e.g. circles, ellipses, triangles, polygons...) and most of the functions we've encountered so far (e.g. polynomials, exponentials, logarithms, $\sin(x)$, $\cos(x)$, ...) are continuous curves. In this worksheet we will determine what the condition is to be a continuous function, and explore some examples that are continuous and some that are not.

1.1 The Definition

Definition: A function $f(x)$ is said to be *continuous at* c if the left and right limits exist as x approaches c, and that

$$\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x).$$

We must have that all three terms exist, are finite, and are equal.

First, notice that the definition requires us to know how to compute limits (specifically one-sided limits). So if you haven't encountered that topic yet, check out the limits worksheet. Second, we could have just as easily written that a function is continuous at c if $\lim\limits_{x\to c} f(x)$ exists and is equal to $f(c)$. But how do we check that this limit exists? We would have to check that the left and right limits exist and are equal, which exactly what is written above. If you can get away with the shortcut of just computing the usual limit $\lim\limits_{x\to c} f(x)$, then you know the two one-sided limits are equal.

Another way to say that a function is continuous is if the limit can "pass trough" the function. In other words, that the following holds

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c).$$

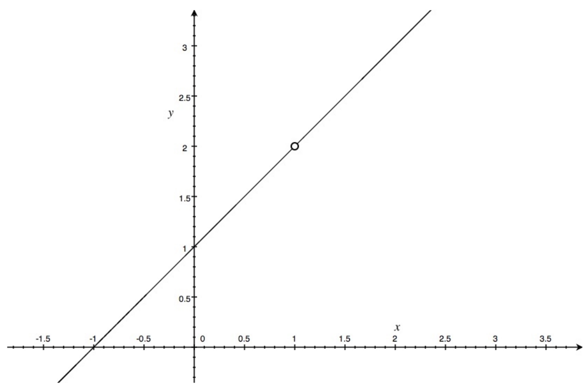
Some of the functions we have already encountered in our mathematical careers are continuous. The family of polynomials are always continuous at every real number. Exponential functions, $\sin(x), \cos(x), |x|$ also have this property of being continuous everywhere. We also have that logarithims, rational functions and $\tan(x)$ are continuous *on their domains*, meaning that as long as we throw out the places where these functions are undefined (for example when you would be dividing by zero), then they are nice and continuous. Knowing this, when ever we reduce a problem down to taking a limit of a polynomial, then because we know they are continuous, we know that we can just plug in the value to get the answer to the limit.

1.2 Where continuity breaks down

The idea with continuity is that a function behaves as we expect it to. Recall that the limit tells us about the behavior of the function as it approaches a point. If the function is continuous, then the value it actually takes on, $f(c)$, is exactly the value that the limit says that we should expect to hit, $\lim_{x \to c} f(x)$. These are generally nice functions to deal with. There are some pathological examples that are continuous but have very, very odd properties, but those will not be discussed in this worksheet.

For now, let's see what types of behaviors functions can have that make it so they are not continuous at certain points. The points where the function fails to be continuous are called *discontinuities*, and they fall into three main categories: holes, jumps, and asymptotes.

- **<u>Holes.</u>** Let's begin with the function $f(x) = \dfrac{x^2 - 1}{x - 1}$. It's graph looks like this:



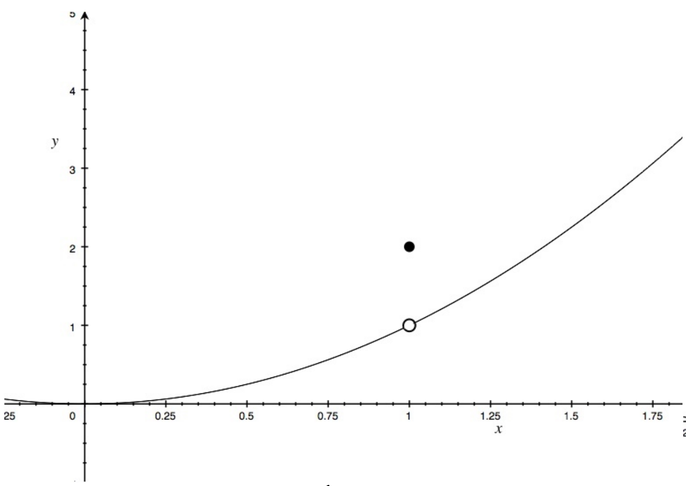
If we did not highlight the point that is missing (aka the hole) in the graph, how would we determine that this function is not continuous at $x = 1$? The answer is that we would have to see where the definition of continuity is not satisfied. So let's compute the pieces that we need for continuity. The first thing you should always try when trying to determine if a function is continuous is to just plug in the value. Here, we want to know what happens when we plug in $x = 1$. Then $f(1) = \dfrac{(1)^2 - 1}{1 - 1} = \dfrac{0}{0}$. We can see that this value does not exist because we would be dividing by zero, so $f(1)$ doesn't exist. Therefore we know right off the bat that this function cannot be continuous at $x = 1$. Just to make sure that we can still do the other parts, let's compute the limit as x gets close to 1 to see that this at least exists. Thus,

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} x + 1 = 1 + 1 = 2,$$

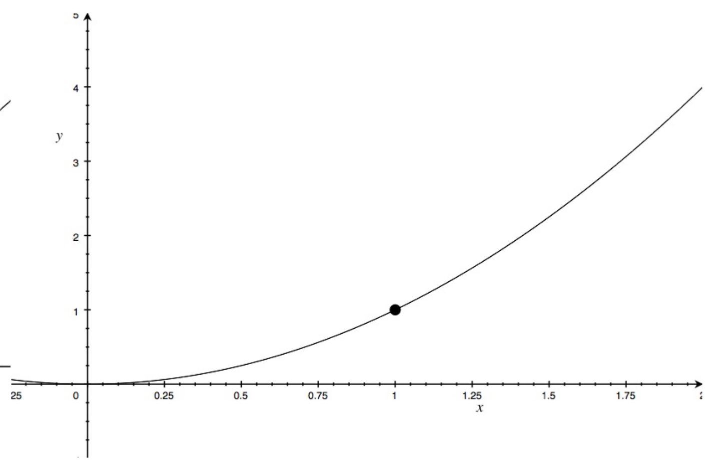
where the second equals sign comes from the fact that we are now using a limit and are not actually plugging in $x = 1$, so we can cancel the $x - 1$ that is on the top and bottom. Since we computed this limit and got the answer of 2, we can say that both the left and right side limits are also 2 and do exist. Still, because $f(1)$ did not exist, this function is discontinuous at $x = 1$. In particular, a hole happens when the left and right hand limits exist and are equal and finite, but the function value $f(c)$ is different than that value.

An interesting observation is that we can actually determine what function $f(x)$ acts like if it wouldn't have had that hole. The answer is in the limit calculation. If we could simplify $\frac{x^2-1}{x-1}$, we would factor the top and cancel the $x-1$ to get the function $g(x) = x+1$. But we **<u>cannot</u>** write $\frac{x^2-1}{x-1} = x+1$. This comes from the fact that we are defining two functions on two different domains. The function $f(x) = \frac{x^2-1}{x-1}$ has domain $(-\infty, 1) \cup (1, \infty)$, whereas $g(x) = x+1$ has domain $(-\infty, \infty)$. We can only say two function are equal if they have the same outputs from the same domain. It is a subtle point, but an important one.

Holes could be put into a function on purpose. Below we have two pictures of very similar functions. The only difference is that at $x = 1$ we have redefined what we think the function value should be. The result of redefining this point makes the function now discontinuous at $x = 1$ because $f(1) = 2$ but $\lim_{x\to 1} f(x) = 1$. However in the picture on the right, we have filled the hole with the value that the limit expects the function to take. Therefore, we do have that $\lim_{x\to 1^-} g(x) = g(1) = \lim_{x\to 1^+} g(x) = 1$.

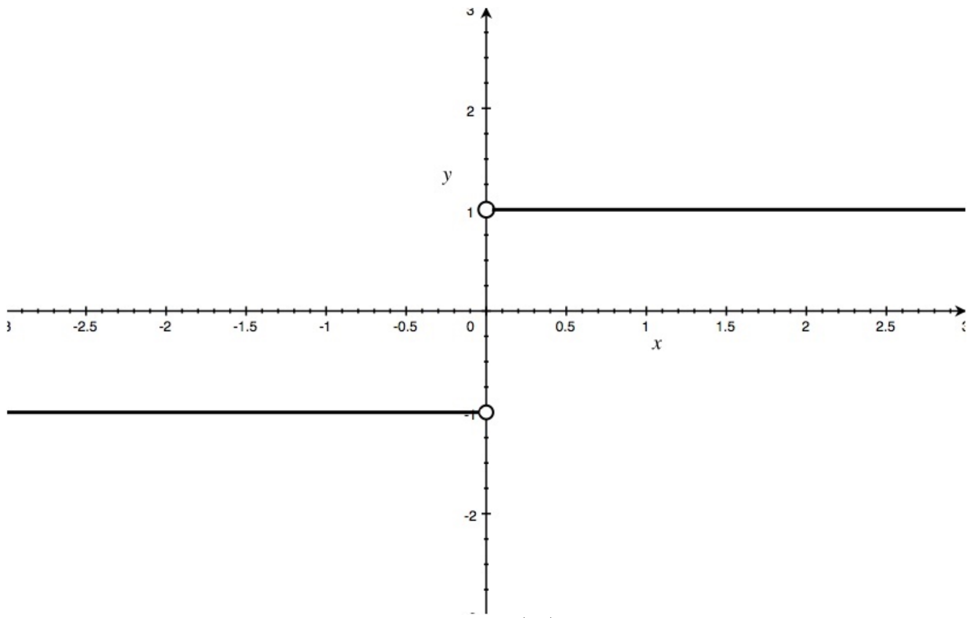


$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



$$g(x) = x^2$$

- **Jumps.** The second kind of discontinuity is a jump, which usually comes from a piecewise defined function. Below we have a picture of the graph of a function that has a jump.



$$f(x) = \frac{|x|}{x}$$

We can see that this function has value -1 for all $x < 0$ and value 1 for all $x > 0$. At $x = 0$ the function is undefined since we would get $f(0) = \frac{0}{0}$, which divides by zero. So we could write this as the piecewise function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

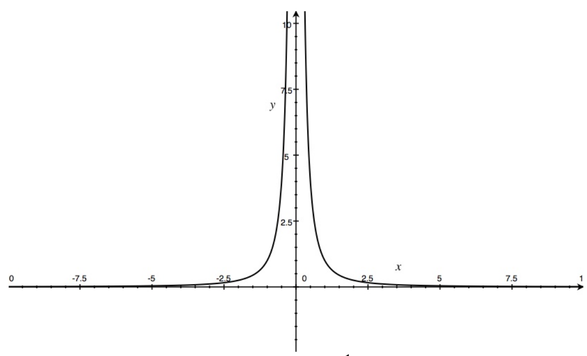
Clearly this function is not continuous at $x = 0$ because $f(0)$ is not defined, but even more intuitively because there is no way you could draw that graph without picking up your pencil. Moving from left to right, it is doing fine at -1 before it suddenly decides it should be at 1 instead. This is why we call it a jump discontinuity.

We do not have a hole at $x = 0$ because even though the function value is not defined there, the left and right hand limits are not equal to each other. Notice that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -1 = -1$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$. Thus, no matter what value would have been given to $f(0)$ we would have a jump discontinuity. A jump discontinuity happens when the left and right limits are both still finite, but have different values.

- **Asymptotes.** We have met asymptotes before in our precalculus and algebra II classes. We would like to say that asymptotes happen when you try to divide by zero, but that isn't quite true. Notice that in our first example above we divided by $x-1$, which means we could expect $x = 1$ to be an asymptote, but it turned out to be a hole instead. This happened because we could also find an $x - 1$ on top to "cancel" with the bottom one. When we don't have anything on top to cancel the bottom is when we get asymptotes. As an example, we can take the function $f(x) = \frac{1}{x^2}$. We see that we are dividing by a power of x, and since we can't divide by zero, we know that we have to throw out $x = 0$ as a value for the domain. But we can still ask for the behavior of the function near by $x = 0$. Thus, we can see that

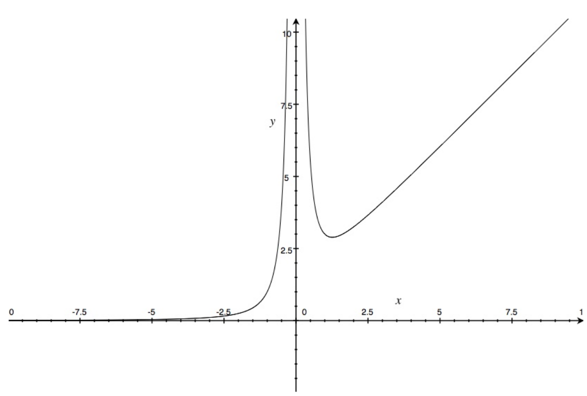
$$\lim_{x\to 0^-} \frac{1}{x^2} = \infty, \qquad \lim_{x\to 0^+} \frac{1}{x^2} = \infty.$$

Thus, as we approach $x = 0$ from both sides, the function values go up to infinity, but $f(0) = \frac{1}{0^2}$ is undefined. Below is a picture of the graph of this function.



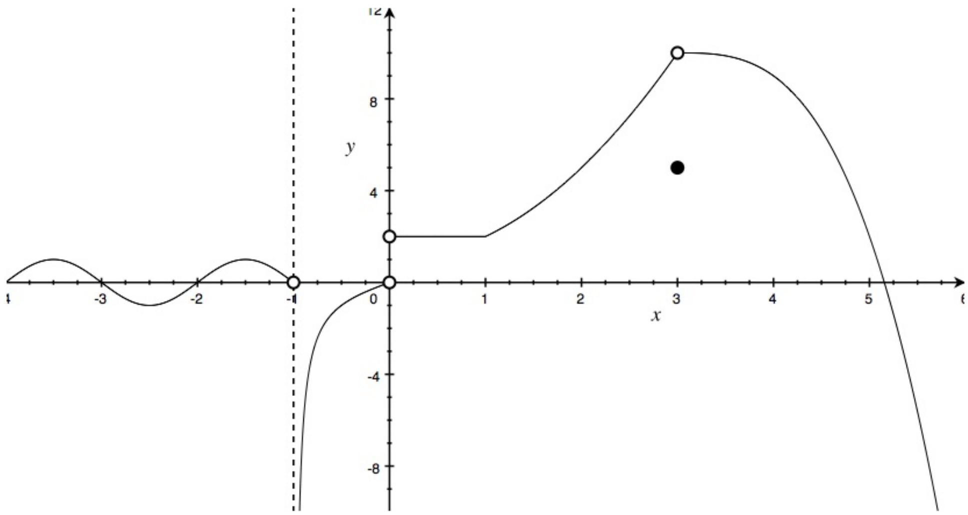
$$f(x) = \frac{1}{x^2}$$

Asymptotes happen when either the left or right limits (or both) are equal to $-\infty$ or ∞. The functions don't have to be symmetric about the asymptote either, as the next graph shows.



This functions has an asymptote at $x = 0$ and we can see that the left and right limits are both ∞.

Some functions can even contain all the types of discontinuities! An exmaple is graphed below



We can see that this function has an asymptote at $x = -1$, a jump at $x = 0$ and a hole at $x = 3$. Everywhere else this function is continuous, but at those three points we have some weird behavior. But now we know how to mathematically detect that odd behavior and how to quantify it.