The Day 1 Calculus Primer

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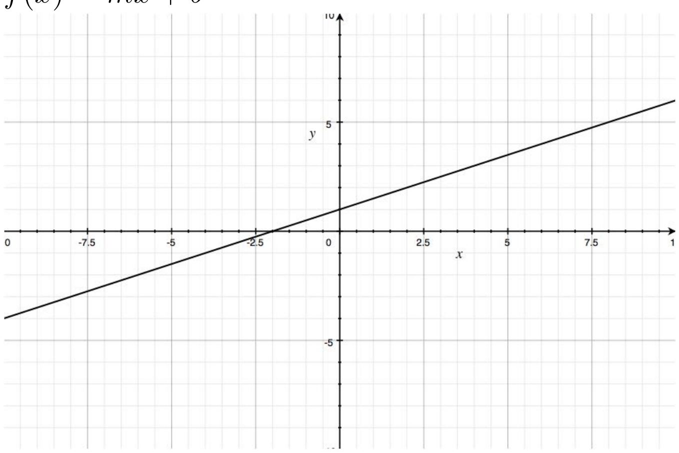
Below is a collection of many of the things that are generally expected that you know off the top of your head when you start a calculus class. This does not constitute *everything*, and every class and professor is different, but if you have the following memorized and ready to answer off in a heartbeat then you should be well prepared for what comes up in a beginning calculus class. This worksheet is meant as a study guide and review of material from geometry, algebra 2, trigonometry, and precalculus, and does not constitute a replacement for taking those classes.

1.1 Parent Functions

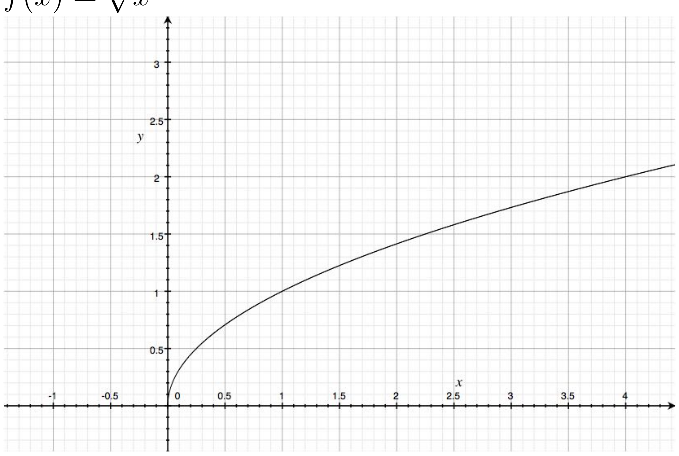
In this section we will list a set of parent functions for which you should know the graph, domain, range, and any special characteristics of (like asymptotes or zeros). In a later section we will talk about transformations of these graphs, but we first need to know the general shape of these standard functions.

- $f(x) = mx + b$



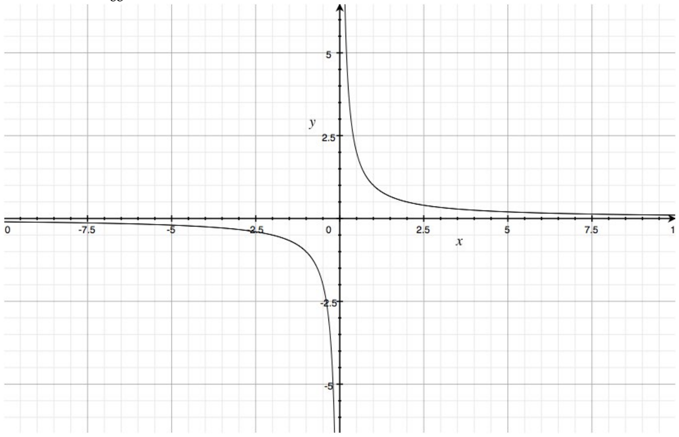
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Characteristics: Line with slope m and y-intercept $(0, b)$.
Increasing when $m > 0$.
Decreasing when $m < 0$.
Odd function when $b = 0$.

- $f(x) = \sqrt{x}$



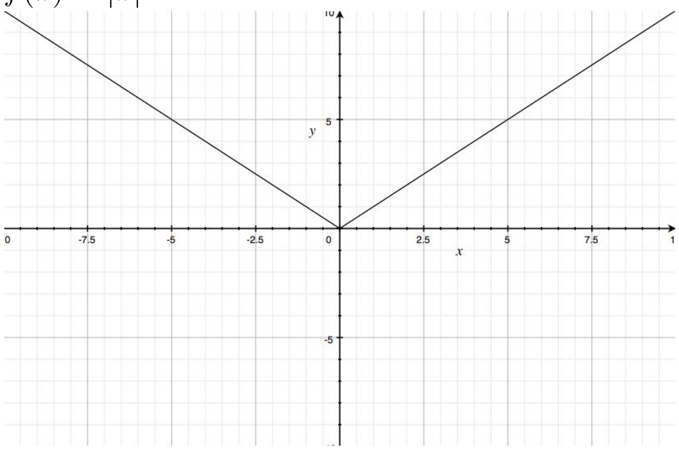
Domain: $[0, \infty)$
Range: $[0, \infty)$
Characteristics: Is equal to zero only when the inside of the root is zero.
Always increasing.

- $f(x) = \dfrac{1}{x}$



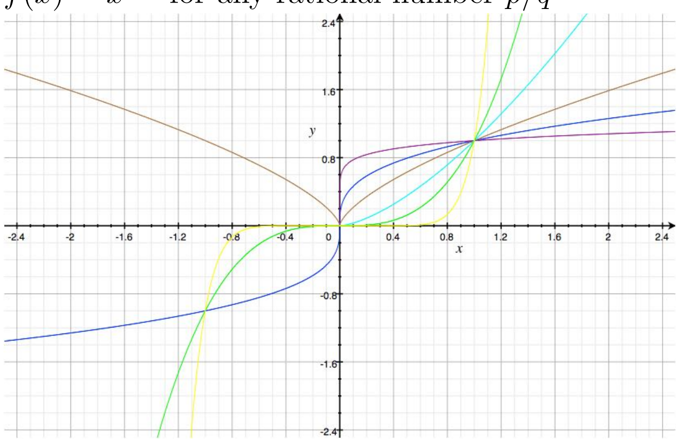
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Characteristics: Vertical asymptote at $x = 0$,
Horizontal asymptote at $y = 0$.
Always decreasing.
Odd function.

- $f(x) = |x|$



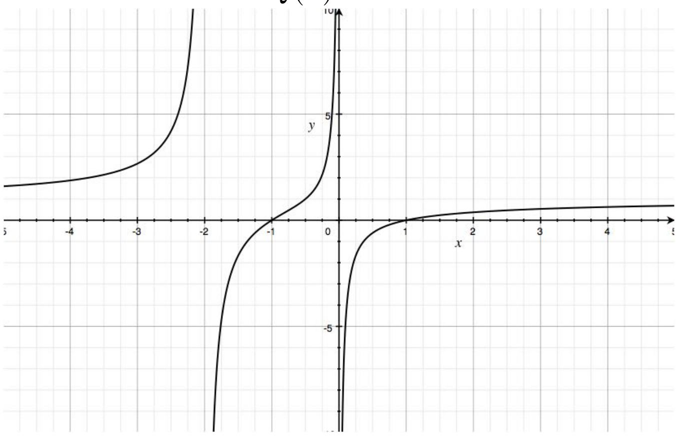
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Characteristics: Technically a piecewise function.
Always positive.
Even function

- $f(x) = x^{p/q}$ for any rational number p/q



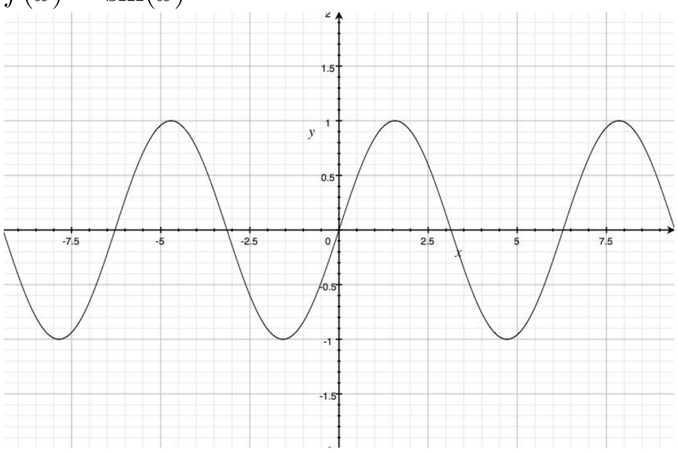
Domain: $[0, \infty)$ if q is even
$(-\infty, \infty)$ if q is odd
Range: $[0, \infty)$ if q is even
$(-\infty, \infty)$ if q is odd
Characteristics: Zero at $x = 0$ when $\frac{p}{q} > 0$.
Vertical asymptote at $x = 0$ when $\frac{p}{q} < 0$.
In particular, know the graphs of x^n for integers n.

- Rational functions $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.



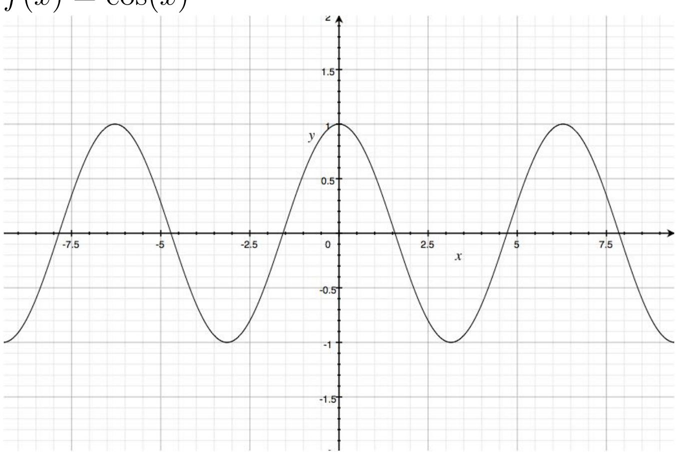
Domain: $\{x : Q(x) \neq 0\}$
Range: Depends on the function
Characteristics: Vertical asymptotes where $Q(x) = 0$ and $P(x) \neq 0$.
Zeroes where $P(x) = 0$ and $Q(x) \neq 0$.
Holes where $P(x) = 0$ and $Q(x) = 0$.

- $f(x) = \sin(x)$



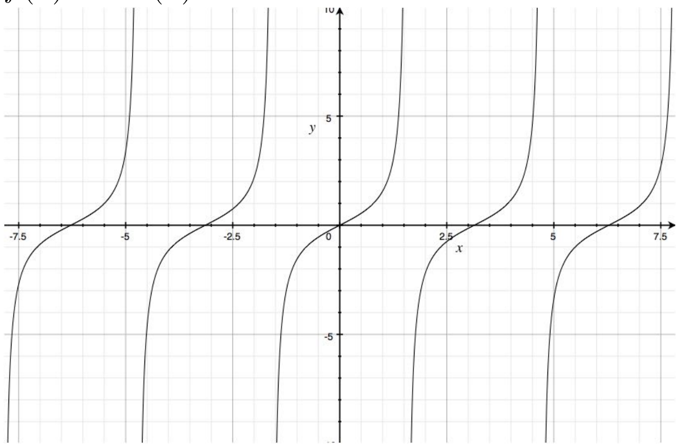
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Characteristics: Oscillates between -1 and 1.
Odd function.

- $f(x) = \cos(x)$



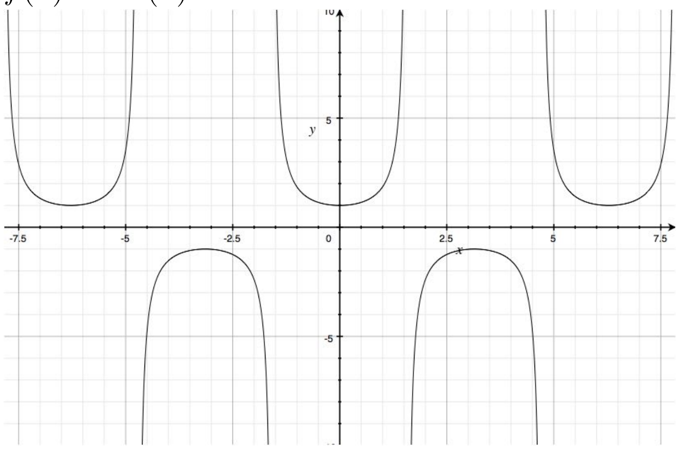
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Characteristics: Oscillates between -1 and 1.
Even function.

- $f(x) = \tan(x)$



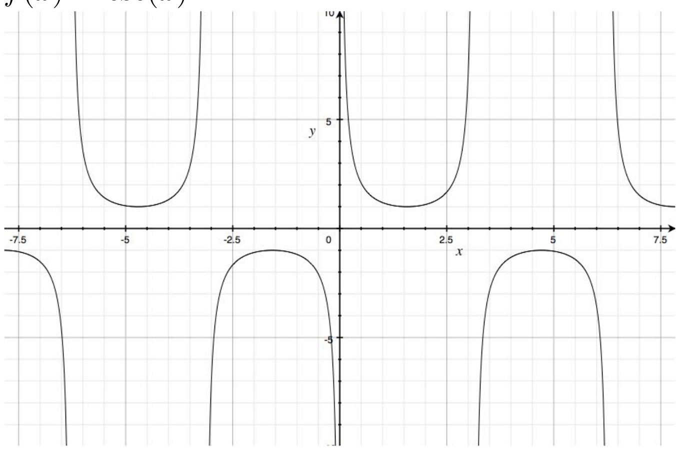
Domain: $\{x : x \neq \frac{\pi(2k+1)}{2}, k \text{ an integer}\}$
Range: $(-\infty, \infty)$
Characteristics: Vertical asymptotes at every odd multiple of $\pi/2$.
Zeros at every multiple of π.
Odd function.

- $f(x) = \sec(x)$



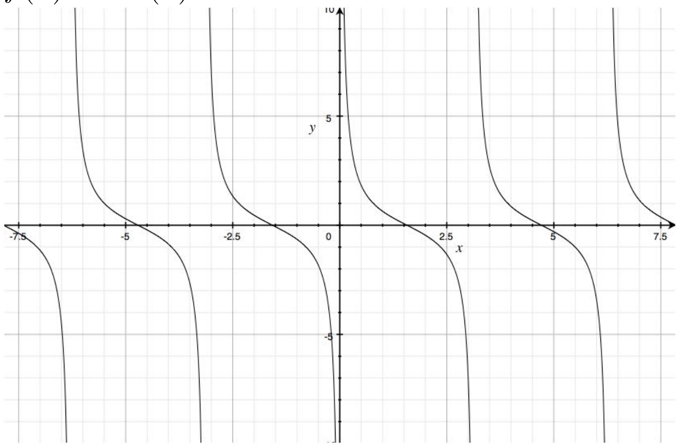
Domain: $\{x : x \neq \frac{\pi(2k+1)}{2}, k \text{ an integer}\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Characteristics: Vertical asymptotes at every odd multiple of $\pi/2$.
Even function.

- $f(x) = \csc(x)$



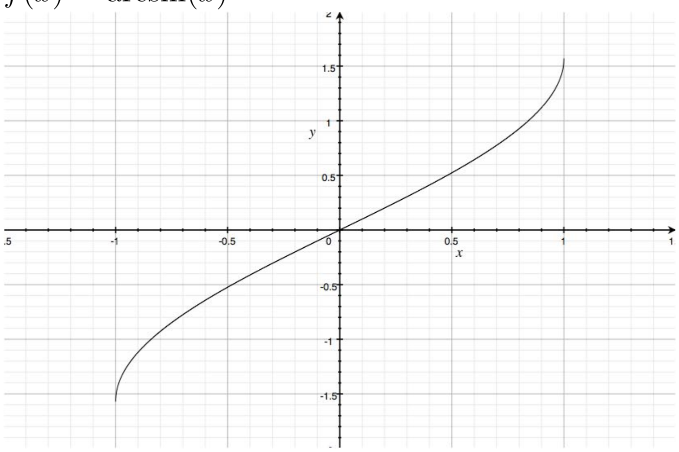
Domain: $\{x : x \neq k\pi, k \text{ an integer}\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Characteristics: Vertical asymptotes at multiple of π.
Odd function

- $f(x) = \cot(x)$



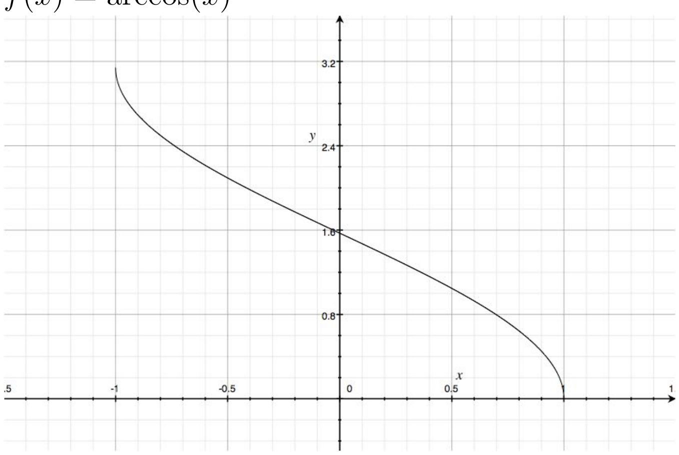
Domain: $\{x : x \neq k\pi, k \text{ an integer}\}$
Range: $(-\infty, \infty)$
Characteristics: Vertical asymptotes at every multiple of π.
Zeros at every odd multiple of $\pi/2$.
Odd function.

- $f(x) = \arcsin(x)$



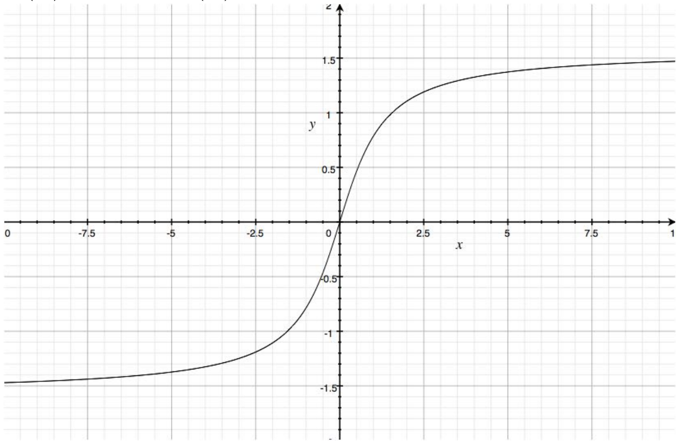
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$
Characteristics: Range can change depending on what strip you want the answer to live in.

- $f(x) = \arccos(x)$



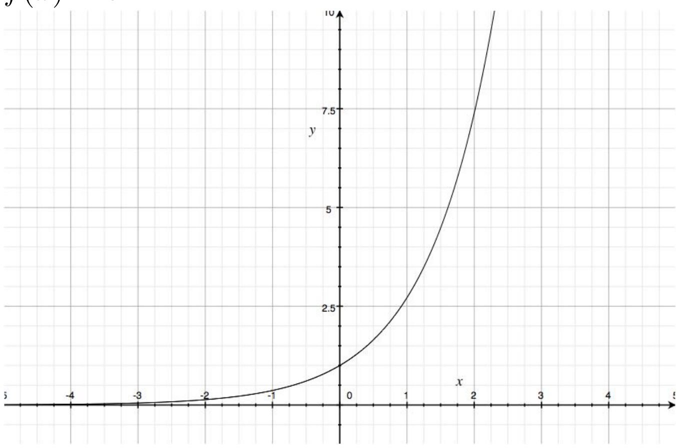
Domain: $[-1, 1]$
Range: $[0, \pi]$
Characteristics: Range can change depending on what strip you want the answer to live in.

- $f(x) = \arctan(x)$



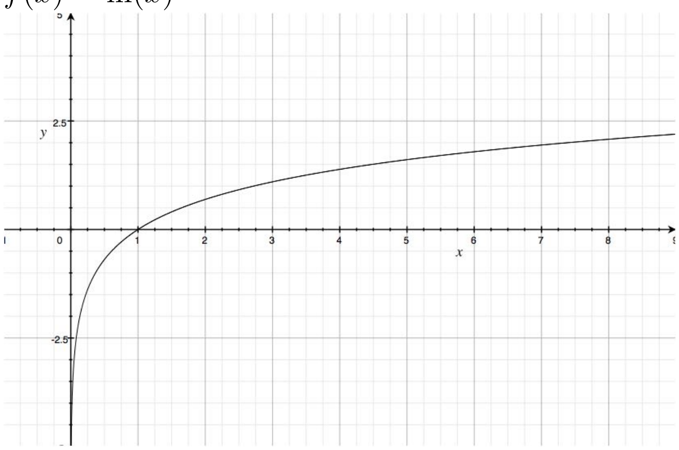
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$
Characteristics: Range can change depending on what strip you want to answer to live in .
Always increasing.
Odd function.
Horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$.

- $f(x) = e^x$



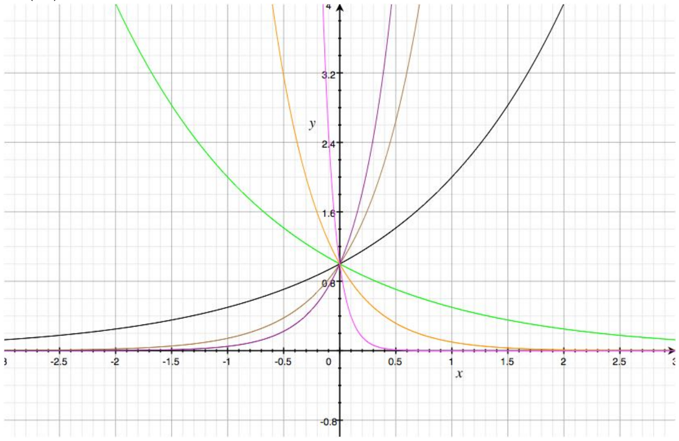
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Characteristics: Always positive.
Horizontal asymptote at $y = 0$.
y-intercept at $x = 0$.
Always increasing.

- $f(x) = \ln(x)$



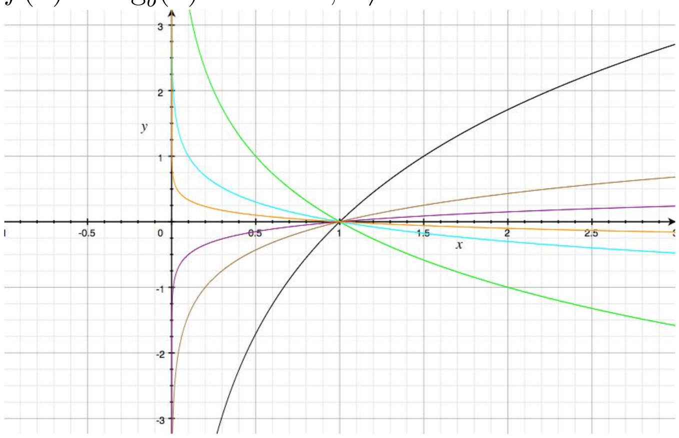
Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Characteristics: Zero at $x = 1$.
Vertical asymptote at $x = 0$.
Always increasing.

- $f(x) = a^x$ for $a > 0, a \neq 1$



Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
Characteristics: If $0 < a < 1$, then the function is always decreasing. (light colors)
If $a > 1$, then the function is always increasing. (dark colors)
y-intercept at $x = 0$.
Horizontal asymptote at $y = 0$.

- $f(x) = \log_b(x)$ for $b > 0, b \neq 1$



Domain: $(0, \infty)$
Range: $(-\infty, \infty)$
Characteristics: If $0 < b < 1$, Then the function is always decreasing. (light colors)
If $b > 1$, then the function is always increasing. (dark colors)
Zero at $x = 1$.
Vertical asymptote at $x = 0$.

1.2 Algebra 2 Concepts

1.2.1 Completing the Square

In section 1.7, we will discuss transformations of a generic function. For this section, though, we focus on getting a quadratic function into what is often called "graphing form". That is, we want to transform a standard quadratic $y = ax^2 + bx + c$ into the form $y = a(x-h)^2 + k$. To do this, we follow the process called "completing the square".

Consider the example $y = 3x^2 + 6x + 5$. First, we make it easier by pulling the "a" term out, so that $y = 3(x^2 + 2x + \frac{5}{3})$. Next, we add zero to the inside of the parentheses in a special form to make it so that we can factor the first two terms. The form of zero is to both add and subtract $(\frac{b}{2})^2$. Thus, we get $y = 3\left(x^2 + 2x + (\frac{2}{2})^2 - (\frac{2}{2})^2 + \frac{5}{3}\right)$. Then we can collapse the first three terms because they form a perfect square (by design), and combine the last two terms to get a single number. Thus, $y = 3\left((x+1)^2 + \frac{2}{3}\right)$. Lastly, we can distribute the 3 that we pulled out originally to get our final graphing form $y = 3(x+1)^2 + 2$.

When doing this from a generic quadratic, we start with $y = ax^2 + bx + c$ and end with $y = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$. The quadratic formula is derived from setting $y = ax^2 + bx + c$ equal to zero and completing the square to solve for x.

1.2.2 Exponent Properties

Let a, b, c be real numbers. Then

- $a^c \cdot b^c = (ab)^c$
- $\frac{a^c}{b^c} = \left(\frac{a}{b}\right)^c$
- $a^b \cdot a^c = a^{b+c}$
- $\frac{a^b}{a^c} = a^{b-c}$

In particuar, roots are one of the most common applications of these exponent rules. For $n, m \in \mathbb{N}$.

- $\sqrt{x^2} = |x|$
- $\sqrt{-1} = i$
- $\sqrt{-x} = i \cdot \sqrt{x}$
- $\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y}$
- $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$ when $y \neq 0$
- $\sqrt{x} = x^{\frac{1}{2}}$
- $\sqrt[n]{x} = x^{\frac{1}{n}}$
- $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

  Things to be careful of:
- $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$
- $\sqrt{x-y} \neq \sqrt{x} - \sqrt{y}$

1.2.3 Function Notation

We often use function notation to emphasize not just the name of the output of a function, but the input that results in said output. For example, we may write the quadratic $y = x^2 + x$, or we may use function notation as $f(x) = x^2 + x$. The advantage of function notation is that if we change the input, it is easy to see how the output is changed. For the same example, we can easily compute $f(6)$ because all we have done is replace the x on the left side with 6, so all we need to do is replace any x on the right side with 6 as well. Thus, $f(6) = (6)^2 + 6 = 42$. This works regardless of what we replace x with in the function on the left. We can see that

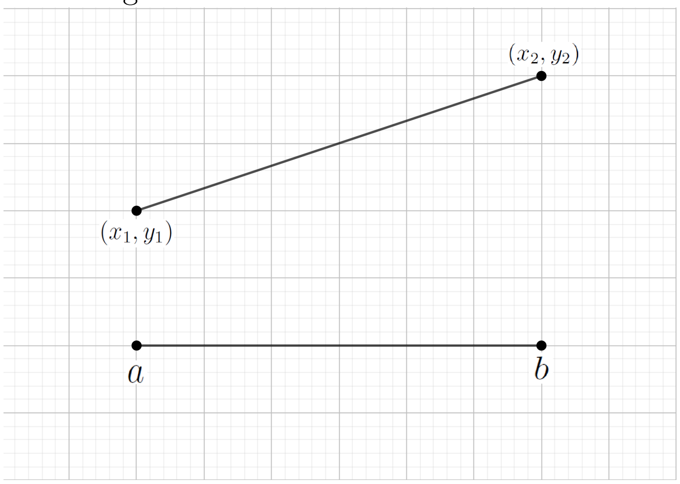
$$\begin{aligned} f(\sin(x)) &= \sin^2(x) + \sin(x) \\ f(t+h) &= (t+h)^2 + (t+h) \\ f(☺) &= (☺)^2 + ☺ \end{aligned}$$

1.3 Geometry Formulas

In this section we discuss some of the classic shapes and solids from geometry.

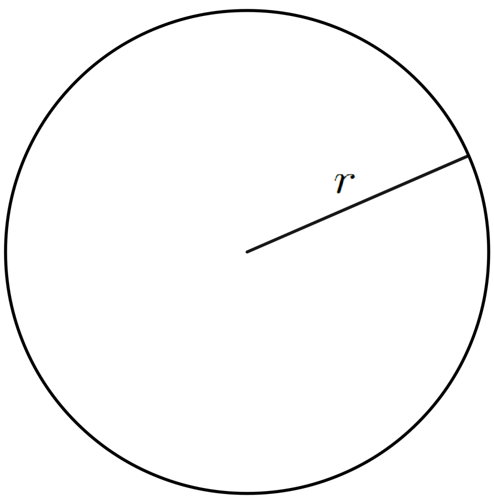
For a circle, let r be the radius of the circle.

- A line segment.



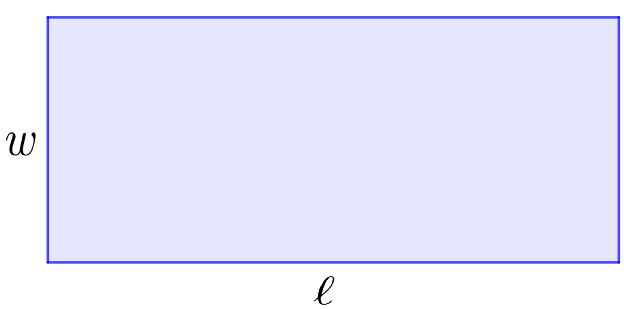
The length of a horizontal line on the interval $[a, b]$ is $b - a$.
The length of a line in the plane defined between two points (x_1, y_1) and (x_2, y_2) is given by $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

- A Circle.



Let the radius of the circle be given by r.
The circumference of the circle is given by $C = 2\pi r$.
The area of the circle is given by $A = \pi r^2$.

- A rectangle.

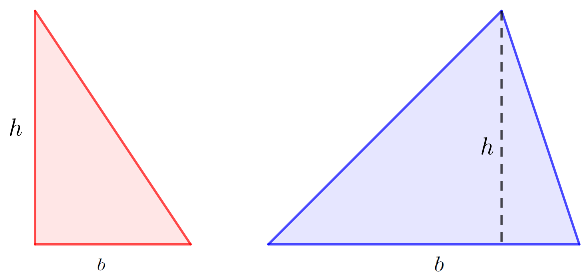


Suppose we call the length ℓ and the width w.
The perimeter of the rectangle is given by $P = 2\ell + 2w$.
The area of the rectangle is given by $A = \ell \cdot w$.

- A trapezoid.

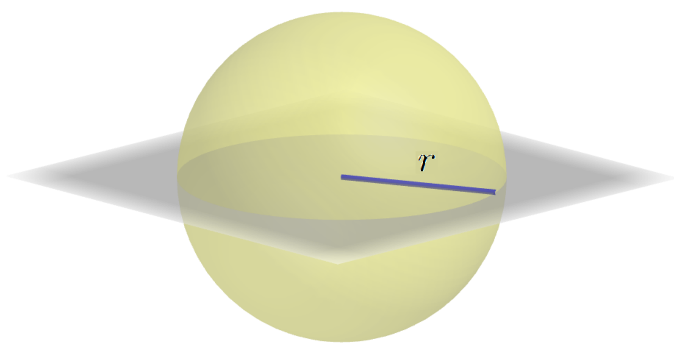
If we orient the trapezoid so that the parallel sides stretch left-to-right, then we can call the bottom side b_1, the top side b_2, and the distance between them h .
The perimeter of the trapezoid is the sum of the lengths of the sides.
The area of the trapezoid is given by
$A = \frac{1}{2}(b_1 + b_2)h$.

- A triangle.



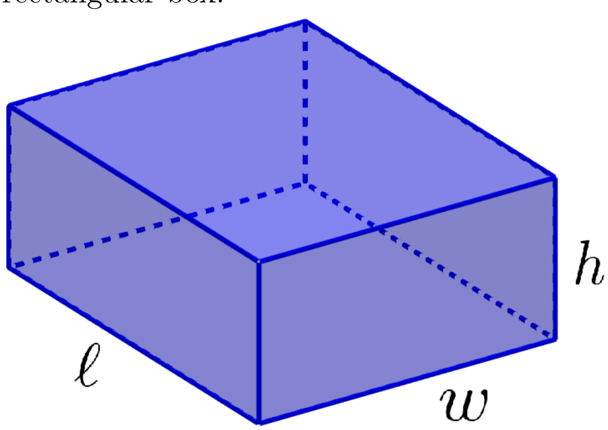
The perimeter of a triangle is the sum of the side lengths.
The area of a triangle is $A = \frac{1}{2}bh$.
If the triangle is a right triangle, the side lengths satisfy the Pythagorean Theorem: $a^2 + b^2 = c^2$.

- A sphere.



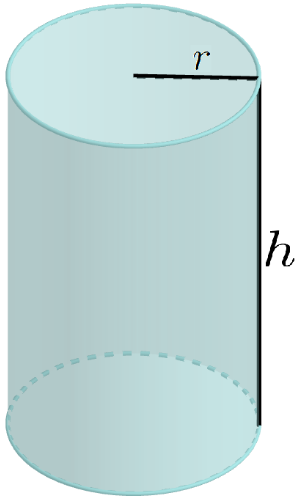
Let the radius of the sphere be given by r.
The surface area of the sphere is given by
$S = 4\pi r^2$.
The volume of the sphere is given by $V = \frac{4}{3}\pi r^3$.

- A rectangular box.



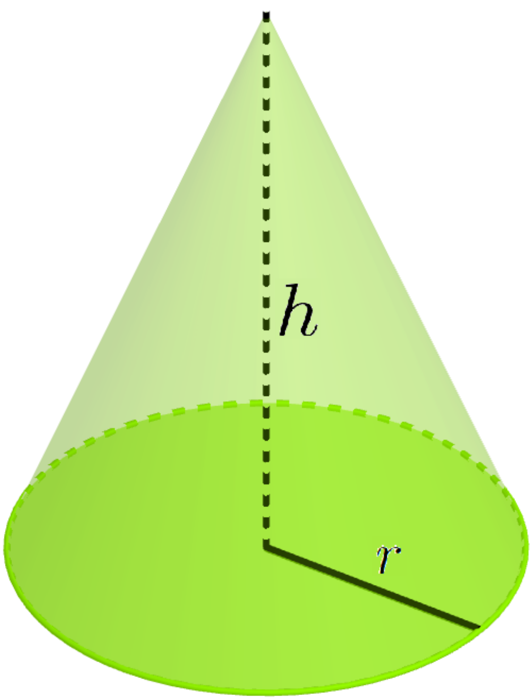
Let the lengths of the three defining sides be $\ell, w,$ and h.
The surface area of the box is
$S = 2(\ell w + \ell h + wh)$.
The volume of the box is $V = \ell wh$.

- A right cylinder.



Let the radius of the base circle be r, and the height be h.
The surface area of the cylinder is
$S = 2\pi rh + 2\pi r^2$.
The volume of the cylinder is $V = \pi r^2 h$.

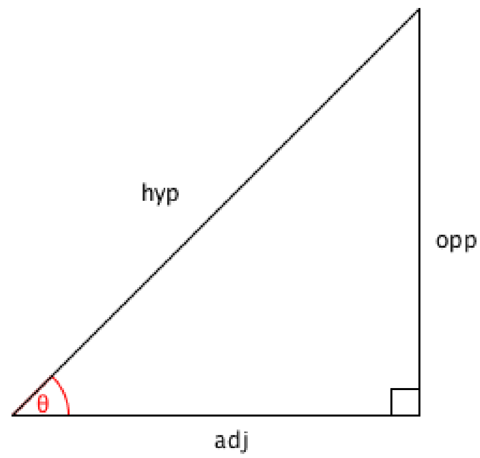
- A right circular cone.



Let the radius of the base circle be r, and the height of the cone h.
The surface area of the cone is
$S = \pi r(r + \sqrt{h^2 + r^2})$.
The volume of the cone is $V = \frac{1}{3}\pi r^2 h$.

1.4 Trigonometric Identities

Suppose we have a right triangle and we choose an angle θ. Then we can define the sides to be opposite, adjacent, and the hypotenuse. The six trigonometric functions can then be defined as follows.



- $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$
- $\cos(\theta) = \dfrac{\text{adj}}{\text{hyp}}$
- $\tan(\theta) = \dfrac{\text{opp}}{\text{adj}}$
- $\csc(\theta) = \dfrac{\text{hyp}}{\text{opp}}$
- $\sec(\theta) = \dfrac{\text{hyp}}{\text{adj}}$
- $\cot(\theta) = \dfrac{\text{adj}}{\text{opp}}$

Since this is for a right triangle, we know that the sides obey the Pythagorean Theorem. That means $(\text{adj})^2 + (\text{opp})^2 = (\text{hyp})^2$, or in other words,

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

We can expand our definition of $\sin(\theta)$ and $\cos(\theta)$ to apply to all real numbers by making the functions periodic with period 2π. That is,

$$\sin(\theta + 2k\pi) = \sin(\theta) \qquad \text{and} \qquad \cos(\theta + 2k\pi) = \cos(\theta),$$

for any integer k. Then we have the graphs as in section 1.1.

1.4.1 Sum and Difference Formulas

- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

1.4.2 Double Angle and Half-Angle Formulas

The double angle formulas can be found by plugging in θ for both α and β in the sum formulas. They are:

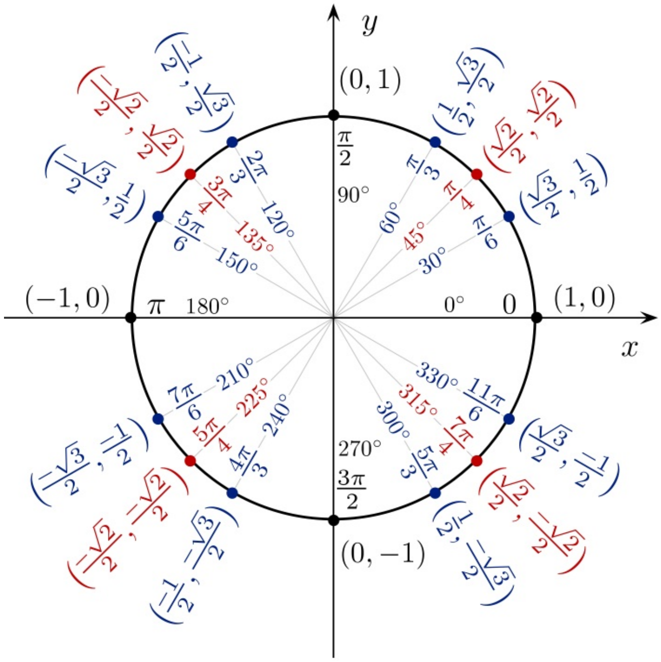
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta) = 2\cos^2(\theta) - 1$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}.$

The half-angle formulas can be found from the double angle formulas by solving for the term that has just θ instead of 2θ. They are:

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos\theta}} = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$.

1.5 The Unit Circle

Below is the unit circle. In most math courses, radians will be used as the units for angles. It is usually sufficient to memorize the first quadrant values for sine and cosine, then apply what you know about the properties of the two functions to find the values for the other three quadrants.



First quadrant values

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	1	DNE

1.6 Inverses

We have been following the same pattern since kindergarten when learning math operations: We learn how to do something, then how to undo it. We started with addition, then learned subtraction. We learned multiplication, then division. We learned about composition, and now we need to know how to undo compositions of functions with inverses. Loosely speaking, an inverse is a function that, when applied to an expression, will "cancel" the outer most operation. For example, if we have the equation $\frac{x+2}{3} = 4$, we could apply the multiplication operator by 3 to both sides to cancel the division that is happening. That would look like $3 \cdot \frac{x+2}{3} = 4 \cdot 3$, or simplified to $x+2 = 12$. Then we could apply the subtraction operator to cancel the 2 that is being added to the x. This would look like $x + 2 - 2 = 12 - 2$, which is $x = 10$. A way to find the graph of the inverse function from the original is to flip the graph of the original function across the line $y = x$. To find a formula for the inverse of a function, we change all the x's to y and all the y's to x and re-solve for y. We now give some of inverses to the parent functions from section 1.1.

1.6.1 Power Functions

Recall the parent functions $f(x) = x^{p/q}$ where p/q is a positive rational number. In order to cancel out the power and get just x, we need to apply another function to $f(x)$. Since composing power functions inside of power functions multiplies their exponents, we can see that the inverse of $x^{p/q}$ is $x^{q/p}$. We check by showing that $(x^{p/q})^{q/p} = x = (x^{q/p})^{p/q}$.

1.6.2 Exponential and Logarithm Properties

Our parent functions a^x and $\log_a(x)$ are inverses of each other. Thus, $a^{\log_a(x)} = x = \log_a(a^x)$.

Recall that exponential functions have the following three properties:

1. $a^b \cdot a^c = a^{b+c}$ multiplication in the base becomes addition in the exponent.
2. $\dfrac{a^b}{a^c} = a^{b-c}$ division in the base becomes subtraction in the exponent.
3. $(a^b)^c = a^{b \cdot c}$ exponentiation of the base becomes multiplication in the exponent.

Logarithms have similar properties. They are:

1. $\log_x(b \cdot c) = \log_x(b) + \log_x(c)$ multiplication in the log becomes addition on the outside.
2. $\log_x\left(\dfrac{b}{c}\right) = \log_x(b) - \log_x(c)$ division in the log becomes subtraction on the outside.
3. $\log_x(b^c) = c\log_x(b)$ exponentiation inside the log becomes multiplication on the outside.

1.6.3 Arc-trig Functions

The arc-trig functions are the inverses of the usual trigonometric functions. However, since the arc-trig functions are only functions for a restricted range, we need to be careful with our definitions. As long as we define the domain of $\sin(x)$ and the range of $\arcsin(x)$ to be the same interval, then we get

$$\sin^{-1}(\sin(x)) = x = \sin(\sin^{-1}(x)).$$

We need this since if we consider the usual strip for $\sin^{-1}(x)$, we would get $\sin^{-1}(\sin(3\pi)) = \sin^{-1}(0) = 0$, and $3\pi \neq 0$. But if we define $\sin^{-1}(x)$ to have range $[5\pi/2, 7\pi/2]$, then $\sin^{-1}(\sin(3\pi)) = \sin^{-1}(0) = 3\pi$.

We can do a similar thing with the domains and ranges for the other two trig and arc-trig pairs to get

$$\cos^{-1}(\cos(x)) = x = \cos(\cos^{-1}(x)),$$
$$\tan^{-1}(\tan(x)) = x = \tan(\tan^{-1}(x)).$$

Sometimes we want the angle that will result in a specific value from a trig function. For instance, what angle would you need to plug into $\cos(x)$ in order to get $\sqrt{3}/2$? This is equivalent to writing $\arccos(\sqrt{3}/2) = \theta$. A way to solve this is by knowing the unit circle and using a guess-and-check process to solve $\cos(\theta) = \sqrt{3}/2$. In particular, since $\cos(\theta)$ represents the x-coordinate of a point on the unit circle, we are looking for an angle that results in the x-coordinate being $\sqrt{3}/2$. Looking back in section 1.1 at our parent function $\arccos(\theta)$, we know the range is $[0, \pi]$. Thus, our answer will be some angle(s) in $[0, \pi]$ that have an x-coordinate of $\sqrt{3}/2$. Using our unit circle from section 1.5, we see that the only angle in $[0, \pi]$ with a corresponding x-coordinate of $\sqrt{3}/2$ is $\theta = \pi/6$.

1.7 Precalculus Transformations

Once we know our parent functions in section 1.1, we also need to know how to move their graphs around and what that does to their expressions. In the following, consider $f(x)$ to be the parent function. We define

$$g(x) = cf(b(x-a)) + d$$

as the transformation of f. The transformations are a combination of reflections, translations, and stretching. In order to find which ones, we work from the inside to the outside with the following rules in mind:

- If $a > 0$, then the graph of f is translated to the right by a. If $a < 0$, then the graph is translated to the left by a.
- If $|b| > 1$, the f is compressed left and right by a factor of $|b|$. If $|b| < 1$, then the graph of f is stretched left and right by a factor of $|b|$. If $b < 0$, then the graph is reflected about the y-axis.
- If $|c| > 1$, the f is stretched up and down by a factor of $|c|$. If $|c| < 1$, then the graph of f is compressed up and down by a factor of $|c|$. If $c < 0$, then the graph is reflected about the x-axis.
- If $d > 0$, then the graph of f is translated up by d. If $d < 0$, then the graph of f is translated down by d.