Julia Sets, the Mandelbrot set, and Complex Dynamics

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1. Introduction

In the complex setting we have many desirable properties that may be lacking in the real setting. For instance, the algebraic closure of $\mathbb{C}$ gives us that every polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$, so if we iterate polynomial maps, we will always have periodic points. We can also leverage the rich theory of complex analysis. Thus, we may want to lift real dynamics to the complex setting where we can apply these powerful tools, and then reduce back to the real setting. For example, below we will study a parameter space which contains the parameter space for the logistic map, and we will use our knowledge in complex setting to better understand the logistic map, which is a real dynamical system. In this paper, our main tools for this process will be the Julia and Mandelbrot set. These also have the merit that they are incredibly interesting in themselves.

2. Julia Sets and the Mandelbrot Set

2.1 Julia Sets. We will be interested in iterating polynomial maps on the complex plane, more specifically, quadratic polynomials. Most of the following results can be extended to polynomials of arbitrary degree, and some of the results extend to rational functions. Of course, a general quadratic looks like

$$az^2 + bz + d,$$

but in fact, we can restrict our attention to quadratics of the form

$$f_c(z) = z^2 + c,$$

because these two maps are conjugate. Let $c = ad + \frac{b}{2} - \left(\frac{b}{2}\right)^2$ and $\varphi(z) = az + \frac{b}{2}$, then

$$
\begin{aligned}
\varphi^{-1} \circ f_c \circ \varphi(z) &= \varphi^{-1} \circ f_c \left(az + \frac{b}{2} \right) \\
&= \varphi^{-1} \left(\left(az + \frac{b}{2} \right)^2 + ad + \frac{b}{2} - \left(\frac{b}{2} \right)^2 \right) \\
&= \varphi^{-1}(a^2 z^2 + baz + ad + \frac{b}{2}) \\
&= \frac{(a^2 z^2 + baz + ad + \frac{b}{2}) - \frac{b}{2}}{a} \\
&= az^2 + bz + d
\end{aligned}
$$

We will define the *Julia set of f, J(F)*, to be the closure of the set of repelling periodic points. It turns out that, an equivalent definition is set of points for which iterates of f behave chaotically. Chaotically, here, means nearby trajectories diverge rapidly. The complement of $J(f)$ will be called the *Fatou set,* $F(f)$; these sets were named after Gaston Julia and Pierre Fatou, pioneers in the study of complex dynamical systems.

Let us briefly illustrate these definitions, and their equivalents, with the following example. Let $f_0(z) = z^2 + 0 = z^2$. We know what this mapping does to the complex plane: it takes a point

$$re^{i\theta} \mapsto r^2 e^{i2\theta}.$$

From this it is clear that the only periodic points can lie on the unit circle, which would be of the form $z = e^{i\theta}$. For these points

$$f_0^n(z) = e^{i2^n\theta}.$$

Periodicity would require that

$$2^n\theta \equiv \theta \pmod{2\pi}$$

$$(2^n - 1)\theta \equiv 0 \pmod{2\pi}$$

From this we see that θ must be a rational multiple of π, and, in reduced form, cannot have denominator divisible by 2. The set of these z is dense in the unit circle, and because the unit circle is a repelling cycle, these points are repelling periodic points. Thus, the Julia set $J(f_0)$ is S^1, and the Fatou set is its complement $\mathbb{C} \setminus S^1$. Visualizing the trajectories of points inside, outside, and near the unit circle, we can also see that the unit circle is the set of points for which nearby points create diverging orbits.

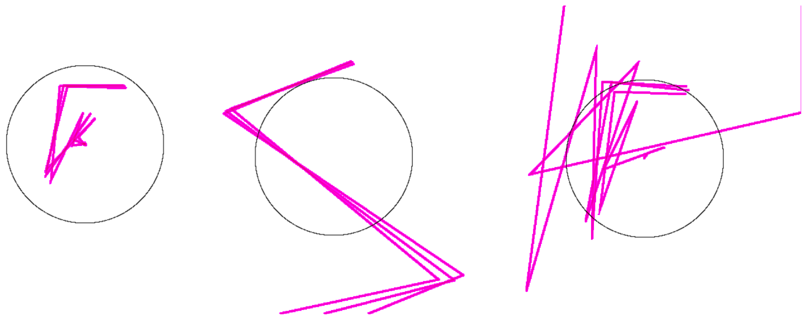

Figure 1: Trajectories of nearby orbits for initial data away from, and then near the Julia set for f_0.

The Julia set for this map is fairly simple, but generally speaking, Julia sets can be very complicated, and are rarely as easy to find or parametrize. They can be simple closed loops, fractals, dust-like, dendrites, or totally disconnected. Figure 2 below shows six different values of c and the Julia sets associated to those parameters. To be precise, the Julia set is the boundary of the set colored in black. Below we will discuss the significance of the other colors.

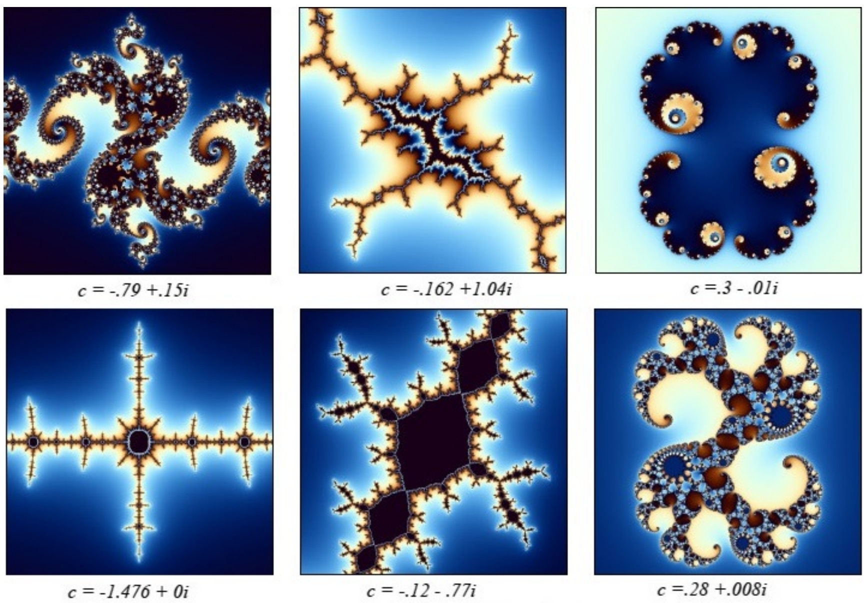

Figure 2: Julia sets for 6 different values of c

2.2 Properties of Julia sets. In this section, we will merely state some important properties of Julia sets, and the implications that these properties have. For proofs of these facts, we refer the reader to [2]. In short, this text proves these properties by studying properties of normal families of analytic functions (the family being f_c and its iterates) and then connects the lack of normality of a family, at a point, to the point not being in the Julia set.

The first of our properties is that the Julia set is f and f^{-1} invariant:

$$J(f) = f(J(f)) = f^{-1}(J(f)).$$

By taking complements, we immediately get that the same is true for the Fatou set as well. Recall the $c = 0$ case; clearly, S^1 and its complement are f and f^{-1} invariant.

Our next property is of significant practical importance. If w is an attractive fixed point (possibly ∞) then $J(f) = \partial A(w)$, where $A(w)$ is w's basin of attraction. Take the circle example again; We had two attractive fixed points $w = 0, \infty$ with basins $A = \{z : |z| < 1\}$ and $A = \{z : |z| > 1\}$, respectively, both of which have boundary S^1. This fact is useful in practice for finding the Julia set, as ∞ is always an attracting fixed point. To numerically estimate the Julia set, one can establish a large threshold M, and interate f for a large number of iterations, and see if $f^n(z) < M$ for all of those iterates. If it does not, then one would say that the point is in $A(\infty)$, and the Julia set would be the boundary of such set of points. This is the basis of the coloring in figure 2; the coloring is indicative of how many interates it took that initial value to surpass the threshold value. Note too that, this implies that the Julia set is bounded, and it is closed by definition, so in fact $J(f)$ is always compact.

Also, the Julia set always has empty interior. Hence, why, in figure 2, the Julia set is the boundary of the set colored in black (the entire set colored black is called the *filled in Julia set* and is the set of points that remain bounded under iteration). However, for a generic c, the Hausdorff dimension of $J(f_c)$ is actually 2.

If $|c| < \frac{1}{4}$, then $J(f_c)$ is a simple closed curve; the parameter c has not been perturbed far enough from 0 to drastically change the topology of the Julia set. However, if $|c| > \frac{1}{4}(5 + 2\sqrt{6})$ then $J(f_c)$ is drastically different, in fact, it is totally disconnected (i.e. $J(f_c)$ is dust).

Our final property, one of great significance, is that $J(F)$ is connected if and only if $0 \notin A(\infty)$. It turns out that the parameters c, for which the Julia set are connected, are most interesting, which brings us to the definition of the Mandelbrot set.

2.3 The Mandelbrot Set. The *Mandelbrot set*, M, is defined as follow

$$M := \{c : J(f_c) \text{ is connected}\}.$$

By the last property we listed about Julia sets, $c \in M$ if and only if $\{f_c^n(0)\}$ is a bounded set, and by the previous properties, we know it contains the circle of radius $\frac{1}{4}$, centered at zero, and is contained in the circle of radius $\frac{1}{4}(5+2\sqrt{6}) \approx 2.4747$. Figure 3 contains a picture of the Mandelbrot set (colored in black).

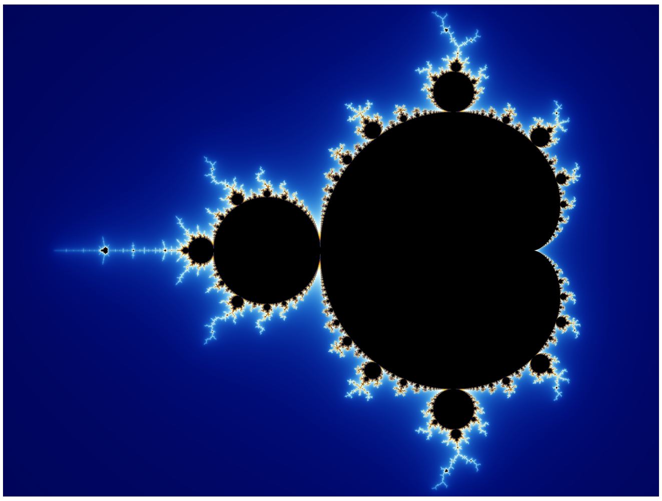

Figure 3: The Mandelbrot set

2.4 Properties of the Mandelbrot Set. Again, we will list properties without formal justification; we refer to references for the proofs. Our first properties are about the topology of M. M is connected, and even simply connected, however, it is a open problem to determine if M is locally connected.

M is self-similar; there are mini-Mandelbrot sets at arbitrarily small scales all along the boundary of M. However, M is not fractal: the Hausdorff dimension of M is 2. What is remarkable, however, is that the Hausdorff dimension of ∂M is 2. Recall that this is the same as the Hausdorff dimension of a generic Julia set. This is related to the following, interesting, qualitative result: On the boundary of M, M looks like the Julia sets associated to nearby values of c. This is demonstrated in figure 4.

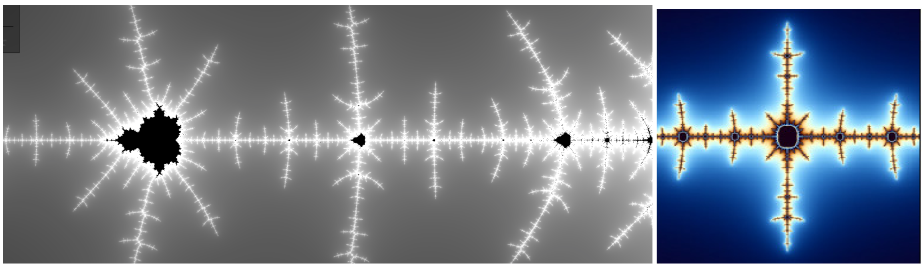

Figure 4: The Mandelbrot set near $c = -1.476$ (left) and the Julia set associated to that parameter (right)

Our final property is of a qualitative nature as well. The parameters c in the same 'bulb' of M have very similar Julia sets. For example, we already know that for $|c| < \frac{1}{4}$ the Julia sets associated to these values are simple closed curves, but even better, for all c in the main cardioid of M (the main bulb, which includes the disk of radius $\frac{1}{4}$), the Julia sets for these values are simple closed curves.

3. Application: Logistic Map

Conisder the following application to the logistic map

$$z \mapsto \lambda z(1-z)$$

for $\lambda \in [1,4]$. Since this is a quadratic map, it is conjugate to f_c for

$$c = 0 + \frac{\lambda}{2} - \left(\frac{\lambda}{2}\right)^2 = \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right)$$

Notice that

$$\frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right) : \lambda \in [1,4] \leftrightarrow c \in \left[-2, \frac{1}{4}\right]$$

So we get a bijective correspondence between parameters c, on an interval of the real axis, and the λ for the logistic map. Further, we can see qualitative aspects of the Mandelbrot set connecting to the period doubling bifurcation of the logistic map in figure 5.

Figure 5: Connection between bulb formation and period doubling

This connection can be made precise through the discussion of bulb formation: when a bulb of the Mandelbrot set forms off another bulb, we get a change in the period of the attracting cycle. In the case of the bulbs forming along the real axis, we get a period doubling with bulb formation.

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