Rational Zeroes to Polynomials with Integer Coefficients

The following is a short note to explain some details of the last example in class, there is a problem at the end which you should hand in on Wednesday 10th October as a quiz assessment.

Let $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1} + c_n x^n$ be a polynomial with $c_i \in \mathbb{Z}$ for all $0 \leq i \leq n$. Suppose that $x = \frac{a}{b}$ is a rational solution of $f(x)$ in its lowest form, i.e. $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$.
Claim: a divides c_0 and b divides c_n.
Proof of Claim: We can write the following equation:

$$0 = f\left(\frac{a}{b}\right) = c_0 + c_1\left(\frac{a}{b}\right) + c_2\left(\frac{a}{b}\right)^2 + \cdots + c_{n-1}\left(\frac{a}{b}\right)^{n-1} + c_n\left(\frac{a}{b}\right)^n$$

We can clear denominators by multiplying by b^n:

$$0 = f\left(\frac{a}{b}\right) = c_0 b^n + c_1 a b^{n-1} + c_2 a^2 b^{n-1} + \cdots + c_{n-1}a^{n-1}b + c_n a^n$$

Now observe that:

$$\underbrace{0}_{\text{divisible by } b} = f\left(\frac{a}{b}\right) = \underbrace{c_0 b^n + c_1 a b^{n-1} + c_2 a^2 b^{n-1} + \cdots + c_{n-1}a^{n-1}b}_{\text{divisible by } b} + c_n a^n$$

so b must divide $c_n a^n$. But b cannot divide a^n because a and b have no common factors, so b must divide c_n.
By a similar argument, a must divide c_0. ■
This is sometimes called the *rational root theorem.*

Consequence: We have an algorithm to determine all rational roots of an integral polynomial by considering the prime factors of c_0 and c_n.

Exercises: Using the rational root theorem, compute all the rational roots to the following polynomials:

1. $5x^2 + 7x + 2$
2. $2x^2 + 5x - 12$
3. $70x^3 + x^2 - 3x + 51$