Selected Solutions for 2.6, 2.7, 2.8, 3.1, 3.2

2.6

This is a problem from the review worksheet that a few people have had problems with:

A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. Express the combined total area of the circle and the square as a function of c, where c represents the length of the wire that is used for the circle.

If c amount of wire is used to make the circle, then $12 - c$ is used to make the square. Each side of the square must then have length $\frac{12-c}{4}$, and so the area of the square is given by $A_{\square} = \frac{(12-c)^2}{16}$.
The area of a circle is given by $A_{\circ} = \pi r^2$, where r is the radius of the circle. We want the $A_{\circ}$ to be given in terms of c, so we can use $c = 2\pi r \implies r = \frac{c}{2\pi}$,

$$A_{\circ}(c) = \pi(\frac{c}{2\pi})^2 = \frac{c^2}{4\pi}$$

And finally the total area is

$$A_{\text{total}}(c) = A_{\square} + A_{\circ} = \frac{(12-c)^2}{16} + \frac{c^2}{4\pi}$$

Remark: Check that the RHS only involves the variable c.

Section 2.7

18. $f(x) = 3x - 5$, $g(x) = 2 - x^2$. Compute:

1. $f(f(4))$
2. $g(g(3))$

There are a couple of ways to attempt this question. You may want to compute the number $f(4) = 3(4) - 5 = 7$ and then input that into $f(x)$ again. So we have $f(f(4)) = f(7) = 3(7) - 5 = 21 - 5 = 16$. Alternatively, you can first compute $f(f(x)) = 3(3x-5) - 5 = 9x - 20$, so $f(f(4)) = 9(4) - 20 = 16$. Similarly, $g(g(3)) = -47$.
Remark: This problem is not difficult but many mistakes are caused by lack of working out. Try not to skip steps in composition of functions because you may miss some details.

36. $f(x) = x - 4$, $g(x) = |x + 4|$.

1. $f \circ f(x) = (x - 4) - 4 = x - 8$.
2. $g \circ g(x) = ||x + 4| + 4| = |x + 4| + 4$. (since $|x + 4| + 4$ is always +ve)
3. $f \circ g(x) = |x + 4| - 4$.
4. $g \circ f(x) = |(x - 4) + 4| = |x|$.

Since the domain of $f(x)$ and $g(x)$ is $\mathbb{R}$, the domains of their compositions are also $\mathbb{R}$.
Remark: $|x + 4| - 4 \neq |x|$, consider $x = -5$ for example.

50. Write $\sqrt{1 + \sqrt{x}}$ as a function of the form $f \circ g$.
There are many answers to this question, one of the simplest and most common one is $f(x) = \sqrt{1 + x}$, $g(x) = \sqrt{x}$.
Remark: Remember to check that your order of composition is correct.

Section 2.8

46. Compute the inverse of $f(x) = (2 - x^3)^5$.
Let $y = f(x)$, then $y = (2 - x^3)^5$.
First switch the letters x and y so $x = (2 - y^3)^5$, and then rearrange to isolate y.
$\implies \sqrt[5]{x} = 2 - y^3 \implies y^3 = 2 - \sqrt[5]{x} \implies y = \sqrt[3]{2 - \sqrt[5]{x}}$.
Remark: A common mistake here was taking the cuberoot of $-y^3$ then getting rid of the negative sign. You should eliminate the negative sign first *then* take the cuberoot.

66. $g(x) = (x - 1)^2$. Give a restricted domain so that the function is one-to-one then write down the inverse function.
The point is that we don't want any y-values to be repeated as our output, in otherwords, we need to restrict the graph so that it passes the horizontal line test. A very natural restriction would be $x \leq 1$ or $x \leq 1$, both are valid answers, but would give a different inverse function.
(i) If $x \geq 1$, then $f^{-1}(x) = \sqrt{x} + 1$, with domain $x \geq 0$.
(ii) If $x \leq 1$, then $f^{-1}(x) = -\sqrt{x} + 1$, with domain $x \geq 0$.
Remark: The inverse function in (i) (resp. (ii)) would work for *any* restriction of $x \geq a$ (resp. $x \leq a$) where $a \geq 1$ (resp. $a \leq 1$) but then the domain of the inverse will not be $x \geq 0$.

72. $V(t) = 100(1 - \frac{t}{40})^2$.

1. Compute V^{-1}. Interpret what the inverse means.
2. Compute $V^{-1}(15)$. Interpret what this means.

(a) First "switch" the letters: $t = 100(1 - \frac{V^{-1}(t)}{40})^2$ and rearranging gives $V^{-1}(t) = 40 - 4\sqrt{t}$.
$V^{-1}(t)$ represents the time required to drain until t amount of water is remaining in the tank.
(b) $V^{-1}(15) = 40 - 4\sqrt{15} \approx 24.5$ minutes. This means it takes ≈ 24.5 minutes for the tank to be drained with 15 gallons of water left.
Remark: If $V(t)$ is a function that takes time as the input and volume as the output, the inverse $V^{-1}(t)$ should have volume as the input and time as the output, so you should at least recognise $V^{-1}(15)$ as a quantity of time.

Section 3.1

34. $P(x) = x^4 - 2x^3 + 8x - 16$. Factorise and graph $P(x)$.
Most of you recognised that $P(x) = (x^3 + 8)(x - 2)$ without much working out. To justify your logic, you can either plot the graph $P(x)$ and spot the root $x = 2$, or just by pure speculation (guessing the roots by plugging in numbers). In either case, you should then compute $\frac{P(x)}{(x-2)}$, which should give you $x^3 + 8$ (note that there is no remainder since $x = 2$ is a root of $P(x) \implies (x - 2)$ divides $P(x)$).
The next step is to try to factorise $x^3 + 8$. Again, you should spot that $x = -2$ is a root of $x^3 + 8$, and then compute $\frac{x^3+8}{x+2} = x^2 - 2x + 4$. You may want to try and factorise $x^2 - 2x + 4$ but a quick application of the quadratic formula shows that $x^2 - 2x + 4$ cannot be factorised.
So finally, $P(x) = (x+2)(x-2)(x^2-2x+4)$, so $P(x)$ has two real roots, and hence only two x-intercepts at $x = -2$ and $x = 2$.
The end behaviour is like x^4, so $x \to \pm\infty \implies y \to +\infty$. You could also compute the y-intercept by setting $x = 0$ for $P(x)$.
Remark: You should see that $P(x) \neq P(-x)$ and so is not an even function, so your graph should not be symmetric about the y-axis.

58. Compute the number of local extrema for $y = 6x^3 + 3x + 1$ by graphing.
Straight forward use of calculator. This graph is increasing at all points so there is no local minimum or maximum.

78. $P(t) = 120t - t^4 + 1000$.

1. Compute maximum population and when it occurs.
2. Compute when the population goes to zero.

(a) Another application of calculator here. The maximum population is represented by the local maximum of the graph. Which is approximately given by the point $(4.22, 1380)$, i.e. maxumum population is about 1380 at about 4.22 months.
(b) The population hits zero when the y value is zero i.e. the x-intercept, which is approximately 8.42 months.

3.2

40. $P(x) = x^3 - x^2 + x + 5$, $c = -1$. Use the synthetic division and the remainder theorem to compute $P(-1)$.
First of all, you *must* use the stated method to compute $P(-1)$ even if direct computation is very easy here.
Synthetic division gives:

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 1 & 5 \\ \hline & & -1 & 2 & -3 \\ \hline & 1 & -2 & 3 & 2 \end{array}$$

The remainder of $P(x)$ when divided by $(x - (-1))$ is 2 so $P(-1) = 2$.
Remark: There has been some confusion about the number on the top left of the table. It is given by c, not $-c$, where $P(x)$ is being divided by $(x - c)$.

56. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6$, $c = \frac{1}{3}, -2$.

1. Verify that $c = \frac{1}{3}$ are -2 are zeroes.
2. Compute all other zeroes of $P(x)$.

(a) The quickest way to check these are zeroes is by plugging the numbers into $P(x)$.
(b) Since $x = \frac{1}{3}$ and $x = -2$ are zeroes of $P(x)$, we know that $(x - \frac{1}{3})$ and $(x + 2)$ both divide $P(x)$, so

$$P(x) = (x - \frac{1}{3})(x + 2)g(x)$$

for some quadratic $g(x)$. Using division algorithm (or repeated use of synthetic division), we find

$$g(x) = \frac{P(x)}{(x - \frac{1}{3})(x + 2)} = 3x - 6x - 9$$

But $3x - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$ So

$$P(x) = 3(x - \frac{1}{3})(x + 2)(x - 3)(x + 1)$$

and all zeroes of $P(x)$ are $x = \frac{1}{3}, -2, 3, -1$.
Remark: If you cannot factorise the quadratic, we may want to use the quadratic formula to compute the remaining zeroes.

64. We can see that there are only two zeroes, namely $x = -1$ and $x = 2$. You should notice that $x = 2$ is a root with even multiplicity. By the factor theorem we must have $P(x) = k(x+1)(x-2)^2$, where k is some constant. (Notice any higher even power of $(x-2)$ makes our polynomial of degree higher that 3.)
To compute the value of k, we can make use of the y intercept. When $x = 0$, $y = 4$, so $4 = P(0) = k(1)(-2)^2 = 4k \implies k = 1$.
So $P(x) = (x+1)(x-2)^2$.