~~Try this~~

1. Zeta Functions

$\mathbb{F}_q$, $q = p^a$, p prime

$f(x_1, \dots x_n) \in \mathbb{F}_q[x_1^{\pm 1}, \dots x_n^{\pm 1}]$

$U_f = \{(x_1, \dots x_n) \in \mathbb{G}_m^n \mid f(x) = 0\}$

$U_f(\mathbb{F}_q) = \{(x_1, \dots x_n) \in \mathbb{F}_q^{*n} \mid f(x) = 0\}$

$U_f(\mathbb{F}_{q^k})$, $k = 1, 2, 3, \dots$

Def. $Z(U_f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \# U_f(\mathbb{F}_{q^k})\right)$

$\in 1 + T\mathbb{Z}[[T]]$

Basic Properties/Questions.

1) $Z(U_f, T) \in \mathbb{Q}(T)$.

$$Z(U_f, T) = \frac{\prod_{i=1}^{d_1}(1 - \alpha_i T)}{\prod_{j=1}^{d_2}(1 - \beta_j T)}, \quad \alpha_i, \beta_j \in \overline{\mathbb{Q}}$$

$$\# U_f(\mathbb{F}_{q^k}) = \sum \beta_j^k - \sum \alpha_i^k,$$

$k = 1, 2, 3 \dots$

2) If f is Δ-regular, $\Rightarrow$

$$Z(U_f, T) = \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i}\binom{n}{i+1}} .$$

$$P(f, T)^{(-1)^n}.$$

$$P(f, T) = \prod_{j=0}^{d(\Delta)-2} (1 - \beta_j T) \in 1 + T\mathbb{Z}[T]$$

of deg $d(\Delta) - 1$.

$$\Rightarrow \# V_f(\mathbb{F}_{q^k}) = \frac{(q^k-1)^n + (-1)^{n+1}}{q^k} + (-1)^{n+1}(\beta_0^k + \beta_1^k + \cdots + \beta_{d(\Delta)-2}^k).$$

$$\Rightarrow \quad |\beta_j| \le q^{\frac{n-1}{2}}, \qquad |\beta_j| = \sqrt{q}^{\,w_j}. \qquad w_j \in [0, n-1] \cap \mathbb{Z}.$$

(weight)

w_j can be determined.

$|\beta_j|_\ell = 1.$ $\qquad \ell$ prime $\neq p$

$\mathrm{ord}_\ell\, \beta_j = 0.$

3). f Δ-regular, $\Rightarrow$

$|\beta_j|_q = q^{-s_j}$ $\qquad s_j \in [0, n-1] \cap \mathbb{Q}$

$s_j = ?$ (q-adic slope of β_j).

(related to Hodge numbers).

4). How s_j and $Z(U_f, T)$ vary as f varies ?

5) What more can be said about $Z(U_f . T)$ if f is a CY.

2. L-function of exp sums.

Let $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^*$

$x \longrightarrow \exp\left(\frac{2\pi i x}{p}\right)$

$$\mathbb{F}_{q^k} \xrightarrow{\mathrm{Tr}} \mathbb{F}_p \xrightarrow{\psi} \mathbb{C}^*$$

$\psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}$:

$$S_k(x_0 f) = \sum_{\substack{x_i \in \mathbb{F}_{q^k}^* \\ 0 \le i \le n}} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 f) \in \mathbb{Z}$$

$$q^k \# U_f(\mathbb{F}_{q^k}) = \sum_{\substack{x_i \in \mathbb{F}_{q^k}^* \\ 1 \le i \le n}} \sum_{x_0 \in \mathbb{F}_{q^k}} \psi \circ \mathrm{Tr}(x_0 f)$$

$$= (q^k - 1)^n + S_k(x_0 f)$$

Def. $L(\chi_0 f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k(\chi_0 f) \right)$

$\Rightarrow$ $Z(U_f, qT) = Z(\mathbb{G}_m^n, T) \cdot L(\chi_0 f, T)$

$\Rightarrow$ enough to study $L(\chi_0 f, T)$.

3. Dwork's p-adic character

Def. The Artin - Hasse series

$$E_p(T) = \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots\right)$$

$$= \prod_{(k,p)=1} (1 - T^k)^{-\frac{\mu(k)}{k}}, \quad \mu = \text{Möbius}.$$

$$\in 1 + T(\mathbb{Z}_p \cap \mathbb{Q})[[T]].$$

$\Rightarrow$ $E_p(T)$ converges in $|T|_p < 1$. NOT on $|T|_p \leq 1$.

Def. Let π be a fixed root of

$$T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots = 0 \quad \text{in } \overline{\mathbb{Q}_p}$$

s.t. $\operatorname{ord}_p(\pi) = \frac{1}{p-1}$ (exactly $p-1$ such roots)

$\mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_p)$, π is a uniformizer. $\pi \sim 1-\zeta_p$

Def. $\theta(T) = E_p(\pi T)$ is convergent in $|T|_p < p^{\frac{1}{p-1}}$.

$= 1 + \pi T + \cdots$

Prop $\theta(1) = 1 + \pi \pmod{\pi^2}$ $\theta(1) \neq 1$.

$\theta(1)^p = 1$.

Def. ψ. $\mathbb{F}_p \longrightarrow \mathbb{C}_p^*$

$\bar{x} \to \theta(1)^{\bar{x}} = \theta(x)$, $x = \mathrm{Teich}(\bar{x})$. $x^p = x$.

$\psi \circ \mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}_p} : \mathbb{F}_{p^k} \xrightarrow{\mathrm{Tr}} \mathbb{F}_p \xrightarrow{\psi} \mathbb{C}_p^*$. $\psi \circ \mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}_p}(\bar{x}) = \theta(x)\theta(x^p)\cdots\theta(x^{p^{k-1}})$

4. p-adic rep of $S_k(x_0 f)$.

Write $x_0 \bar{f} = \sum_{j=1}^{J} \bar{a}_j x_0 x^{V_j}$, $\bar{a}_j \in \mathbb{F}_q$.

$\in \mathbb{F}_q[x_0, x_1^{\pm 1}, \cdots x_n^{\pm 1}]$. $\quad q = p^a$.

$$S_k(x_0 f) = \sum_{\bar{x}_i \in \mathbb{F}_{q^k}^*} \psi_0 \, \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 \bar{f}).$$

$$= \sum_{\bar{x}_i \in \mathbb{F}_{q^k}^*} \prod_{j=1}^{J} \psi_0 \, \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}\left(\bar{a}_j x_0 x^{V_j}\right)$$

$$= \sum_{x_i^{q^k-1}=1} \prod_{j=1}^{J} \prod_{i=0}^{ak-1} \theta\left((a_j x_0 x^{V_j})^{p^i}\right), \quad q^k = p^{ak}$$

Write $F(f,x) = \prod_{j=1}^{J} \theta(a_j x_0 x^{V_j}) \in \mathbb{Z}_q[\pi][[x_0, x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$

$$F_a(f,x) = \prod_{i=0}^{a-1} \prod_{j=1}^{J} \theta\left((a_j x_0 x^{V_j})^{p^i}\right)$$

$$= F(f,x)\, F^{\tau}(f, x^p) \cdots F^{\tau^{a-1}}(f, x^{p^{a-1}}), \quad \tau \in \mathrm{Gal}(\mathbb{Z}_q/\mathbb{Z}_p)$$

Note. $\theta(a_j; x_\circ x^{v_j}) = E_p(\pi a_j; x_\circ x^{v_j}) \in S_{\Delta, q}$

$$F(f.x) = \prod_{j=1}^{J} \theta(a_j; x_\circ x^{v_j}) \in S_{\Delta.q}$$

$$F_a(f,x) = F(f.x)\, F(f.x^q) \cdots F(f.x^{q^{a-1}}) \notin S_{\Delta.q}$$

$(a > 1)$.

6. Frob end.

Def. $\psi_p : \mathbb{Z}_q[\pi][[X_0^{\pm 1}, \ldots X_n^{\pm 1}]]$ additive

$$\psi_p(\Sigma A_u X^u) = \Sigma A_{pu}^{\tau^{-1}} X^u, \ \tau^{-1}\text{-linear}.$$

$$\psi_q = \psi_p^a = \underbrace{\psi_p \circ \cdots \circ \psi_p}_{a}$$

$$\psi_q(\Sigma A_u X^u) = \Sigma A_{qu} x^u, \ \text{linear}.$$

<u>Prop</u> $S_k(x,f) = \sum_{x_i^{q^k-1}=1} F_a(f,x)\, F_a(f,x^q) \cdots F_a(f,x^{q^{k-1}})$.

5. p-adic Banach space.

Write $f = \sum_{j=1}^{J} a_j x^{V_j}$, $a_j \in \mathbb{Z}_q$, $a_j^{q-1}=1$.

$\Delta = \Delta(f) = \mathrm{Conv}(V_j)$. in $\mathbb{R}^n$

$\bar{\Delta} = \mathrm{Conv}(0, (1, V_j))$ in $\mathbb{R}^{n+1}$.

$C(\bar{\Delta})$ = Cone gen'ed by $\bar{\Delta}$.

$L(\bar{\Delta}) = C(\bar{\Delta}) \cap \mathbb{Z}^{n+1}$. a f.g. monoid.

<u>Def.</u> $S_{\Delta,q} = \left\{ \sum_{u \in L(\bar{\Delta})} A_u \pi^{u_0} x^u \mid A_u \in \mathbb{Z}_q[\pi] \right\}$. a p-adic Banach alg.

$u = (u_0, u_1, \ldots u_n)$. $x^u = x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$.

Formal basic $\Gamma = \{ \pi^{u_0} x^u \mid u \in L(\bar{\Delta}) \}$.

<u>Def.</u> $\phi_1 = \psi_p \circ F(f, x)$. $\qquad \tau^{-1}$- linear.

$$\phi_a = \psi_q \circ F_a(f, x) = \phi_1^{\,a} \quad , \text{ linear},$$

$\Rightarrow$ $d\,\phi_a$ is a compact operator on $S_{\Delta,q}$.

$\mathrm{Tr}(\phi_a)$ is defined

$\det(I - T\phi_a)$ is p-adic entire.

$$\phi_1(\Gamma) = \Gamma A_1(f).$$

$$A_1(f) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ p\,a_{10} & p\,a_{11} & p\,a_{12} & \cdots \\ p^2 a_{20} & p^2 a_{21} & p^2 a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

a_{ij}.

$\ell(i\Delta)$ - rows

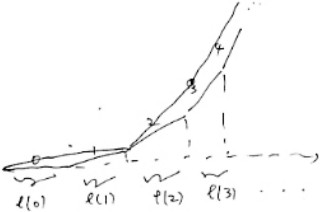
$\ell(j\Delta)$ - columns

entries in $\mathbb{Z}_q[\pi]$.

Prop. q-adic NP of $\det(I - \phi_a T)$

$\geq$ the polygon in $\mathbb{R}^2$ with vertices

$$\left(\sum_{k=0}^{m} \ell(k),\ \sum_{k=0}^{m} k\,\ell(k) \right);\quad m = 0, 1, 2, \ldots$$



$$\ell(i) = \#(i\Delta \cap \mathbb{Z}^n)$$

7. Dwork trace formula.

<u>Thm</u> $S_k(x_0 f) = (q^k - 1)^{n+1} \operatorname{Tr}(\phi_a^k)$

$$= \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} q^{ki} \operatorname{Tr}(\phi_a^k)$$

$$\Rightarrow L(x_0 f, T) = \prod_{i=0}^{n+1} \exp\Big(\sum_{k=1}^{\infty} \frac{T^k}{k} \operatorname{Tr}\big((q^i \phi_a)^k\big)\Big)^{(-1)^{n+1-i}\binom{n+1}{i}}$$

$$= \prod_{i=0}^{n+1} \det(I - T q^i \phi_a)^{(-1)^{n-i}\binom{n+1}{i}}$$

<u>Cor</u>. $L(x_0 f, T)$ is p-adic mero. in T.

$\cap$

$1 + T\mathbb{Z}[[T]]$

8. Rationality.

Lemma (Borel – Dwork). Let $g(T) \in \mathbb{Z}[[T]]$.

Then $g(T) \in \mathbb{Q}(T) \Longleftrightarrow$

$$\begin{cases} 1) & g(T) \text{ analytic } \cancel{\text{near}}\ \text{at } 0 \text{ in } \mathbb{C}. \\ 2) & g(T) \text{ is } p\text{-adic mero for some } p > 0. \end{cases}$$

$\Rightarrow$

<u>Thm</u> (Dwork). $L(\chi \circ f, T) \in \mathbb{Q}(T)$.

$Z(U_f, T) \in \mathbb{Q}(T)$

9. p-adic coh formula for $L(\chi_0 f, T)$.

<u>Def</u> $G(x) = F(f,x)\, F^{\tau}(f,x^p)\, F^{\tau^2}(f,x^{p^2}) \cdots \in \mathbb{Z}_q[\pi][[X_0, X_1^{\pm 1}, \ldots]]$

$$F(f,x) = \frac{G(x)}{G^{\tau}(x^p)}.$$

$$\psi_p(x^u) = \begin{cases} x^{\frac{u}{p}}, & \text{if } p \mid u \\ 0, & \text{if } p \nmid u. \end{cases}$$

$$\phi_1 = \psi_p \circ F(f,x) = \psi_p \circ \frac{G(x)}{G^{\tau}(x^p)} = G(x)^{-1} \circ \psi_p \circ G(x)$$

$$\phi_1^a = \phi_a = G(x)^{-1} \cdot \psi_p^a \circ G(x) = G(x)^{-1} \circ \psi_q \circ G(x).$$

For $0 \le i \le n$. let $D_i = G(x)^{-1} \circ x_i \frac{\partial}{\partial x_i} \cdot G(x)$ acts on $S_{\Delta,q}$.

$$D_i D_j = D_j D_i, \qquad \phi_a \circ D_i = q D_i \circ \phi_a$$

Koszul complex $K_\bullet(S_{\Delta,q}, D_0, D_1, \dots D_n)$

$$
\begin{tikzcd}
0 \arrow[r] & S_{\Delta,q}^{\binom{n+1}{n+1}} \arrow[r, "d"] \arrow[d, "q^{n+1}\cdot\phi_a"'] & S_{\Delta,q}^{\binom{n+1}{n}} \arrow[r, "d"] \arrow[d, "q^{n}\cdot\phi_a"'] & \cdots \arrow[r, "d"] & S_{\Delta,q}^{\binom{n+1}{1}} \arrow[r, "d"] \arrow[d, "q\cdot\phi_a"'] & S_{\Delta,q} \arrow[r] \arrow[d, "\phi_a"] & 0 \\
0 \arrow[r] & S_{\Delta,q}^{\binom{n+1}{n+1}} \arrow[r] & S_{\Delta,q}^{\binom{n+1}{n}} \arrow[r] & \cdots & S_{\Delta,q}^{\binom{n+1}{1}} \arrow[r] & S_{\Delta,q} \arrow[r] & 0
\end{tikzcd}
$$

$$\Rightarrow L(x_0 f, T)^{(-1)^n} = \prod_{i=0}^{n+1} \det\left(I - T q^i \phi_a \mid S_{\Delta,q}\right)^{(-1)^i \binom{n+1}{i}}$$

$$= \prod_{i=0}^{n+1} \det\left(I - T q^i \phi_a \mid H_i(K_\bullet)\right)^{(-1)^i}.$$

If f is Δ-regular $/\mathbb{F}_q$ $\Rightarrow$ $\begin{cases} H_i = 0 & \forall\, i > 0 \\ H_0 = S_{\Delta,q} \Big/ \sum_{i=0}^{n+1} D_i(S_{\Delta,q}) \end{cases}$

is a free $\mathbb{Z}_q[\pi]$-module of rk $d(\Delta)$

From now on, assume f is Δ-regular $\Rightarrow$

<u>Thm</u> $\quad L(x_0 f, T)^{(-1)^n} = \det(I - T\phi_a \mid H_0)$

$\in 1 + T\mathbb{Z}[T]$ of deg $d(\Delta)$

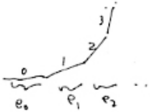
10). $\quad L(x_0 f, T)^{(-1)^n} = \prod_{i=1}^{d(\Delta)} (1 - \alpha_i T), \quad \alpha_i \in \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$

$\Rightarrow |\alpha_i| = q^{\frac{w_i}{2}}, \quad w_i \in \mathbb{Z} \cap [0, n+1].$

(mixed of weight $\in n+1$).

<u>Def</u>. $\quad e_j = \#\{1 \le i \le d(\Delta) \mid w_i = j\}, \quad j = 0, 1, \cdots, n+1$

The weight polygon of Δ is $\quad (e_0 = 1)$.

$WP(\Delta)$: 

$wp(\Delta)$ can be determined.

<u>Eg</u> Δ a simplex. let

$$C_0 = 1, \qquad c_i = \sum_{\substack{\sigma \subset \Delta \\ \dim\sigma = i-1}} \mathrm{vol}(\sigma), \quad i \geq 1$$

$$\Rightarrow e_0 = 1, \qquad e_j = \sum_{i=0}^{j} (-1)^{j-i}\, i! \binom{n+1-i}{n+1-j} c_i, \quad j \geq 1.$$

<u>Ex</u>. $f(a,x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda$. Δ - regular.

11. Newton polygon.

$$L(x_0 f, T)^{(-1)^n} = \prod_{i=1}^{d(\Delta)} (1-\alpha_i T), \quad \alpha_i \in \overline{\mathbb{Q}_p}$$

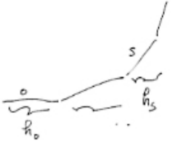
$$|\alpha_i|_q = q^{-s_i}, \qquad s_i = \mathrm{ord}_q(\alpha_i).$$

$$s_i \in \mathbb{Q} \cap [0, n+1].$$

<u>Def</u>. $h_s = \#\{1 \le i \le d(\Delta) \mid s_i = s\}$. $\quad s \in \mathbb{Q} \cap [0, n+1)$

q-adic NP.

NP(f):



$\Sigma h_s = d(\Delta)$

<u>Question</u>. $NP(f) = ?$

Prop. write

$$L(x_0 f, T)^{(-1)^n} = \sum_{m=0}^{d(\Delta)} A_m T^m, \quad A_m \in \mathbb{Z}.$$

$\Rightarrow NP(f)$ is convex closure in $\mathbb{R}^2$ of

the pts $(m, \operatorname{ord}_q(A_m))$. $m = 0, 1, \ldots d(\Delta)$

Prop. Vertices of $NP(f) \in \mathbb{Z}^2$.

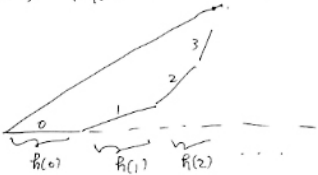
(2). Hodge polygon.

$\Delta \subset \mathbb{R}^n$, n-dim integral convex

$$W(k) = \#(\mathbb{Z}^n \cap k\Delta).$$

$$\sum_{k=0}^{\infty} W(k) T^k = \frac{\sum_{k=0}^{n} h(k) T^k}{(1-T)^{n+1}}.$$

Def. The Hodge polygon of Δ is the polygon



$\sum h(k) = d(\Delta)$.

Thm. $NP(f) \geq HP(\Delta)$ with endpts coincide.

<u>Def.</u> If $NP(f) = HP(\Delta)$, $\Rightarrow$ f is ordinary

i.e. $L^{(-1)^n}$ has exactly $h(k)$ recip root α_i's

s.t $\mathrm{ord}_q(\alpha_i) = k$, $k = 0, 1, \dots n$.

13). Variation of $NP(f)$ with p.

Conj. Let $f(X_1, \cdots, X_n) \in \mathbb{Q}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$, Δ-regular.

1) $\exists$ infinitely many p s.t. $NP(f \otimes \mathbb{F}_p) = HP(\Delta)$.

2) $$\delta(f) = \lim_{t \to \infty} \frac{\#\{p \le t \mid f \otimes \mathbb{F}_p \text{ is ordinary}\}}{\#\{p \le t\}}$$

exits and $\delta(f) > 0$.

Eg. $f = X_1 + X_2 + \frac{1}{X_1 X_2} - \lambda \in \mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}]$ Δ-regular.

(elliptic curve $/\mathbb{Q}$)

$$\Rightarrow \delta(f) = \begin{cases} \frac{1}{2}, & f \text{ has CM} \quad (\text{Deuring}) \\ 1, & f \text{ has non-CM} \quad (\text{Serre}) \end{cases}$$

Eg. $f(\lambda, x) = x_1 + \cdots + x_n + \dfrac{1}{x_1 \cdots x_n} - \lambda$, Δ-regular.

$n \geq 4$.

$n = 3$ (surface. K_3)

14. Geometric variation.

$$M_p(\Delta) = \{ f \in \overline{\mathbb{F}}_p[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \mid \Delta(f) = \Delta, \quad f \ \Delta\text{-regular} \}.$$

Question: when $M_p(\Delta) \neq \phi$?

Ex. $p > d(\Delta) \Rightarrow M_p(\Delta) \neq \phi$.

$f \in M_p(\Delta)(\overline{\mathbb{F}}_p) \Rightarrow f \in M_p(\Delta)(\mathbb{F}_q)$ for some $\mathbb{F}_q/\mathbb{F}_p$.

$\Rightarrow$ q-adic $NP(f)$ defined.

indep of $\mathbb{F}_q$.

The relative GR $H_0(K_\cdot)$ is "locally free" over $M_p(\Delta)$.

$\Rightarrow$ an F-crystal over $M_p(\Delta)$.

By G-K.

Thm. 1) $\{ NP(f) \mid f \in M_p(\Delta)(\bar{\mathbb{F}}_p) \}$

$\exists$ a unique initial element, $GNP(\Delta, p)$ wrt the partial ordering "$\geq$".

2) $\exists$ $U_p(\Delta) \hookrightarrow M_p(\Delta)$

open dense

s.t. $NP(f) = GNP(\Delta, p) \iff f \in U_p(\Delta)$.

Newton stratification of $M_p(\Delta)$.

$$NP(f) \underset{\text{generically}}{\overset{\geq}{=}} GNP(\Delta, p) \underset{?}{\overset{\geq}{=}} HP(\Delta)$$

Def. If $GNP(\Delta, p) = HP(\Delta)$,

$\Rightarrow$ Δ is ordinary at p

(or p is ordinary for Δ).

Question: which p is ordinary for Δ?

§7 ordinary primes.

Conj (AS). Δ is ordinary for all $p \gg 0$.

Thm A. 1) $\exists$ $D(\Delta) > 0$ s.t. if $p \equiv 1 \pmod{D(\Delta)}$

$\Rightarrow$ p is ordinary for Δ.

2) If $n \leq 3$ $\Rightarrow$ ($D(\Delta) = 1$)

p is ordinary $\forall$ $p > d(\Delta)$.

3) If $n \geq 4$. $\exists$ n-dim Δ s.t.

Δ is NOT ordinary for all

p in a residue class of some $D(\Delta)$.

16. Local theory.

<u>Lemma 1</u>: If Δ is indecomp (no lattice pts $\neq$ vertices) and $p \equiv 1 \ (d(\Delta)) \Rightarrow \Delta$ is ordin at p.

Pf. Gauss sums + Stickelberger.

$(d(\Delta) = n! \, \mathrm{Vol}(\Delta).)$

<u>Cor 1</u>. If $n \leq 2$ and Δ indecomp

$\Rightarrow d(\Delta) = 1 \quad \Rightarrow \quad \Delta$ is ordin $\forall p$.

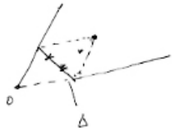
Lemma 2: Let $\Delta = \langle V_0, V_1, \dots V_n \rangle$ be indecomp.

$p \nmid d(\Delta)$. Then

p is ording for Δ

$\iff$ degree of pts in $\mathbb{Z}^{n+1} / \langle (1.V_0), \cdots (1.V_n) \rangle \mathbb{Z}^{n+1}$

is stable under mult by p.



$\deg(u) = \deg(\overline{pu})$.

Cor 2. If $n=3$, Δ indecomp

$\Rightarrow$ degree is p-stable

$\Rightarrow$ p is ordinary.

Cor 3. If $n=4$, Δ indecomp. $0 \in \Delta$ vertex

$\Rightarrow \Delta$ is ording for $p \nmid d(\Delta)$.

17). Global Collapsing decomp.

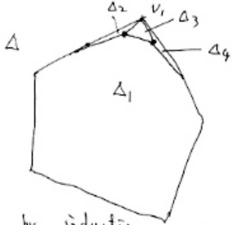
Thm. $\Delta = \bigcup_{i=1}^{k} \Delta_i$ be

a complete Collapsing decomp.

If p is ording for each Δ_i.

$\Rightarrow$ p is ordinary for Δ.

$(p > d(\Delta))$.



Continue by induction ...

$\Rightarrow$ decompse Δ into indecomp pieces

Cor. Thm A 1) follows.

Thm A 2) follows $(n \leq 3)$

$GNP(\Delta, p) \overset{?}{=} HP(\Delta)$.

<u>Conj</u>. $\exists$ a positive integer ~~[illegible]~~ $\mu(\Delta)$

s.t. $\{ p > d'(\Delta) \mid p \text{ ordinary} \}$

$=$ disjoint union of some congruence classes mod $\mu(\Delta)$.

slope:

$$Z(U_{f_\lambda}, T) \sim \prod (1-\alpha_i(\lambda)T)$$

$$\alpha_i(\lambda) \in \overline{\mathbb{Q}}.$$

1) $\alpha_i(\lambda) \hookrightarrow \mathbb{C}.$

$$\alpha_i(\lambda) = |\alpha_i(\lambda)| \cdot e^{i\theta(\lambda)}$$

$$= q^{w_i(\lambda)} \cdot e^{i\theta(\lambda)}$$

$w_i(\lambda)$ is constant. $\leftrightarrow$ Hodge numbers $h^{p,q}$.

2) $\alpha_i(\lambda) \hookrightarrow \mathbb{C}_\ell, \quad \ell \neq p.$

$$|\alpha_i(\lambda)|_\ell = 1.$$

3). $\alpha_\phi(\lambda) \hookrightarrow \mathcal{O}\,\mathbb{C}_p$.

$$\alpha_\phi(\lambda) = q^{-s_\phi(\lambda)} \cdot u_\phi(\lambda). \qquad |u(\lambda)|_p = 1.$$

slope $s(\lambda) = \mathrm{ord}_q\, \alpha(\lambda) \in \mathbb{Q}$.

depends on λ, in mild way.

(only finite # of possibility)

$\Rightarrow$ Newton polygon $NP(f)$ is a finer invariant than $HP(\Delta)$,

but cruder than zeta.

18). ordinary primes for reflexive Δ.

<u>Def</u>. $\Delta \subseteq \mathbb{R}^n$, n-dim integ. convex

$$\Delta^* = \left\{ (x_1, \dots x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i y_i \geq -1, \ \forall\, y \in \Delta \right\}.$$

$\Rightarrow \Delta^*$ is convex, but not integral.

<u>Def</u>. Δ is reflexive if Δ^* is also integral.

$(\Delta^*)^* = \Delta$.

<u>Def</u>. Δ is Fano if reflexive.

Δ_i is a simplex with $d(\Delta_i) = 1$.

Thm B Let Δ be reflexive.

1) If $n \le 4 \Rightarrow \Delta$ is ordinary $\forall\, p > d(\Delta)$.

2) If Δ is Fano, $\Rightarrow \Delta$ is ordinary

for all $p > d(\Delta) = \cancel{n+1}$.

$= \#\{$of codim 1 faces$\}$

Thm (Star decomp) Δ reflexive.

$$\Delta = \bigcup_{i=1}^{k} \Delta_i$$

If p is ordinary $\forall\, \Delta_i$.

$\Rightarrow p$ is ordinary $\forall\, \Delta$.

? If Reflexive Δ has $\dim = n \geq 5$,

Is Δ ordin for $p > d(\Delta)$?

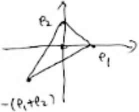
19. Basic example.

$$f(\lambda, X) = X_1 + X_2 + \cdots + X_n + \frac{1}{X_1 \cdots X_n} - \lambda.$$

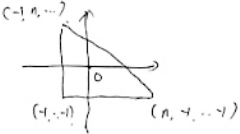
$$\Delta = \Delta(f) = \langle e_1, e_2, \cdots, e_n, -(e_1 + \cdots + e_n) \rangle.$$

$$\Delta^* = \langle (n, -1, \cdots, -1), \cdots, (-1, -1, \cdots, n), (-1, -1, \cdots, -1) \rangle.$$

Δ.



Δ^*.



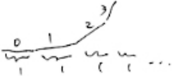
reflexive.

Δ is Fano. Δ^* is NOT Fano $(n > 1)$

$f(\lambda, X)$ is Δ-regular, $\iff \lambda \neq (n+1)\alpha$, $\alpha^{n+1} = 1$.

$d(\Delta) = n+1$. $\quad h(0) = h(1) = \cdots = h(n) = 1$.

$Hp(\Delta)$.



$\Sigma h(i) = n+1$.

Thm. ~~Let $p \nmid (n+1)$~~. $f(\lambda, x)$ is Δ-regular $/\mathbb{F}_q$ $\Rightarrow$

1) $L(x.f, T)^{(-1)^n} = \prod_{i=0}^{n} (1 - \alpha_i(\lambda) T)$.

2) $\alpha_0(\lambda) = 1$. $|\alpha_i(\lambda)| = q^{\frac{n+1}{2}}$ $(1 \le i \le n)$.

3) Generically ordinary $\forall\, p \nmid (n+1)$.
For all but finitely many λ, $\Rightarrow$

$$\mathrm{ord}_q(\alpha_i(\lambda)) = i.$$

Question: 1) How $Z(U_{f_\lambda}, T)$ varies with λ?

2) How $\alpha_i(\lambda)$ varies with λ?

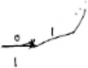
20). Zeta functions.

$$f \in \mathbb{F}_q[X_1^{\pm 1}, \dots, X_n^{\pm 1}], \quad \Delta\text{-regular} / \overline{\mathbb{F}_q}.$$

$$L(x_0 f, T)^{(-1)^n} \Big|_{T=1} = 0.$$

$$\Rightarrow \frac{L(x_0 f, T)^{(-1)^n}}{1-T} = P(f, qT),$$

$P(f, T) \in 1 + T\mathbb{Z}[T]$ of deg $d(\Delta) - 1$.



$$Z(U_f, qT) = \prod_{i=0}^{n} (1 - q^i T)^{(-1)^{n-i-1}\binom{n}{i}} L(x_0 f, T)$$

$$= \prod_{i=1}^{n} \cdots \left(\frac{L(x_0 f, T)^{(-1)^n}}{1-T} \right)^{(-1)^n}$$

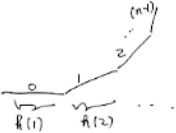
$$\Rightarrow Z(U_f, T) = \prod_{i=0}^{n-1} (1 - q^i T)^{(-1)^{n-i}\binom{n}{i+1}} P(f, T)^{(-1)^n}.$$

$$P(f, T) = \prod_{i=0}^{d(\Delta)-2} (1 - \beta_i T)$$

$$\Rightarrow \quad \# U_f(\mathbb{F}_{q^k}) = \frac{(q^k-1)^n + (-1)^{n+1}}{q^k} + (-1)^{n+1}\left(\beta_0^k + \beta_1^k + \cdots + \beta_{d(\Delta)-2}^k\right).$$

$$(k = 1, 2, 3, \ldots)$$

Def. The primitive Hodge polygon $PHP(\Delta)$ is



$$NP(P(f, T)) \geq PHP(\Delta).$$

Generic slope sym:

1) $GNP(P(M_\lambda, T)) = NP\left((1-T)(1-\xi T)^{101}(1-\xi^2 T)^{101}(1-\xi^3 T)\right)$

$= HP(M_\lambda).$

2) $GNP(P(W_\lambda, T)) = NP\left((1-T)(1-\xi T)(1-\xi^2 T)(1-\xi^3 T)\right)$

$= HP(W_\lambda).$

Thm. 2) is true for all p. (Δ^* is Fano)

1) is true if $p \equiv 1 \pmod 5$.

?: 1) is true for all p ?

21). Slope mirror symmetry.

Δ. M_λ : $x_1^5 + \cdots + x_5^5 - \lambda x_1 \cdots x_5 = 0 \quad /\mathbb{F}_q$

Smooth proj.

Δ^*. W_λ mirror Smooth proj $/\mathbb{F}_q$.

$$Z(M_\lambda, T) = \frac{P(M_\lambda, T)}{(1-T)(1-qT)(1-q^2T)(1-q^3T)}$$

$\deg P(M_\lambda, T) = 204$, pure of weight 3.

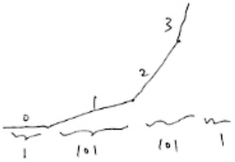
$$Z(W_\lambda, T) = \frac{P(W_\lambda, T)}{(1-T)(1-qT)^{101}(1-q^2T)^{101}(1-q^3T)}$$

$\deg P(W_\lambda, T) = 4$.

$HP(\Delta)$:

$\|$

$HP(M_\lambda)$

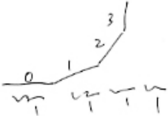


$= HP\left(\begin{matrix}\text{denomin} \\ \text{of } \cancel{HP(W_\lambda)} \\ Z(W_\lambda, T)\end{matrix}\right)$

$HP(\Delta^*)$

$\|$

$HP(W_\lambda)$



$= HP\left(\text{denom. of } Z(M_\lambda, T)\right)$

$HP(W_\lambda, T)$ and $HP(M_\lambda, T)$ switch under mirror.

<u>Question</u>. $NP(M_\lambda)$ and $NP(W_\lambda)$ switch ? under mirror.

(numer. of $Z(M_\lambda, T)$) (denom. of $Z(W_\lambda, T)$).

22). p-adic analytic formula for zeta.

Let $f(\bar{\lambda}, x) \in M_p(\Delta)(\bar{\mathbb{F}}_q)$. $q = p^a$.

$\Rightarrow Z(U_f, T) = * \cdot P(f(\bar{\lambda}, x), T)^{(-1)^n}$.

$P(f(\bar{\lambda}, x), T) \in 1 + T\mathbb{Z}[T]$ of deg $d(\Delta) - 1$.

$\qquad\parallel$

$\det(I - F(\bar{\lambda})T \mid H_0(K_\cdot))$.

<u>Thm</u>. 1) Zariski locally on $M_p(\Delta)$, $\Rightarrow$

$F(\bar{\lambda}) = A(\lambda^{p^{a-1}}) \cdots A(\lambda^p) A(\lambda)$, $\lambda = \mathrm{Teich}(\bar{\lambda})$.

$A(\lambda)$ is a p-adic analytic matrix $/\mathbb{Z}_p$.

2) $p > 2$. One can take $A(\lambda) = C(\lambda^p)^{-1} A(0) C(\lambda)$, 0 is a regular pt.

$C(\lambda)$ = fund sol. matrix of Picard-Fuchs.

Thm. 1) Zariski locally.

$$P_k(\bar{\lambda}, T) = \det\left(I - F_k(\bar{\lambda})\, q^k T\right).$$

$$F_k(\bar{\lambda}) = A_k(\lambda^{p^{a-1}}) \cdots A_k(\lambda^p)\, A_k(\lambda) .$$

$\bar{\lambda} \in \mathbb{F}_{p^a}$.

$\lambda = \mathrm{Teich}(\bar{\lambda})$.

$$A_k(\lambda) \in GL_{h(k+1) \times h(k+1)}(\qquad)$$

24). p-adic rep's and unit root L-function.

Let Δ be ordinary at p.

$A_k(\lambda)$ is the Frob matrix of a p-adic Galois rep.

$$\rho_k : \pi_1^{arith}\left(H_p(\Delta)/\mathbb{F}_p\right) \longrightarrow GL_{k(k+1)}(\mathbb{Z}_p)$$

$$\rho_k(\mathrm{Frob}_{\bar\lambda}) = F_k(\bar\lambda) = A_k(\lambda^{p^{a-1}}) \cdots A_k(\lambda)$$

$$(\bar\lambda \in \mathbb{F}_{p^a}).$$

$$L(\rho_k, T) = \prod_{\substack{\bar\lambda \in H_p(\Delta) \\ \text{closed pt}}} \frac{1}{\det\left(I - \rho_k(\mathrm{Frob}_{\bar\lambda}) T^{\deg(\bar\lambda)}\right)}$$

$$\in 1 + T(\mathbb{Z}_p^{\Lambda\bar{\mathbb{Q}}})[[T]]$$

25) Dwork's Conj.

Thm. $L(P_k, T)$ is p-adic mero in T.

$$\parallel \quad \frac{\prod_{i=1}^{\infty}(1-\alpha_i T)}{\prod_{j=1}^{\infty}(1-\beta_j T)}, \quad \alpha_i \to 0, \ \beta_j \to 0.$$

Question: $\mathrm{ord}_q(\alpha_i) = ?$ $\mathrm{ord}_q(\beta_j) = ?$