Instanton Sums and Monodromy

String theory

10-dim'l spacetimes

$X^6 \times M^4$ ← Minkowski

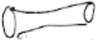
want Supersymmetry (SUSY) in 4D.

$\Rightarrow$ Covariantly constant spinor on X^6.

holonomy is reduced: $SU(3)$

$\Rightarrow$ Ricci flat Kähler,

$\Rightarrow$ X^6 is a projective alg. variety.



$\Sigma \longrightarrow X^6$

worldsheet

- Maps (Σ, X^6)

quantum field theory

Expansion in q, coefficients are "instanton numbers".

want all maps in a fixed coho. class with minimum action. $n_{\vec{d}}$

$$n_{\vec{0}} + \sum n_{\vec{d}}\, q^{\vec{d}} = \text{physical quantity}$$

e.g. $n_{\vec{d}}$ = Gromov-Witten invariants of X.

$X^6 \subseteq \mathbb{CP}^4$

$X^6 = \{ x_1^5 + x_2^5 + \cdots + x_5^5 = 0 \}$

QFT assoc. to X^6
can sometimes be described by a
"gauged linear sigma model".

Instantons are easy to calculate.

Related to original QFT by "renormalization".

renormalization → need a parameter change
(in q) when comparing answers.

Superspace

P. Deligne, D. Freed
"Supersolutions" in: Quantum Fields & Strings, vol 1,
ed by Deligne et al, AMS 1998?

A superspace is a supermanifold ($\mathbb{Z}_2$-graded) whose
"body" is Minkowski space.

$$M^{2|(2,2)}$$

x^0, x^1 : coords on M^2
$(x^0)^2 - (x^1)^2$

$$\partial_\pm = \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \text{, even vector fields}$$

odd vector fields: $D_+, \bar{D}_+, \bar{D}_-, D_-$

$$[D_+, D_+] = \cdots = 0$$
$$[D_-, D_-] = \cdots = 0$$
$$[D_+, \bar{D}_+] = -2\partial_+$$
$$[D_-, \bar{D}_-] = -2\partial_-$$

$\mathcal{L}$ = Super Lie algebra = extension of Poincaré alg by these odd v.f.

$M^{2|(2,2)}$ is acted upon by $\exp(\mathcal{L})$.

$D_\pm, \partial_\pm$ are left-invariant v.f. on $M^{2|(2,2)}$

$$\Phi : M^{2|(2,2)} \longrightarrow \mathbb{C}^n \quad \text{Super field.}$$

e.g. $D_+\Phi\big|_{M^2}$, $D_-\Phi\big|_{M^2}$ this produces "usual" fields

$$\Phi\big|_{M^2} = \varphi : M^2 \to \mathbb{C}^n.$$

<u>chiral super field</u>: $\bar{D}_+\Phi_i = \bar{D}_-\Phi_i = 0.$

(holomorphic)

Need compact abelian group G.

$$\rho : G \to U(1)^n \subseteq GL(n,\mathbb{C}).$$

We will also need

$$\tau \in \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$$

$$W = G\text{-invariant poly on } \mathbb{C}^n.$$

$$\mathcal{L} = \int d^4\theta \left(\underbrace{\| e^{-i\, d\rho \circ V} \|^2}_{\substack{G\text{-invariant} \\ (\text{kinetic term})}} - \frac{1}{4e^2}\|\Sigma\|^2 \right.$$

$d^4\theta = d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^-$

V = gauge multiplet: $M^{2|(2,2)} \to \mathfrak{g}$

$$\Sigma = \bar{D}_+ D_- V.$$

"twisted chiral" $\quad \bar{D}_+\Sigma = D_-\Sigma = 0.$

Need norm on $\mathfrak{g}$.

$$\mathcal{L} = \text{kinetic term}$$
$$+ \operatorname{Re} \int d\theta^+ d\theta^- \, W(\Phi_1, \dots, \Phi_n)$$
$$+ \operatorname{Im} \int d\theta^+ d\bar{\theta}^- \, (\tau, \Sigma)$$

$\tau \in \operatorname{Hom}(\mathfrak{g}, \mathbb{C})$

$W = $ poly. coefs of W. = params

$\rho : G \to U(1)^n$.

$\tau \in \operatorname{Hom}(\mathfrak{g}, \mathbb{C})$ more params.

<u>Example</u> $W(X_0, X_1, \dots, X_5)$
$$= X_0 X_1^5 + \dots + X_0 X_5^5$$
$$+ X_0 X_1 X_2 X_3 X_4 X_5$$

$G = U(1)$

X_0	X_1	$\dots$	X_5
-5	1	$\dots$	1

$\leftarrow Q_i^a$

<u>Low-energy analysis</u>

Pick a basis for $\mathfrak{g}$
$$V = (V_1, \dots, V_{n-d}).$$
group action $e^{V_a} : \Phi_i \to (e^{V_a})^{Q_i^a} \Phi_i$

Derivatives of certain component fields do not occur in action: can solve:

$$\tau = ir + \frac{1}{2\pi}\theta.$$

$$\overset{\text{comp of } V_a}{D_a} = -e^2 \left(\sum Q_i^a |\varphi_i|^2 - r_a \right)$$

$$F_i = -\frac{\partial W}{\partial \varphi_i}$$

$\uparrow$ comp of Φ_i

Potential for bosonic fields:
$$U = \frac{1}{2e^2} \sum D_a^2 + \sum |F_i|^2 + \dots$$

To minimize U, must have

$$D_a = 0, \quad F_i = 0.$$

Example

$$-5|\varphi_0|^2 + |\varphi_1|^2 + \dots + |\varphi_5|^2 - r = 0.$$

$$W(\vec{x}) = x_0 P_5(\vec{x})$$

$$\frac{\partial W}{\partial x_0} = P_5(\vec{x}) = 0$$

$$\frac{\partial W}{\partial x_i} = x_0 \frac{\partial P_5}{\partial x_i}(\vec{x}) = 0$$

if $r < 0$ then not all $\varphi_1, \dots, \varphi_5$ can vanish.

$$\mathbb{C}^\times \quad \mathbb{C}^5 \setminus \{0\} \cong \mathbb{C}^\times \boxed{S^9 / U(1)}$$

$$\mathbb{CP}^4$$

get a complex line bundle $\mathcal{O}(-5)$ over $\mathbb{CP}^4$.

If P is generic,

$P, \frac{\partial P}{\partial x_i}$ will share no common zero outside $\vec{0} \in \mathbb{C}^5$.

$$\Rightarrow P = 0, \quad x_0 = 0.$$

$$\mathbb{CP}^4$$

(quintic hyp) $\subseteq \mathbb{CP}^4$

(Calabi-Yau).

Constraint on choices

$\rho: G \to U(1)^n$, $W(x_1, \ldots, x_n)$, G-invariant

family of theories with coefs varying

{monomials in W}.

$$x^{t_a} = \prod x_i^{t_{ai}}$$

t_a = row of a nonnegative integer matrix T.

G determines a set of G-invariant Laurent monomials.

P has rows P_{ai} integers.

$$x^{P_a} = \prod x_i^{P_{ai}}$$

$G \subseteq U(1)^n$.

$$T = SP.$$

$\uparrow$

nonnegative int

rank d

Mirror theory:

$$^tT = {}^tP\,{}^tS.$$

new monomials

new group

---

conformally invariant theory in IR.:

$\exists\, \vec{\mu}, \vec{\nu}$ rational vectors

$$T\vec{\mu} = (1\,1\,1\,1 \cdots)^T$$

$$\vec{\nu}^T T = (1\,1\,1\,1 \cdots 1)$$

Instanton Sums and Monodromy
Lecture 2

$\rho : G \to U(1)^n$ acts on $\mathbb{C}^n$, G-invariant $W(x_1, \ldots, x_n)$

G = compact abelian

$\mu : \mathbb{C}^n \longrightarrow \mathfrak{g}^*$

$\mu^{-1}(r)/G$ = a Kähler space (sometimes a manifold)

top. depends on location of $r \in \mathrm{Im}(\mu)$.

$X = \mu^{-1}(r)/G$ is a toric variety.

Coordinate charts: Say rank $G = n-d$.

$\Rightarrow \dim_{\mathbb{C}} X = d$.

Consider subsets of d of the homog. coords

$x_{i_1}, \ldots, x_{i_d}$.

does open set in $\mathbb{C}^d$ containing $\vec{0}$ descend to a subset of X?

Example $\quad x_0 \; x_1 \; \cdots \; x_5$

$G = U(1) \quad -5 \;\; 1 \; \cdots \; 1$

$\mu = -5|x_0|^2 + |x_1|^2 + \cdots + |x_5|^2$

$x_0, x_1, \ldots, x_4$ a coord chart?

$$|x_5|^2 = r + 5|x_0|^2 - |x_1|^2 - \cdots - |x_4|^2$$

if $r > 0$, open set containing $\vec{0}$ on which can be solved.

use $U(1)$ to make x_5 real.

In the example,

$$\mu^{-1}(r)/G = \text{total space of } \mathcal{O}(-5) \text{ over } \mathbb{CP}^4$$

$$\frac{\partial W}{\partial x_i} = 0.$$

$$W = x_0 (x_1^5 + \cdots + x_5^5 + x_1 x_2 x_3 x_4 x_5)$$

$$\Rightarrow x_0 = 0, \quad x_1^5 + \cdots + x_5^5 + x_1 \cdots x_5 = 0.$$

on a quintic hypersurface in $\mathbb{CP}^4$.

Families of CY manifolds:

2 types of params: coefs in W.

Kähler classes on X (complexified)

τ.

$\prod x_i^{p_{ir}}$ $\qquad (p_{ir}) = P$ nonnegative integers

Laurent basis for fermions $\prod x_i^{t_{ia}}$ $\qquad (t_{ia}) = T$ integer matrix

$$P = ST$$

$$\prod x_i^{p_{ir}} = \prod \left(\prod x_i^{t_{ia}}\right)^{S_{ar}}$$

In example: $x_0 x_1^5, \ldots, x_0 x_5^5, x_0 x_1 \cdots x_5$

$$P = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Laurent basis: $x_0 x_1^5$, $\quad x_i / x_0$.

$$\underset{S}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}} \underset{T}{\begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

Mirror:

replace ρ, S, T by

$${}^t\rho,\ {}^tT,\ {}^tS$$

$${}^t\rho = {}^tT\,{}^tS$$

In example for mirror group is

$$U(1) \times \mathbb{Z}_5^3$$

(Briefly skip analysis of 1-loop effects)

<u>Instanton expansions:</u>

Cohomology of toric varieties.

(applies if toric variety is compact)

For $x_{i_1}, \ldots x_{i_d}$ to be coords in some chart, need $\rho|_{\mathrm{span}(x_{i_1}, \ldots x_{i_d})}$ has at worst a finite kernel.

if there is a finite kernel,

$\mathbb{C}^d/\mathrm{Ker.}$ singularities.

$H^*(X, \mathbb{Q})$ can still be calculated.

<u>Fact</u>: $H^*(X, \mathbb{Q})$ is generated as $\mathbb{Q}$-algebra by $H^2(X)$.

$H^2(X)$ gen. by $c_1(\{x_i = 0\}) =: \xi_i$

$G \to U(1)^n$ $\quad X = \mu^{-1}(r)/G.$

any character of G determines a $U(1)$-bundle (or $\mathbb{C}^*$-bundle).

can identify $H^2(X,\mathbb{Q}) = \mathrm{char}(G)\otimes\mathbb{Q}$.

Specifically,

if G acts on x_i by χ_i,

$$\chi_i = \prod (e^{\eta_a})^{Q^i_a}$$

exponents describing group action.

e^{η_a} basis of $\mathrm{char}(G)\otimes\mathbb{Q}$.

η_a basis for $H^2(X,\mathbb{Q})$

$$\xi_i = \sum Q^i_a \eta_a$$

$\xi_{i_1}\cdot\ldots\cdot\xi_{i_d}$ = intersection number.

$$\#\,(x_{i_1}=0)\cap\ldots\cap(x_{i_d}=0) = \begin{cases} 0 & \text{if not a coord. chart} \\ \dfrac{1}{\#(\ker\rho)} \end{cases}$$

Relations in ring

1) linear relations

$$\xi_i = \sum Q^i_a \eta_a$$

2) nonlinear relation

$$\xi_{i_1}\cdot\ldots\cdot\xi_{i_k} = 0$$

if $(x_{i_1}=0)\cap\ldots\cap(x_{i_k}=0) = \emptyset$ on space.

(Stanley–Reisner ideal)

<u>Thm</u> this is a complete set of relations.

$$[\xi_{i_1}\cdot\ldots\cdot\xi_{i_d}]\Big|_{[X]} = ?$$

Correlation functions in (topological) GLSM.

$$\Sigma \longrightarrow \mathbb{C}^n$$

governed by $\mathcal{L}$.

$\mu^{-1}(r)/G$

$\frac{\partial W}{\partial \varphi_i} = 0$.

φ_i = low components of chiral fields.

$\Phi : M^{2|(2,2)} \to \mathbb{C}^N$

$\Phi|_{M^2} = \varphi : M^2 \to \mathbb{C}^N$

~~Classical~~

$\langle\langle \sigma_1 \cdots \sigma_k \rangle\rangle$

σ_j = a certain point P_j on Σ should be mapped to a divisor $\xi_j := \{x_j = 0\}$

Classical term:

insist that map $\Sigma \to \mathbb{C}^n$ be top. trivial, and minimize energy.

$\Rightarrow$ map is constant.

$\Sigma = \mathbb{CP}^1$.

$\mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^*$

$\mu_\Sigma : \mathbb{C}^2 \to \mathbb{R}$ for $U(1)$

$\mathbb{CP}^1 = \mu_\Sigma^{-1}(r)/U(1)$.

$\mathbb{C}^2 \longrightarrow \mathbb{C}^n$ equivariant.

$\lambda : U(1) \longrightarrow G$

λ specifies the homology class of image $\mathbb{CP}^1$.

$U(1) \xrightarrow{\lambda} G \xrightarrow{\chi} U(1)$ $\quad \deg(\chi \circ \lambda) = \# D_\chi \cdot \Sigma$.

$\mathbb{C}^2 \to \mathbb{C}^n$ $\quad \lambda : U(1) \to G$

$X_j = f_j(s,t)$ hom. of deg d_j

where $d_j = \deg(\cancel{\lambda \circ X_j})$ $(X_j \circ \lambda)$

try. trivial: $d_j = 0 \quad \forall j$.

X_j = constant on $\mathbb{C}^2$.

σ_k restricts us to maps f s.t.

$f(P_k) \in (X_k = 0)$

if f is constant, means the image point lies on $(X_k = 0)$.

$\langle\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle\rangle_{class} = \#(X_{i_1} = 0) \cap \cdots \cap (X_{i_k} = 0)$

$\langle\langle \sigma_{i_1} \cdots \sigma_{i_d} \rangle\rangle_{class}$ can be calculated as $\frac{1}{\#(\ker)}$ or 0.

$\frac{\partial W}{\partial x_i} = 0.$

$-5 \quad 1 \cdots 1$

Restrict to $W = x_0 P(x_1, \ldots, x_n)$

$d_j \geq 0 \quad \forall j \geq 1$

$d_0 < 0$

poly of neg. degree is $\equiv 0$.

$\boxed{X_0 \equiv 0.}$

$P(f_1(s,t), \ldots, f_n(s,t)) \equiv 0$

poly. of degree $-d_0 > 0$.

This will vanish identically if it vanishes at $-d_0+1$ points.

divisor class of P = anticanonical class of X.

$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \Big|$ $\square$

$$\langle\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle\rangle_{GLSM}$$

$$\|$$

$$\sum_{\vec{d}} (-1)^{d_0} \langle \sigma_{i_1} \cdots \sigma_{i_k} (-K)^{-d_0+1} \boxed{} \rangle_{\vec{d}}\, q^{\vec{d}}$$

$\vec{d} = \vec{0}$ is classical.

$$q = e^{2\pi i \tau}$$

\# of maps $\mathbb{P}^1 \to \mathbb{C}^n$ of degrees d_j.

$\sigma_{i_1}(P_{x_1}) \in \{X_{i_1} = 0\}$

also conditions for $P(x_0, \dots, x_n) = 0$.

Quintic

$H^2(X) = \mathbb{Z}$.

$\eta \dots$

$$\langle\langle \eta\,\eta\,\eta \rangle\rangle_{GLSM} = \sum c_n q^m$$

$$c_n = (-1)^{d_0} \langle \eta\,\eta\,\eta\,(-K)^{-d_0+1} \rangle_m$$

$$-d_0 = 5m$$

$X_j = f_m(s,t) \quad j = 1 \dots n$

X_0 has deg $-5m$.

$P(\quad)$ has degree $5m$

$K = -5\eta$.

$$-\langle \eta^3 (-5\eta)^{5m+1} \rangle_m = -\langle \eta^{5m+4} (-5)^{5m+1}$$

$$= -(-5)^{5m+1}.$$

$$c_m = -(-5)^{5m+1}$$

$$\sum_{m \geq 0} c_m q^m = \frac{5}{1+5^5 q} = \langle\langle \eta\,\eta\,\eta \rangle\rangle$$

$q=0$ $\quad 5.$

Instanton Sums and Monodromy
Lecture 3

$\rho: G \to U(1)^n$, $W(x_1,\dots,x_n)$ G-invariant.

Example 1

$G = U(1)$

x_0	x_1	x_2	$\cdots$	x_5
-5	1	1	$\cdots$	1

$$W(x_0,\dots,x_n) = x_0\, P_5(x_1,\dots,x_5)$$

Example 2 $\left(Bl_{sing}\, \mathbb{P}^{1,1,2,2,2}\right)$

$G = U(1)^2$.

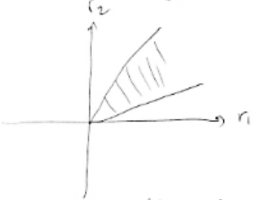
x_0	x_1	x_2	x_3	x_4	x_5	x_6
-4	0	0	1	1	1	1
0	1	1	0	0	0	-2

$$W(x_0,\dots,x_n) = x_0\, P(x_1,\dots,x_6)$$
G-invariant

$\mu^{-1}(r)/G$.

$r \in \mathfrak{g}^*$

example 2



topology of $\mu^{-1}(r)/G$, set of coordinate charts } constant within regions.

$$r_1 = -4|x_0|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2$$

"Secondary fan" = the fan structure on $\mathfrak{g}^*$.

$$\mu^{-1}(r)/G \cong \mathbb{C}^n \setminus Z(r) / G_{\mathbb{C}}$$

$$Z(r) = \bigcup_{I \in \mathcal{C}(r)} \bigcap_{i \in I} \{x_i = 0\}$$

$$I \subset \{1, \dots, n\}$$

$I \in \mathcal{C}(r) \iff \{x_i\}_{i \in I}$ is not a subset of coordinates in any coordinate.

One feature from physics:
1-loop correction,
complete determines location of singular locus.

$$\text{parameters} \begin{cases} \tau \in \mathrm{Hom}(\mathfrak{g}, \mathbb{C}) \\ \text{coeffs of } W \end{cases}$$

not relevant

Bosonic potential:

$$U = \frac{1}{2e^2} \sum (D_a)^2 + \sum \left| \frac{\partial W}{\partial \varphi_i} \right|^2 + 2 \sum \bar{\sigma}_a \sigma_b \sum Q_i^a Q_i^b |\varphi_i|^2$$

basis V_a of gauge fields

$$M^{2|(2,2)} \to \mathfrak{g}$$

$$\Sigma_a = \bar{D}_+ D_- V_a$$

$$\sigma_a = \text{lowest component } \Sigma_a|_{M^2}.$$

For generic values of $|\varphi_i|$'s (of parameters)

we get pos def Hermitian form on the space of σ's

to minimize, $\sigma_a = 0$.

For special values, pos. semi-definite.

Divergence

regulated with a cutoff:

→ effective description of theory in σ's.

conclusion about parameter values:

$$2\pi i \tau_a = \sum_{i=1}^{n} Q_i^a \log\left(\sum Q_i^b \sigma_b\right)$$

(assoc. to group G)

can be singularities assoc. to subgroups $G' \subseteq G$.

~~Example 1~~

$$q_a = e^{2\pi i \tau_a} = \prod_{i=1}^{n} \left(\sum Q_i^b \sigma_b\right)^{Q_i^a}$$

Example 1

x_0	x_1	...	x_5
-5	1	...	1

$$q = (-5\sigma)^{-5}\sigma^1\sigma^1\sigma^1\sigma^1\sigma^1 = (-5)^{-5}$$

Predicted singularity of physics at

$$q = (-5)^{-5}.$$

Correlation function (last time)

$$\frac{5}{1+5^5 q}.$$

Example 2

G)

X_0	X_1	X_2	X_3	X_4	X_5	X_6
-4	0	0	1	1	1	1
0	1	1	0	0	0	-2

$$q_1 = (-4\sigma_1)^{-4}(\sigma_1)^1(\sigma_1)^1(\sigma_1)^1(\sigma_1-2\sigma_2)$$

$$q_2 = (\sigma_2)^1(\sigma_2)^1(\sigma_1-2\sigma_2)^{-2}$$

$$q_1 = (-4)^{(-4)}\left(\frac{\sigma_1-2\sigma_2}{\sigma_1}\right) = (-4)^{-4}\left(1-2\frac{\sigma_2}{\sigma_1}\right.$$

$$q_2 = \frac{\sigma_2^2}{(\sigma_1-2\sigma_2)^2} = \frac{(\sigma_2/\sigma_1)^2}{\left(1-2\frac{\sigma_2}{\sigma_1}\right)^2}$$

$$\frac{\sigma_2}{\sigma_1} = \frac{1-4^4 q_1}{2}.$$

$$q_2 = \frac{\left(\frac{1-4^4 q_1}{2}\right)^2}{(4^4 q_1)^2}$$

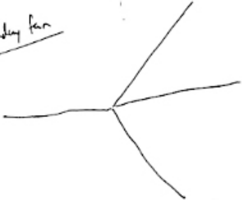
$$\boxed{4^9 q_1^2 q_2 = (1-4^4 q_1)^2}$$

$\sigma' =$ (gen. by exp q_2)

$$q_2 = \sigma_\phi^1 \sigma_\phi^2 (-2\sigma)^{-2} = (-2)^{-2}.$$

$$\boxed{q_2 = 1/4}$$

Secondary fan



Instanton sums

Instanton moduli spaces.

$$x_i = f_i(s,t) \quad \text{c-homog} \quad \mathbb{C}^2$$

$$\mathbb{CP}^1$$

$$\deg f_i = d_i.$$

Labeling:

$n_a, \quad a = 1, \ldots, \dim \mathfrak{g}.$

$$e^{2\pi i \sum \tau_a n_a} = \prod q_a^{n_a}$$

$$\deg f_i = \sum Q_i^a n_a = d_i.$$

Requirement: $\vec{n} \in$ dual cone to the cone in 2nd fan in which r lies.

$x_i \longleftrightarrow f_i(s,t)$ of degree d_i

$$f_i(s,t) = \sum_{j=0}^{d_i} f_i^{(j)} s^j t^{d_i - j}$$

$\{ f_i^{(j)} \}$ are homogeneous coords on instanton moduli space.

G acts on $f_i^{(j)}$ as it acts on x_i.

$G_{\mathbb{C}}$ acts in same way.

$$X = \mathbb{C}^n \setminus Z(r) / G_{\mathbb{C}}.$$

$$Z(r) = \bigcup_{I \in \mathcal{A}} \bigcap_{i \in I} \{x_i = 0\}$$

$$Z(r)_{\vec{n}} = \bigcup_{I \in \mathcal{A}} \bigcap_{i \in I} \{ f_i^{(j)} = 0 \}$$

$$\mathcal{M}_{\vec n} = \mathbb{C}^{n(\vec n)} \setminus Z(r)_{\vec n} \big/ G_{\mathbb{C}}.$$

$$= \mu_{\vec n}^{-1}(r) \big/ G.$$

where $\mu_{\vec n} : \mathbb{C}^{n(\vec n)} \to \mathfrak{g}^*$

replace $|X_i|^2$ with $\sum |f_i^{(j)}|^2$.

Provided that $d_i \geq 0$, get contributions to correlation functions.

$$\langle \eta_{i_1} \eta_{i_2} \eta_{i_3} (-K)^{-d_i+1} \chi_{\vec n} \rangle_{\vec n}$$

$\chi_{\vec n}$ only need if $d_i < 0$ for some i.

$$\chi_{\vec n} = \prod_{d_i < 0} (\xi_i)^{-d_i - 1}.$$

$$\xi_i^{(j)} = c_1(\{ f_i^{(j)} = 0 \})$$

this coho class is indep of j.

<u>linear relations</u>

$$\cancel{\eta_a = \sum Q_i^a \xi_i}$$

$$\xi_i = \sum Q_i^a \eta_a$$

<u>nonlinear relations</u>

previous: $\xi_{i_1} \cdots \xi_{i_k} = 0$

new: $(\xi_{i_1})^{d_{i_1}+1} \cdots (\xi_{i_k})^{d_{i_k}+1} = 0$

Example 2

$$X_j^{\vec{n}} = \langle \eta_1^{3-j} \eta_2^{j} K^{4n_1+1} x_{\vec{n}} \rangle_{\vec{n}}$$

$$(K = -4\eta_1)$$

$$X_j^{\vec{n}} = \cancel{\pm} 4^{4n_1+1} \langle \eta_1^{4n_1+4-j} \eta_2^{j} \rangle_{\vec{n}}.$$

using relations

$$(\xi_1)^{d_1+1} (\xi_2)^{d_2+1} = \eta_2^{d_1+d_2+2} = 0$$

$$(\xi_3)^{d_3+1} (\xi_4)^{d_4+1} (\xi_5)^{d_5+1} (\xi_6)^{d_6+1}$$

$$= \eta_1^{d_3+d_4+d_5+3} (\eta_1 - 2\eta_2)^{d_6+1}$$

and some combinatorics.

$$X_j = \sum_{\substack{n_1 \geq 0 \\ n_2 \geq 0}} 2^{8n_1+2n_2+3-j} \binom{n_1+1-j}{2n_2+1-j} q_1^{n_1} q_2^{n_2}$$

$$X_0 = \frac{8}{\Delta} \qquad \Delta = (1-2^8 q_1)^2 - 2^{18} q_1^2 q_2$$

$$X_1 = \frac{4(1-2^8 q_1)}{\Delta}$$

$$X_2 = \frac{8 q_2 (2^9 q_1 - 1)}{(1-4q_2)\Delta}$$

$$X_3 = \frac{}{(1-4q_2)^2 \Delta}$$

Mirror symmetry

$\rho : G \to U(1)^n$, W.

$\rightsquigarrow \quad \hat{\rho} : \hat{G} \to U(1)^{\hat{n}}$, $\hat{W}$

Another set of correlation functions:
related to VHS defined by $\hat{W} = 0$.
discriminant locus for $\hat{W}$, as a function of parameters, is

$$\Delta (1-4q_2)^2 \quad \text{in example 2.}$$

(Γ-description $\leftrightarrow$ "Horn uniformization")
Kapranov

Instanton Sums and Monodromy
Lecture 4

$G \to \mathbb{C}^n$

$U(1)^{n-d} \to U(1)^n$

$\mathbb{Z}^{n-d} \to \mathbb{Z}^n \to \mathbb{Z}^d$

$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \mapsto v_i \in \mathbb{Z}^d$

GLSM: convex hull of v_i = Gorenstein cone.

x_0

unit x_0: reflexive polytope

$P = S\,T$

$\big\downarrow$ "mirror to"

${}^tP = {}^tT\,{}^tS$ $\qquad \hat{W} \qquad \hat{G}$

polar of a reflexive polytope.

Parameters in mirror symmetry:

$\tau_a \longleftrightarrow$ basis of $\mathfrak{g}$.

$q_a = e^{2\pi i \tau_a}$

$\hat{W} = \sum \hat{c}_j x^{m_j}$

$U(1)^d$ acts on $\hat{X}$
form invariant quot under $U(1)^d$ action

$$r_a = -\frac{1}{2\pi} \sum \hat{Q}_i^a \log|\hat{c}_i|$$

$$\frac{1}{\pi}\theta_a + i r_a = \tau_a$$

$$q_a = e^{2\pi i \tau_a} = \prod |\hat{c}_i|^{Q_i^a} \cdot \text{phase}$$

$$\boxed{q_a = \prod \hat{c}_i^{\hat{Q}_i^a}}$$

Quintic case

mirror thay.

$$\hat{c}_0 \hat{x}_1 \cdots \hat{x}_5 + \hat{c}_1 \hat{x}_1^5 + \cdots + \hat{c}_5 \hat{x}_5^5 = 0$$

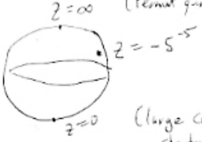
Single invariant quantity

$$z = \frac{\hat{c}_1 \cdots \hat{c}_5}{\hat{c}_0^5}$$

VHS Periods and their monodromy.

period for mirror quintic

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$



(Fermat quintic)

(large complex structure limit)

$$\mathcal{D}\Phi = 0.$$

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\cdots\left(5z\frac{d}{dz}+4\right)$$

$$\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(5\alpha+1)(5\alpha+2)(5\alpha+3)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5} z^{\alpha+n}$$

$$\mathcal{D}\Phi_\alpha(z) = \alpha^4 z^\alpha.$$

$$\mathbb{C}[\alpha]/(\alpha^4)$$

$$\log z \to \log z + 2\pi i$$

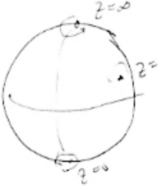
$$\Phi_\alpha(z) \mapsto e^{2\pi i \alpha}\,\Phi_\alpha(z)$$

$$\Phi_\alpha(z) = \sum_{m=0}^{\infty} \frac{[\alpha(\alpha+1)\cdots(\alpha+(m-1))]^5}{(5\alpha)(5\alpha+1)\cdots(5\alpha+5m-1)}\, z^{-\alpha-m}$$

$$\mathcal{D}\,\Phi_\alpha(z) = \underbrace{-5(-5\alpha+1)(-5\alpha+2)(-5\alpha+3)(-5\alpha+4)}_{\overset{\shortparallel}{0}}\, z^{-\alpha+1}$$

to get solutions

$\times\, e^{2\pi i \alpha}$ = mult. by a 5^{th} root 1.

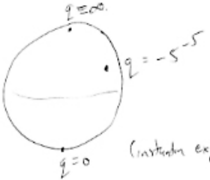


order 5

(no good formal p.s.)

$e^{2\pi i \alpha}$ on $\mathbb{C}[\alpha]/\alpha^4$

<u>periods</u> span $H^3(\hat{X}_z)$



(instanton exp.)

$$H^3(\hat{X}_q) \longleftrightarrow H^{ev}(X_q)$$

natural bundle with fiber $H^{ev}(X)$

near $q=0$:

$(D_1\cdot D_2\cdot D_3) \Longleftrightarrow$ multiplication on coh, given Poincaré duality pairing

$$\langle D_1\cdot D_2, D_3\rangle$$

$$D_1\cdot D_2\cdot D_3 + \sum_{n\geq 0} c_n q^n$$

is a deformation of ring structure.

Quintic ~~Quantum coh. ring~~

Classical coh ring

$$\mathbb{Q}[\alpha]/(\alpha^4)$$

$e^{2\pi i\alpha}$ on $\mathbb{Q}[\alpha]/(\alpha^4)$

$$T = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ & 1 & 1 & \frac{1}{2} \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

2-dim'l quantum field theory:

string theory has additional features including "D-branes".

D-branes $\xleftrightarrow{bd}$ Derived category of coherent sheaves on X.

Automorphisms of derived category:

$$T = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ & 1 & 1 & \frac{1}{2} \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Kontsevich:

$$\Delta \subseteq X \times X, \quad \mathcal{O}_\Delta.$$

$$\mathcal{E} \longmapsto (p_2)_* (p_1^* \mathcal{E} \otimes \mathcal{O}_\Delta)$$

Calabi-Yau is exactly what's required for this to be an automorphism:

$$\gamma \longmapsto \gamma - \left(\int \gamma \wedge \mathrm{Todd}\, T_X \right) 1_X.$$

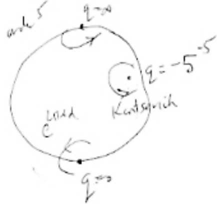
1_X generates $H^0(X)$.

Calculate for quintic:

$$S = \begin{pmatrix} 1 & & & \\ -\frac{25}{6} & 1 & & \\ 0 & 0 & 1 & \\ -5 & 0 & 0 & 1 \end{pmatrix}$$

$$TS = \begin{pmatrix} -4 & 1 & \frac{1}{2} & \frac{1}{6} \\ -\frac{20}{3} & 1 & 1 & \frac{1}{2} \\ -5 & 0 & 1 & 1 \\ -5 & 0 & 0 & 1 \end{pmatrix}$$

$(TS)^5 = $ identity.



Kontsevich verified this proposal for all known 1-param examples with a Fermat point.

There is a bundle over moduli, represents D-branes in physics, and there should geometric transformations corresponding to monodromy around loops.

Start with a sheaf $\mathcal{F}$.

$$\mathcal{I}_{\mathcal{F}} = \mathrm{Ker}\left(\mathcal{F} \boxtimes \mathcal{F}^* \to \mathcal{O}_\Delta\right)$$

$\mathcal{I}_{\mathcal{F}}$ is on $X \times X$.

$$\mathcal{E} \mapsto (p_2)_*\left(p_1^* \mathcal{E} \otimes \mathcal{I}_{\mathcal{F}}\right)$$

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{F}) = \begin{cases} \mathbb{C} & \text{if } k = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

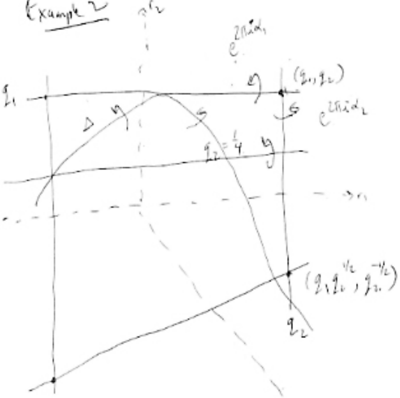
"homology sphere for ext"

$\mathcal{F} = \mathcal{O}_X$. $\quad \mathrm{Ext}^k(\mathcal{O}, \mathcal{O}) = H^k(\mathcal{O}_X)$

Seidel + Thomas, Horja:

many examples have been checked.

Example 2



$$|q_j| = e^{-2\pi r_j}$$

Lecture 5

2-parameter example
"octic"

$\mathbb{P}^{1,1,2,2,2}$ $\quad P(x_1,\dots,x_5)$ = wtd degree 8.

Introduce x_6 (homog. coord used to resolve sing.)

Introduce x_0 (for GLSM)

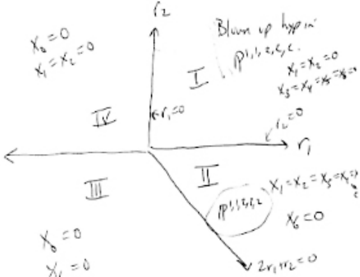
$$W(x_0,x_1,\dots,x_6) = x_0 P(x_1,\dots,x_6)$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$U(1)$	-4	0	0	1	1	1	1
$U(1)$	0	1	1	0	0	0	-2

$[\,-8 \;\; 1 \;\; 1 \;\; 2 \;\; 2 \;\; 2 \;\; 0\,]$

$$\mu^{-1}(r)/G \cong \mathbb{C}^n - Z(r)/G_{\mathbb{C}}.$$

($\mathfrak{g}^*$ has coords r_1, r_2)



(can eliminate x_6, $\mathbb{C}^5/\Gamma$)

Instanton sums:

$\mathbb{CP}^1 \to X$

$f: \mathbb{C}^2 \to \mathbb{C}^n$

$$\frac{\partial W}{\partial x_i} = 0 \quad \Rightarrow \quad X_0 = 0$$

$$P(x_1, \ldots, x_6) = 0.$$

$$X_0 = 0 \Leftarrow f_0(s,t) \text{ has degree } d_0 < 0$$

$$P(f_1(s,t), \ldots, f_6(s,t)) = \text{poly in } s,t \text{ of degree } d_0.$$

make it vanish at $-d_0+1$ points

$$(+K)^{-d_0+1}$$

$$-\sum \xi_i = K.$$

$X_0 = 0$ can be imposed on the moduli spaces of instantons.

$\mathbb{C}^2 \to \mathbb{C}^{n-1}$ setting $X_0 = 0$.

toric analysis: $\mu_0^{-1}(r)/G$ is compact if we're in region I.

$$\langle\langle D_1\, D_2\, D_3 \rangle\rangle = \sum_{\vec{n}} \langle D_1 \cdot D_2 \cdot D_3 \cdot (K)^{-d_0+1} \rangle q^{\vec{n}}$$

makes sense because instanton moduli space are compact

if $d_j < 0$, need $X_{\vec{n}} = \prod_{d_j < 0} \xi_j^{-d_j - 1}$

Instanton sum calculations in regions II, III, IV ...

Method: proposed by DRM + Ronen Plesser

(not a complete derivation).

Observation: form of

$$(K)^{-d_0+1} = (K)^{-d_0-1+2}$$

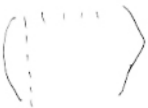
and of

$$\prod_{d_j<0} \xi_j^{-d_j-1} \qquad d_1, d_2, \dots, d_n$$

are very similar.

$$\boxed{\xi_0 = -\sum_{j>0} \xi_j}$$

P = ST



$$\Rightarrow \sum_{j>0} \xi_j = 0$$

$$\left(\prod_{\substack{d_j<0 \\ j>0}} \xi_j^{-d_j-1} \right) \xi_0^2$$

$$\sum \left\langle D_1 \cdot D_2 \cdot D_3 \cdot \left(\prod_{\substack{d_j<0 \\ j>0}} \xi_j^{-d_j-1} \right) \xi_0^2 \right\rangle_{\vec{n}} q^{\vec{n}}$$

Coh. of compact toric variety:

- algebra with an extra mapping

$$\langle\ \rangle : H^*(X) \to \mathbb{C}$$
$$\gamma \mapsto \gamma|_{[X]}$$

"Frobenius algebra"

$H^2(X)$ generates $H^*(X)$

$$\langle D_1 \cdot D_2 \cdot D_3 \cdot \dots \cdot D_k \rangle|_{[M_{?}]} \in \mathbb{Q}$$

~~Given a class ξ_0 in H~~

Formulate instanton calculation using

generators $\xi_1, \dots, \xi_n$ of the algebra

relations: linear relations from G.

$$\left(\text{including } \xi_0 = -\sum \xi_j\right)$$

nonlinear relations

$$\xi_{i_1} \cdots \xi_{i_j} = 0$$

if $(x_{i_1} = \cdots x_{i_j} = 0)$ is a compact Z.

This does not determine a Frobenius algebra.

New ring structure: $\overline{D_1 \cdot D_2} = D_1 \cdot D_2 \cdot \xi_0 \in$ algebra

algebra $/$ Ker (mult. by ξ_0).

Leads to formal calculations for instanton sums in every region, agree with original in geometric regions (e.g. region I).

Quintic case:

$$\frac{5}{1+5^5 q} = \sum (-5)^{-5} q^n$$

$$q = e^{-2\pi r} \qquad r>0$$

(geometric region)

Can expand in q^{-1} $(r<0)$

$$\frac{5q^{-1}}{q^{-1}+5^5} = \text{Series in } q^{-1}$$

coeffs are calculated as above

In example 2

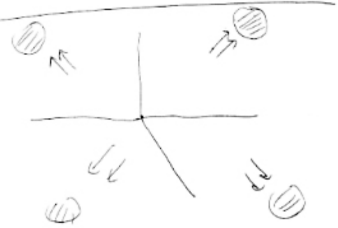
calculations can be made in any region.

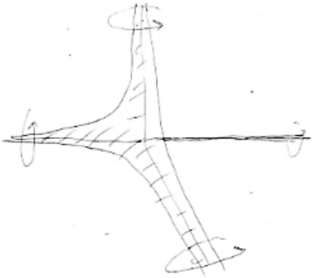
easiest to make in region 3

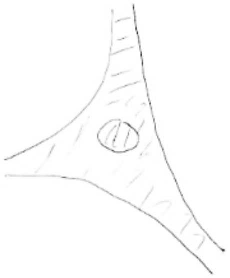
$$X_j^{\vec{n}} = -2^{8n_1+2n_2+3j}\binom{j-2-2n_2}{j-2-n_1}$$

$\vec{n} \in$ dual cone to region III

weighted with appropriate q's







$\cancel{|\Delta| \geq 0}$

$\left\{ |\Delta|_{(|q|,|2|)} : \Delta^{(q,2)}_{c \cup} \right\}$

<u>Mysterious aspect</u>: How to analyze the behavior in the shaded region