CRYSTALLINE SHEAVES, SYNTOMIC COHOMOLOGY AND p-ADIC POLYLOGARITHMS

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In [BD92] (see also [HW98]), A. A. Beilinson and P. Deligne constructed the motivic polylogarithmic sheaf on $\mathbb{P}^1_{\mathbb{Q}}\backslash\{0,1,\infty\}$. Its specializations at primitive d-th roots of unity give the Beilinson's elements of $H^1_{\mathcal{M}}(\mathbb{Q}(\mu_d),\mathbb{Q}(m)) = K_{2m-1}(\mathbb{Q}(\mu_d))\otimes\mathbb{Q}$ $(m\geq 1)$, whose images under the regulator maps to Deligne cohomology are the values of m-th polylogarithmic functions at primitive d-th roots of unity. The polylogarithmic functions appear as the (complex) period functions. In these notes, we will show that the following two p-adic realizations correspond to each other via the theory of crystalline sheaves by G. Faltings [Fal89] V f).

One is the realization in the category of smooth $\mathbb{Q}_p$-sheaves on $(\mathbb{P}^1_{\mathbb{Q}_p}\backslash\{0,1,\infty\})_{\text{ét}}$. Its specializations at primitive d-th roots of unity give the images of the Beilinson's elements in $H^1(\mathbb{Q}_p(\mu_d),\mathbb{Q}_p(m)) = \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p,\mathbb{Q}_p(m))$ $(m\geq 1)$ under the regulator maps. It is known that they also coincide with the Soulé's cyclotomic elements.

The other is the realization in the category of (log) filtered convergent F-isocrystals on $\mathbb{P}^1_{\mathbb{Z}_p}$ endowed with the log structure associated to the divisor $\{0,1,\infty\}$. In [Ban00a], K. Bannai constructed the realization in the category of filtered overconvergent F-isocrystals on $\mathbb{P}^1_{\mathbb{Z}_p}\backslash\{0,1,\infty\}$ using rigid syntomic cohomology, gave an explicit description of it in terms of p-adic polylogarithmic functions, and then proved that the specializations of the crystalline polylogarithmic sheaf at primitive d-th roots of unity for d prime to p give the values of p-adic polylogarithmic functions at primitive d-th roots of unity. His construction and calculation also work for log filtered convergent F-isocrystals and log syntomic cohomology, and we prefer the log version to the overconvergent one because we have the theory of crystalline sheaves by G. Faltings for the former.

We will see that some functions which live in a big ring $\mathcal{B}_{\text{crys}}$ and satisfy the same differential equations as the complex polylogarithmic functions, appear as the p-adic period functions of these p-adic realizations (Proposition 6.8).

By combining the result of K. Bannai explained above with our comparison theorem, we immediately obtain a new proof of the following fact: The images of the Beilinson's elements in $H^1(\mathbb{Q}_p(\mu_d),\mathbb{Q}_p(m))$ $(m\geq 1)$ under the regulator maps for d prime to p coincide with the images of the values of m-th p-adic polylogarithmic functions at primitive d-th roots of unity under the map:

$$\begin{aligned}K(m) = K(m)/(1-\varphi)(Fil^0K(m)) &\xrightarrow{\cong} \mathrm{Ext}^1_{MF_K(\varphi)}(K,K(m))\\ &\xrightarrow{\subset} Ext^1_{Rep_{\mathbb{Q}_p}(G_K)}(\mathbb{Q}_p,\mathbb{Q}_p(m)) = H^1(K,\mathbb{Q}_p(m)).\end{aligned}$$

Here $K=\mathbb{Q}_p(\mu_d)$.

In [Ban00c], K. Bannai also studied the crystalline realization of the motivic elliptic polylogarithm on a CM elliptic curve minus 0 and its relation with the values of the p-adic L-function at positive integers (a p-adic analogue of the Beilinson conjecture). Our argument also works in this case.

I would like to thank K. Bannai for kindly explaining me some basic facts on the motivic polylogarithm and for useful discussions.

1. Filtered convergent F-isocrystals

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k, and let O_K denote the ring of integers. let K_0 be the field of fractions of the ring of Witt vectors $W(k)$. For a scheme X_0 of finite type over $\mathrm{Spec}(k)$, P. Berthelot and A. Ogus ([Ber86], [Ogu84]) defined the notion of convergent isocrystals on X_0/K, which can be regarded as p-adic local systems in characteristic p at least in the case X_0 is proper over k; they are defined in an abstract way using the notion of enlargements of X_0 in general, but if a global embedding of X_0 into a p-adic smooth formal scheme Y over $\mathrm{Spf}(O_K)$ is given, then they can be described more explicitly as modules with connections on the tubular neighborhood $]X_0[_{Y_K}$ of X_0 in the rigid analytic space Y_K associated to Y.

Suppose that X_0 is the special fiber of a proper smooth scheme X over $\mathrm{Spec}(O_K)$. In this case, G. Faltings ([Fal89] V f)) considered filtered convergent F-isocrystals on X, which are just convergent isocrystals on X_0/K_0 endowed with Frobenius automorphisms Φ and some kind of descending filtrations after the base change to K, and generalized the theory of Fontaine on crystalline p-adic representations and filtered φ-modules ([Fon82], [Fon94a], [Fon94b]) to crystalline p-adic sheaves on $(X_K)_{\text{ét}}$ and filtered convergent F-isocrystals. See §3 for more details.

Using the theory of logarithmic structures in the sense of Fontaine and Illusie ([Kat89]), one can generalize, in an almost straightforward way, the notion of convergent isocrystals to fine log schemes of finite type over $\mathrm{Spec}(k)$, and the notion of filtered convergent F-isocrystals to fine log formal schemes smooth over $\mathrm{Spf}(O_K)$. Here we endow $\mathrm{Spec}(k)$ and $\mathrm{Spf}(O_K)$ with the trivial log structures. These generalizations for a proper smooth scheme X over $\mathrm{Spec}(O_K)$ with the log structure defined by a divisor D with relative normal crossings, were also studied by G. Faltings ([Fal89], [Fal90]). See also [Shi00] 5.1.

In this section, we assume $K = K_0$, consider a p-adic smooth formal scheme X over $\mathrm{Spf}(O_K)$ endowed with the log structure M defined by a divisor D with relative normal crossings and explain the local description of filtered convergent F-isocrystals on (X, M) in terms of modules with connections. To understand this description, we do not need any knowledge on log structures.

First assume X is affine and set $A = \Gamma(X, \mathcal{O}_X)$. We further assume that there exist $t_1, t_2, \ldots, t_d \in A$ such that D is defined by the equation $t_1 t_2 \cdots t_d = 0$, the divisor defined by $t_i = 0$ is empty or smooth over $\mathrm{Spf}(O_K)$ for each i, and $\{dt_i | 1 \le i \le d\}$ is a basis of Ω^1_{X/O_K}. We call such set of elements $t_1, t_2, \ldots, t_d$ a log coordinate of (X, M).

Set

$$\Omega_A(\log) := \Gamma(X, \Omega_{X/O_K}(\log D)) \cong \bigoplus_{1 \le i \le d} A \cdot \frac{dt_i}{t_i}$$

and

$$\Omega^i_A(\log) := \wedge^i_A \Omega_A(\log) \quad (i \in \mathbb{Z},\ i \ge 0).$$

Choose a lifting of the absolute Frobenius of $(X,M)\otimes_{O_K} k$ to (X,M) which is equivalent to giving a lifting $F_X\colon X\to X$ of the absolute Frobenius of $X\otimes_{O_K} k$ such that $F_X(t_i)=t_i^p\cdot(\text{unit})$ for each i such that t_i is not invertible. We denote by φ the endomorphism of A induced by F_X.

Giving a filtered convergent F-isocrystal on (X,M) is equivalent to giving the following data $(\mathcal{E}, Fil^i, \nabla, \Phi)$:

- a finite projective $K\otimes A$-module $\mathcal{E}$.
- a decreasing filtration $Fil^i\mathcal{E}$ $(i\in\mathbb{Z})$ by direct factors of $\mathcal{E}$ as $K\otimes A$-modules such that $Fil^i\mathcal{E}=0$ $(i>>0)$ and $Fil^i\mathcal{E}=\mathcal{E}$ $(i<<0)$.
- an integrable connection $\nabla\colon \mathcal{E}\to\mathcal{E}\otimes_A\Omega_A(\log)$ satisfying:
  - Griffiths transversality: $\nabla(Fil^i\mathcal{E})\subset Fil^{i-1}\mathcal{E}\otimes_A\Omega_A(\log)\quad(i\in\mathbb{Z})$
  - Convergence: For any $x\in\mathcal{E}$ and any $\delta\in\mathbb{R}_{>0}$,
  $$p^{[(n_1+n_2+\cdots+n_d)\delta]}\frac{1}{n_1!n_2!\cdots n_d!}\left(\prod_{1\le i\le d}\prod_{0\le j<n_i}(\nabla_i^{\log}-j)\right)(x),$$
  $$(n_1,n_2,\dots,n_d)\in\mathbb{N}^d,$$
  converges to 0 as $n_1+n_2+\cdots+n_d\to\infty$. Here, for $a\in\mathbb{R}$, $[a]$ denotes the maximal integer $\le a$, the endomorphisms $\nabla_i^{\log}$ on $\mathcal{E}$ are defined by the formula $\nabla(y)=\sum_{1\le i\le d}\nabla_i^{\log}(y)d\log(t_i)$ $(y\in\mathcal{E})$, and $\mathcal{E}$ is endowed with the p-adic topology induced by a finitely generated A-submodule $\mathcal{E}^0$ of $\mathcal{E}$ such that $K\otimes\mathcal{E}^0=\mathcal{E}$, which is independent of the choice of $\mathcal{E}^0$.
- an isomorphism $\Phi\colon A_\varphi\otimes_A\mathcal{E}\to\mathcal{E}$ which is horizontal with respect to the connection ∇.

If you choose another lifting of Frobenius F'_X, then the integrable connection induces a canonical isomorphism:

$$A_\varphi\otimes_A\mathcal{E}\cong A_{\varphi'}\otimes_A\mathcal{E}, \tag{1.1}$$

which sends $1\otimes x$ to

$$\sum_{(n_1,\dots,n_d)\in\mathbb{N}^d}(n_1!\cdots n_d!)^{-1}\Big(\prod_{1\le i\le d}(\varphi(t_i)\varphi'(t_i)^{-1}-1)^{n_i}\Big)\otimes\Big(\prod_{1\le i\le d}\prod_{0\le j<n_i}(\nabla_i^{\log}-j)\Big)(x).$$

Here we need the convergence of ∇. Hence the definition of Φ is independent of the choice of F_X. The convergence condition on the connection ∇ is independent of the choice of $t_1,t_2,\dots,t_d$, but the author does not know a direct proof without using enlargements which involves the theory of log structures.

For a general (X,M), there exists an étale covering $\{X_\alpha\to X\}$ such that each X_α satisfies the above conditions. If we choose such an étale covering, then giving a filtered convergent F-isocrystal on (X,M) is equivalent to giving a data $(\mathcal{E}_\alpha, Fil^i_\alpha, \nabla_\alpha, \Phi_\alpha)$ for each (X_α, M_α) and a descent data in the obvious sense.

In the case that K is ramified i.e. $K\neq K_0$, to describe filtered convergent F-isocrystals similarly, we need to choose a lifting (X',D') of $(X\otimes k, D\otimes k)$ over O_{K_0} and an isomorphism $(X,D)\cong(X',D')\otimes_{O_{K_0}}O_K$. Once we choose them, the description is almost the same as in the case $K=K_0$. We denote by

$$MF^{\nabla}_{(X,M)/K}(\Phi)$$

the category of filtered convergent F-isocrystals on X with the log structure M defined by D.

2. Syntomic cohomology

The syntomic cohomology with constant p-torsion coefficients was introduced by J.-M. Fontaine and W. Messing ([FM87]) as the cohomology groups of some sheaves called S_n^r on the syntomic sites in their proof of the crystalline conjecture, which compares the p-adic étale cohomology and the crystalline cohomology for a proper smooth scheme over O_K. The projections of S_n^r on the étale site, which are sometimes called the syntomic complexes, can be described explicitly in terms of the de Rham complexes with coefficients in some PD-envelopes and the Frobenius endomorphisms on them; the syntomic cohomology is given as the étale cohomology of the syntomic complex. This construction using de Rham complexes can be generalized to non-constant coefficients. Such a generalization is studied by W. Niziol ([Niz97], [Niz]) and K. Bannai ([Ban00b]).

Assume $K = K_0$ and let (X, M) and D be as in §1. First assume X is affine and (X, M) has a log coordinate globally as in §1. Choose a lifting of Frobenius F_X on (X, M) and denote F_X^* on A and $\Omega_A^q(\log)$ by φ. In this case, for a filtered convergent F-isocrystal $\mathcal{E}$ corresponding to $(\mathcal{E}, Fil^i, \nabla, \Phi)$, we define the de Rham complex $\mathrm{DR}_{X,F_X}((X, M), \mathcal{E})$ to be $\mathcal{E} \otimes_A \Omega_A^\bullet(\log)$. We define the filtration Fil^i ($i \in \mathbb{Z}$) on it by $Fil^i(\mathcal{E} \otimes_A \Omega_A^q(\log)) = Fil^{i-q}\mathcal{E} \otimes_A \Omega_A^q(\log)$. The isomorphism Φ induces a semi-linear endomorphism on $\mathcal{E}$, which we denote by φ. Note that φ depends on the choice of the lifting of Frobenius F_X. Since Φ is horizontal, $\varphi \otimes \varphi$ defines an endomorphism on $\mathrm{DR}_{X,F_X}((X, M), \mathcal{E})$, which we again denote by φ.

We define the complex $\mathcal{S}_{X,F_X}((X, M), \mathcal{E})$ to be the mapping fiber of

$$1 - \varphi \colon Fil^0(\mathrm{DR}_{X,F_X}((X, M), \mathcal{E})) \longrightarrow \mathrm{DR}_{X,F_X}((X, M), \mathcal{E}).$$

The degree q-part of the complex $\mathcal{S}_{X,F_X}((X, M), \mathcal{E})$ is the direct sum:

$$(Fil^{-q}\mathcal{E}) \otimes_A \Omega_A^q(\log) \bigoplus \mathcal{E} \otimes_A \Omega_A^{q-1}(\log)$$

and the differential map is given by

$$d(x, y) = (\nabla(x), (1 - \varphi)(x) - \nabla(y)).$$

We define the crystalline cohomology and the syntomic cohomology of $\mathcal{E}$ by

$$H^m_{\mathrm{crys}}((X, M), \mathcal{E}) := H^m(\mathrm{DR}_{X,F_X}((X, M), \mathcal{E})),$$
$$H^m_{\mathrm{syn}}((X, M), \mathcal{E}) := H^m(\mathcal{S}_{X,F_X}((X, M), \mathcal{E})).$$

The endomorphism φ on the de Rham complex induces a semi-linear endomorphism φ of the crystalline cohomology. A priori, the definitions of the syntomic cohomology and φ on the crystalline cohomology depend on the choice of the lifting of Frobenius F_X. To prove the independence, we need a more general construction which will be explained later.

If we define $Fil^i H^m_{\mathrm{crys}}((X, M), \mathcal{E})$ to be $H^m(Fil^i(\mathrm{DR}_X((X, M), \mathcal{E})))$, which may not be a submodule of $H^m_{\mathrm{crys}}((X, M), \mathcal{E})$, we have a long exact sequence:

$$\text{(2.1)} \quad \to H^m_{\mathrm{syn}}((X, M), \mathcal{E}) \to Fil^0 H^m_{\mathrm{crys}}((X, M), \mathcal{E}) \xrightarrow{1-\varphi} H^m_{\mathrm{crys}}((X, M), \mathcal{E}) \to \cdots.$$

For $m = 1$, we have the following interpretation of syntomic cohomology, which generalize [BK90] Lemma 4.4.

Proposition 2.2 (cf. [Ban00a] §4). *There exists a canonical isomorphism:*

$$H^1_{syn}((X, M), \mathcal{E}) \cong Ext^1_{MF^\nabla_{(X,M)/K}(\Phi)}(\mathcal{O}, \mathcal{E}),$$

where $\mathcal{O}$ denotes the constant filtered convergent F-isocrystal, which corresponds to $K \otimes A$ with $Fil^i(K \otimes A) = K \otimes A$ $(i \leq 0)$, 0 $(i > 0)$ and the obvious ∇ and Φ.

Proof. We only give a construction of the map from the LHS to the RHS. The proof of the isomorphism is straightforward and is left to the reader. Let $x \in (Fil^{-1}\mathcal{E}) \otimes_A \Omega_A(\log)$ and $y \in \mathcal{E}$ and assume $(1-\varphi)(x) = \nabla(y)$ and $\nabla(x) = 0$. Then the extension corresponding to the class of the cocycle (x, y) is defined to be $\mathcal{F} := (K \otimes A) \oplus \mathcal{E}$ with $Fil^i = Fil^i \oplus Fil^i$, $\nabla(1,0) = (0,x)$ and $\Phi(1,0) = (1,-y)$. The integrability of ∇ on $\mathcal{F}$ is equivalent to $\nabla(x) = 0$ and the condition that Φ on $\mathcal{F}$ is horizontal is equivalent to $(1-\varphi)(x) = \nabla(y)$. $\square$

For a general (X, M), the construction becomes more complicated because a lifting of Frobenius does not exist globally in general. We choose an étale hypercovering $X_\bullet \to X$, a closed immersion of $(X_\bullet, M|_{X_\bullet})$ into a simplicial smooth fine log formal scheme $(Y_\bullet, N_\bullet)$ over $\mathrm{Spf}(O_K)$ and a lifting of Frobenius $F_{Y_\bullet}$ of $(Y_\bullet, N_\bullet)$ such that, for each $i \in \mathbb{N}$, X_i and Y_i are affine. Such a hypercovering, a closed immersion and a lifting of Frobenius always exist. We consider the "closed tube of radius $p^{-\frac{1}{n}}$" $(Z_\bullet^{(n)}, L_\bullet^{(n)})$ of $(X_\bullet, M|_{X_\bullet})$ in $(Y_\bullet, N_\bullet)$. Let A_i, B_i and $C_i^{(n)}$ denote the rings of coordinates of X_i, Y_i and $Z_i^{(n)}$ respectively. If $M|_{X_i}$ is the pull-back of N_i, then $C_i^{(n)}$ is the p-adic completion of $B_i\left[\frac{(I_i)^n}{p}\right]$, where I_i denotes the kernel of $B_i \to A_i$, and $L_i^{(n)}$ is the pull-back of N_i. The log formal scheme $(Z_i^{(n)}, L_i^{(n)})$ can be naturally regarded as an enlargement of (X, M) over $\mathrm{Spf}(O_K)$. Hence one can evaluate a filtered F-isocrystal $\mathcal{E}$ on it and obtains a $K \otimes C_i^{(n)}$-module $\mathcal{E}_i^{(n)}$ endowed with an integrable connection, a filtration and a Frobenius. We define the de Rham complex $\mathrm{DR}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E})$ to be the projective limit with respect to n of the complex associated to the double complex:

$$\mathcal{E}_0^{(n)} \otimes_{B_0^{(n)}} \Omega^\bullet_{B_0^{(n)}}(\log) \longrightarrow \mathcal{E}_1^{(n)} \otimes_{B_1^{(n)}} \Omega^\bullet_{B_1^{(n)}}(\log) \longrightarrow \mathcal{E}_2^{(n)} \otimes_{B_2^{(n)}} \Omega^\bullet_{B_2^{(n)}}(\log) \longrightarrow \cdots,$$

which is filtered and has a Frobenius endomorphism φ. The differential maps are, as usual, defined to be alternating sums of the pull-backs by the projections. We define the syntomic complex $\mathcal{S}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E})$ to be the mapping fiber of

$$1 - \varphi \colon Fil^0(\mathrm{DR}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E})) \longrightarrow \mathrm{DR}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E}).$$

The image of the syntomic complex in the derived category $D^+(\mathbb{Q}_p\text{-Vect})$ of the category $\mathbb{Q}_p$-Vect of $\mathbb{Q}_p$-vector spaces is unique up to canonical isomorphisms: For another $(X'_\bullet, M|_{X_\bullet}) \hookrightarrow (Y'_\bullet, N'_\bullet)$, $F_{Y'_\bullet}$, we take the fiber product $X''_\bullet$ of $X_\bullet$ and $X'_\bullet$ over X and the fiber product $(Y''_\bullet, N''_\bullet)$ of $(Y_\bullet, N_\bullet)$ and $(Y'_\bullet, N'_\bullet)$ over $\mathrm{Spf}(O_K)$. Then the natural morphisms from the syntomic complexes for $(X_\bullet, Y_\bullet)$ and $(X'_\bullet, Y'_\bullet)$ to that for $(X''_\bullet, Y''_\bullet)$ are both quasi-isomorphisms. Here we use the Poincaré lemma and the vanishing of $R^1 \varprojlim_n \mathcal{E}_i^{(n)}$. For the Poincaré lemma holds, we need to work for an "open tube" and that is the reason why we take the projective limit with respect to n in the definition of the de Rham and syntomic complexes. The above argument also works for the de Rham complex with $Fil^\bullet$ and φ.

We define the crystalline cohomology and the syntomic cohomology by

$$H^m_{\mathrm{crys}}((X, M), \mathcal{E}) := H^m(\mathrm{DR}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E}))$$
$$H^m_{\mathrm{syn}}((X, M), \mathcal{E}) := H^m(\mathcal{S}_{X_\bullet, Y_\bullet, F_{Y_\bullet}}((X, M), \mathcal{E})).$$

Proposition 2.2 is still valid for a general (X,M). If we define $Fil^iH^m_{\mathrm{crys}}((X,M),\mathcal{E})$ to be the cohomology of Fil^i of the de Rham complex, then we have the exact sequence (2.1).

Assume (X,M) is the p-adic formal completion of a proper smooth scheme $\mathcal{X}$ over $\mathrm{Spec}(O_K)$ with the log structure defined by a divisor with relative normal crossings. Set $\mathcal{X}_K := \mathcal{X}\times_{\mathrm{Spec}(O_K)}\mathrm{Spec}(K)$. Then a filtered convergent F-isocrystal on (X,M) defines a locally free $\mathcal{O}_{\mathcal{X}_K}$-module E endowed with an integrable connection with log poles ∇ and a decreasing filtration $Fil^\bullet$ by direct factors, and we have canonical isomorphisms:

$$H^m_{\mathrm{crys}}((X,M),\mathcal{E})\cong H^m(\mathcal{X}_K, E\otimes_{\mathcal{O}_{\mathcal{X}_K}}\Omega^\bullet_{\mathcal{X}_K}(\log)),$$
$$Fil^iH^m_{\mathrm{crys}}((X,M),\mathcal{E})\cong H^m(\mathcal{X}_K, Fil^{i-\bullet}E\otimes_{\mathcal{O}_{\mathcal{X}_K}}\Omega^\bullet_{\mathcal{X}_K}(\log)).$$

Hence these are finite dimensional K-vector spaces and $Fil^i=0$ $(i>>0)$. Furthermore, if the spectral sequence with respect to the filtration $Fil^{i-\bullet}E\otimes\Omega^\bullet_{\mathcal{X}}(\log)$ degenerates at E_1, then the natural map $Fil^iH^m_{\mathrm{crys}}((X,M),\mathcal{E})\to H^m_{\mathrm{crys}}((X,M),\mathcal{E})$ $(i\in\mathbb{Z})$ is injective and the crystalline cohomology can be regarded as a filtered φ-module (= a filtered convergent F-isocrystal on $\mathrm{Spec}(O_K)$ over K). In this case, the long exact sequence (2.1) is rewritten as short exact sequences:

$$(2.3)\quad 0\longrightarrow H^1_{\mathrm{syn}}(O_K, H^{m-1}_{\mathrm{crys}}((X,M),\mathcal{E}))\longrightarrow H^m_{\mathrm{syn}}((X,M),\mathcal{E})$$
$$\longrightarrow H^0_{\mathrm{syn}}(O_K, H^m_{\mathrm{crys}}((X,M),\mathcal{E}))\to 0.$$

We can define a syntomic cohomology even over a ramified base similarly noting that giving a filtration on a convergent crystal $\mathcal{E}$ on X/K_0 is equivalent to giving a filtration on the inverse image of $\mathcal{E}$ to X/K. In [Niz], W. Niziol gave a different construction in the case $D=\emptyset$ generalizing the method of of Bloch-Kato in [BK90] §3, where they treat the case $X=\mathrm{Spec}(O_K)$.

3. Crystalline sheaves

Let X be a proper smooth scheme over $\mathrm{Spec}(O_K)$ and let M be the log structure on X defined by a divisor D with relative normal crossings. In [Fal89] V f), G. Faltings defined the notion "crystalline" for smooth $\mathbb{Q}_p$-sheaves on the étale site of $X_{\mathrm{triv}} := X_K\backslash D_K$ generalizing the Fontaine's theory of crystalline p-adic representations of $\mathrm{Gal}(\overline{K}/K)$ ([Fon82], [Fon94a], [Fon94b]). Similarly as crystalline p-adic representations, we have a fully-faithful functor from the category of crystalline sheaves to the category of filtered convergent F-isocrystals on (X,M) over K. Note that, in the case $X=\mathrm{Spec}(O_K)$, the latter category coincides with the category $MF_K(\varphi)$ of filtered φ-modules over K considered by Fontaine.

In this section, we will explain crystalline sheaves in the case $K=K_0$. Our approach is slightly different from Faltings'; we use the description of filtered convergent F-isocrystals as modules with connections, which is reviewed in §1, while Faltings consider their values on every enlargement.

Assume $K=K_0$. Let X be a smooth affine formal scheme over $\mathrm{Spf}(O_K)$ endowed with the fine log structure M defined by a divisor D with relative normal crossings. We set $A=\Gamma(X,\mathcal{O}_X)$. We assume that (X,M) has a log coordinate $t_1,t_2,\ldots,t_d\in A$ globally as in §1. We choose a lifting of Frobenius F_X of the log formal scheme (X,M). We further assume that X is connected and hence A is a regular domain. Choose an algebraic closure $\overline{\mathrm{Frac}(A)}$ of the field of fractions $\mathrm{Frac}(A)$ of A, and define $\overline{A}$ to be the union of the normalization of A in L, where L ranges over all finite extensions of $\mathrm{Frac}(A)$ contained in $\overline{\mathrm{Frac}(A)}$ such that the normalizations of $A[(pt_1t_2\cdots t_d)^{-1}]$ in L are étale. We set $G := \mathrm{Gal}(\mathrm{Frac}(\overline{A})/\mathrm{Frac}(A))$, which is canonically isomorphic to the fundamental group of $\mathrm{Spec}(A[(pt_1t_2\cdots t_d)^{-1}])$ for the base point $\mathrm{Spec}(\overline{\mathrm{Frac}(A)})$.

Associated to the ring $\overline{A}$ with the natural action of G and (X, M), F_X, there exists a $K\otimes A$-algebra $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ endowed with the following structures:

- an action of G.
- a decreasing filtration $Fil^i\mathcal{B}_{\mathrm{crys}}(\overline{A})$ $(i\in\mathbb{Z})$ by $K\otimes A$-submodules such that $\cup_i Fil^i\mathcal{B}_{\mathrm{crys}}(\overline{A}) = \mathcal{B}_{\mathrm{crys}}(\overline{A})$ and $\cap_i Fil^i\mathcal{B}_{\mathrm{crys}}(\overline{A}) = 0$.
- an integrable connection: $\nabla\colon \mathcal{B}_{\mathrm{crys}}(\overline{A}) \to \mathcal{B}_{\mathrm{crys}}(\overline{A})\otimes_A \Omega_A(\log)$ satisfying Griffiths transversality: $\nabla(Fil^i)\subset Fil^{i-1}\otimes_A\Omega_A(\log)$ $(i\in\mathbb{Z})$.
- an injective endomorphism (called the Frobenius) $\varphi\colon \mathcal{B}_{\mathrm{crys}}(\overline{A}) \to \mathcal{B}_{\mathrm{crys}}(\overline{A})$ horizontal with respect to the connection and semi-linear with respect to F_X.

The filtration is stable under the action of G and the connection and the Frobenius commute with the action of G. If we set $B_{\mathrm{crys}}(\overline{A}) := \mathcal{B}_{\mathrm{crys}}(\overline{A})^{\nabla=0}$ and endow it with the induced filtration, then $\mathbb{Q}_p(i)$ $(i\in\mathbb{Z})$ is contained in $Fil^iB_{\mathrm{crys}}(\overline{A})$ and $\mathrm{gr}^iB_{\mathrm{crys}}(\overline{A})$ is canonically embedded into $(K\otimes_{O_K}\widehat{\overline{A}})(i)$. If we use the surjectivity of the absolute Frobenius of $\overline{A}/p\overline{A}$, which follows from of the theory of almost étale extensions by Faltings, then we can prove that this embedding is an isomorphism.

The ring $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ is defined along the following lines. We first define the ring $R_{\overline{A}}$ to be the perfection of $\overline{A}/p\overline{A}$ i.e. the projective limit of $\overline{A}/p\overline{A} \leftarrow \overline{A}/p\overline{A} \leftarrow \overline{A}/p\overline{A} \leftarrow \cdots$ where the transition maps are the absolute Frobenius. We have a canonical embedding $\mathbb{Z}_p(1)(\overline{A}) \hookrightarrow R^*_{\overline{A}}$ sending $\varepsilon = (\varepsilon_n)_{n\geq 0}$, $\varepsilon_0 = 1$, $\varepsilon_{n+1}^p = \varepsilon_n$ to $\underline{\varepsilon} := (\varepsilon_n \mod p)_{n\geq 0}$. We define the ring $\mathcal{A}_{\mathrm{crys}}(\overline{A})$ to be the p-adic completion of some subring of $\mathbb{Q}_p\otimes(A\otimes_{O_K}W(R_{\overline{A}}))$ containing $A\otimes W(R_{\overline{A}})$. For any $\varepsilon\in\mathbb{Z}_p(1)$, $\log(1\otimes[\underline{\varepsilon}]-1)$ converges in $\mathcal{A}_{\mathrm{crys}}$ p-adically and we obtain a canonical embedding $\mathbb{Z}_p(1)\hookrightarrow\mathcal{A}_{\mathrm{crys}}$. We regard $\mathbb{Z}_p(1)$ as a submodule of $\mathcal{A}_{\mathrm{crys}}$. The ring $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ is defined to be $\mathcal{A}_{\mathrm{crys}}[p^{-1},\varepsilon^{-1}]$, where ε is any non-zero element of $\mathbb{Z}_p(1)$.

The following two properties are important.

Proposition 3.1. (1) *The G-invariant part of $\mathcal{B}_{crys}(\overline{A})$ is $K\otimes_{O_K}A$.*

(2) *$\mathbb{Q}_p$ with the trivial action of G is quasi-isomorphic to the mapping fiber of the morphism of complexes of G-modules:*

$$1-\varphi\otimes\varphi\colon Fil^0(\mathcal{B}_{crys}(\overline{A})\otimes_A\Omega_A^\bullet(\log)) \to \mathcal{B}_{crys}(\overline{A})\otimes_A\Omega_A^\bullet(\log).$$

Here the degree q-part of Fil^0 is defined to be $Fil^{-q}\mathcal{B}_{crys}(\overline{A})\otimes_A\Omega_A^q(\log)$.

Let V be a p-adic representation of G, i.e. a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of G, which corresponds to a smooth $\mathbb{Q}_p$-sheaf on the étale site of $\mathrm{Spec}(A[(pt_1t_2\cdots t_d)^{-1}])$. Similarly as the Fontaine's theory, we define $D_{\mathrm{crys}}(V)$ and a filtration on it by

$$D_{\mathrm{crys}}(V) := (\mathcal{B}_{\mathrm{crys}}(\overline{A})\otimes_{\mathbb{Q}_p}V)^G,$$
$$Fil^iD_{\mathrm{crys}}(V) := (Fil^i\mathcal{B}_{\mathrm{crys}}(\overline{A})\otimes_{\mathbb{Q}_p}V)^G \quad (i\in\mathbb{Z}).$$

Here an element $g\in G$ acts on $\mathcal{B}_{\mathrm{crys}}(\overline{A})\otimes_{\mathbb{Q}_p}V$ by $g\otimes g$. The connection and the Frobenius on $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ induce those on $D_{\mathrm{crys}}(V)$ and they satisfy $\nabla(Fil^i)\subset Fil^{i-1}\otimes_A\Omega_A(\log)$ and $\nabla\circ\varphi = (\varphi\otimes\varphi)\circ\nabla$.

Definition 3.2. We say that V is *crystalline* if it satisfies the following two conditions:

(1) The canonical map $\mathcal{B}_{\mathrm{crys}}(\overline{A}) \otimes_{K\otimes A} D_{\mathrm{crys}}(V) \to \mathcal{B}_{\mathrm{crys}}(\overline{A}) \otimes_{\mathbb{Q}_p} V$ is a filtered isomorphism.

(2) $D_{\mathrm{crys}}(V)$ and $\mathrm{gr}^i D_{\mathrm{crys}}(V)$ ($i \in \mathbb{Z}$) are finite projective $K \otimes_{O_K} A$-modules.

If we use the flatness of $K \otimes_{O_K} \widehat{\overline{A}}$ (where $\widehat{\overline{A}}$ denotes the p-adic completion of $\overline{A}$) over $K \otimes_{O_K} A$, which follows from the theory of almost étale extensions by Faltings, then we can prove (1) implies (2). However we avoid to use this fact here because it is very hard to understand all the details of the Faltings' theory.

If V is crystalline, then $(D_{\mathrm{crys}}(V), Fil^i D_{\mathrm{crys}}(V), \nabla, \varphi)$ defines a filtered convergent F-isocrystal on (X, M) over K, which we also denote by $D_{\mathrm{crys}}(V)$.

Theorem 3.3 (G. Faltings [Fal89] Lemma 5.5). *The functor*

$$D_{crys}\colon Rep_{crys}(G) \longrightarrow MF^{\nabla}_{(X,M)/K}(\Phi)$$

from the category of crystalline representations of G to the category of filtered convergent F-isocrystals is fully faithful and exact. Furthermore the essential image is stable under extensions.

Proof of fully faithfulness. By Proposition 3.1 (2), we have

$$V = Fil^0(\mathcal{B}_{\mathrm{crys}}(\overline{A}) \otimes_{K\otimes A} D_{\mathrm{crys}}(V))^{\nabla=0,\varphi=1},$$

which immediately implies the fully faithfulness. □

Let V be a crystalline p-adic representation of G and let $\mathcal{E}$ be the corresponding filtered convergent F-isocrystal on $(X, M)/K$. Then we can construct canonical homomorphisms:

$$H^m_{syn}((X, M), \mathcal{E}) \longrightarrow H^m_{\mathrm{cont}}(G, V) \quad (m \in \mathbb{Z}) \tag{3.4}$$

functorial on V as follows. See [Niz] for the case $D = \emptyset$. By tensoring V to the resolution of $\mathbb{Q}_p$ of Proposition 3.1 (2) and using the filtered isomorphism in Definition 3.2 (1), we obtain the resolution of V by the mapping fiber $\overline{\mathcal{S}}_{X,F_X}((X, M), \mathcal{E})$ of the morphism of complexes:

$$1 - \varphi\colon Fil^0(\mathcal{B}_{\mathrm{crys}}(\overline{A}) \otimes_{K\otimes A} \mathcal{E} \otimes_A \Omega^\bullet_A(\log)) \longrightarrow \mathcal{B}_{\mathrm{crys}}(\overline{A}) \otimes_{K\otimes A} \mathcal{E} \otimes_A \Omega^\bullet_A(\log).$$

We define the homomorphisms (3.4) by taking the cohomology of the following maps:

$$\begin{aligned}\mathcal{S}_{X,F_X}((X, M), \mathcal{E}) &\to \Gamma(G, \overline{\mathcal{S}}_{X,F_X}((X, M), \mathcal{E})) \\ &\to \text{“}R\Gamma_{\mathrm{cont}}(G, \overline{\mathcal{S}}_{X,F_X}((X, M), \mathcal{E}))\text{”} \xleftarrow{\sim} R\Gamma_{\mathrm{cont}}(G, V)\end{aligned}$$

The ring $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ can be described as a union $\cup_{m\geq 0} t^{-m}\mathcal{A}_{\mathrm{crys}}(\overline{A})$ of p-adically complete and separated G-stable submodules $t^{-m}\mathcal{A}_{\mathrm{crys}}(\overline{A})$ (where $0 \neq t \in \mathbb{Z}_p(1) \subset \mathcal{A}_{\mathrm{crys}}(\overline{A})$) and it induces a description of $\overline{\mathcal{S}}_{X,F_X}((X, M), \mathcal{E})$ as a union of complexes of G-modules $\mathcal{S}_m$ ($m \in \mathbb{N}$) p-adically complete and separated. The third term “$R\Gamma_{\mathrm{cont}} \ldots$” is defined to be the inductive limit of the inhomogeneous continuous cochain complexes of $\mathcal{S}_m$ with the p-adic topology.

Lemma 3.5. *The homomorphism* (3.4) *for $m = 1$ coincides with the map:*

$$H^1_{syn}((X, M), \mathcal{E})) \cong Ext^1_{MF^{\nabla}_{(X,M)/K}(\Phi)}(\mathcal{O}, \mathcal{E}) \to Ext^1_{Rep_{\mathbb{Q}_p}(G)}(\mathbb{Q}_p, V) \cong H^1_{cont}(G, V)$$

induced by D_{crys}.

Proof. Choose $x \in Fil^{-1}\mathcal{E} \otimes_A \Omega_A(\log)$ and $y \in \mathcal{E}$ and assume $(1-\varphi)(x) = \nabla(y)$ and $\nabla(y) = 0$. Then there exists $a \in Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes_{K\otimes A} \mathcal{E}) \cong Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)$ unique up to addition by an element of V such that $y = (1-\varphi)(a)$ and $x = \nabla(a)$ and the map (3.4) sends the class of (x, y) to the class of the cocycle $G \to V; g \mapsto -(g(a)-a)$. On the other hand, if we define the extension $\mathcal{F}$ of $\mathcal{O}$ by $\mathcal{E}$ as in the proof of Proposition 2.2, then the element $(1, -a) \in \mathcal{B}_{\mathrm{crys}} \otimes \mathcal{F} = \mathcal{B}_{\mathrm{crys}} \oplus (\mathcal{B}_{\mathrm{crys}} \otimes \mathcal{E})$ is contained in the extension of $\mathbb{Q}_p$ by V corresponding to $\mathcal{F}$. □

We still assume $K = K_0$ and let X be a proper smooth scheme over $\mathrm{Spec}(O_K)$ endowed with the fine log structure M defined by a divisor D with relative normal crossings. Set $X_{\mathrm{triv}} := X_K \backslash D_K$. For a $\mathbb{Q}_p$-smooth sheaf V on $(X_{\mathrm{triv}})_{\text{ét}}$, we say that V is *crystalline* if, for each affine étale scheme $U = \mathrm{Spec}(A)$ of X having a global log coordinate $t_1, t_2, \ldots, t_d$, the pull-back of V on $\mathrm{Spec}(\hat{A}[(pt_1t_2\cdots t_d)^{-1}])$ is crystalline. Here $\hat{A}$ denotes the p-adic completion of A. By gluing the functor D_{crys} for $\mathrm{Spf}(\hat{A})$ with the inverse image of M, we obtain a functor D_{crys} from the category of crystalline sheaves on $(X_{\mathrm{triv}})_{\text{ét}}$ to the category $MF^{\nabla}_{(X,M)/K}(\Phi)$ of filtered convergent F-isocrystals on $(X, M)/K$ and Theorem 3.3 holds. By taking $X_\bullet$, $(X_\bullet, M|_{X_\bullet}) \hookrightarrow (Y_\bullet, N_\bullet)$, $F_{Y_\bullet}$ as in the definition of the syntomic cohomology in the general case and using $\mathcal{B}_{\mathrm{crys}}$ associated to these data, one can construct canonical homomorphisms:

$$H^m_{\mathrm{syn}}((X, M), D_{\mathrm{crys}}(V)) \to H^m_{\text{ét}}(X_{\mathrm{triv}}, V) \tag{3.6}$$

similarly as (3.4). Here the RHS is the continuous étale cohomology ([Jan88]). See [Niz] in the case $D = \emptyset$. Lemma 3.5 is still valid.

4. $\mathcal{L}og$

In this section, we review the realizations of the motivic pro-sheaf $\mathcal{L}og = (\mathcal{L}og^{(n)})_{n\geq 1}$ on $\mathbb{G}_{m,\mathbb{Q}} = \mathbb{P}^1_{\mathbb{Q}} \backslash \{0, \infty\}$ in the category of smooth $\mathbb{Q}_p$-sheaves on $(\mathbb{G}_{m,\mathbb{Q}_p})_{\text{ét}}$ and the category of filtered convergent F-isocrystals on $\mathbb{P}^1_{\mathbb{Z}_p}$ with the log structure defined by $\{0, \infty\}$. The realization in filtered overconvergent F-isocrystals on $\mathbb{G}_{m,\mathbb{Z}_p}$ was studied by K. Bannai in [Ban00a], but from his explicit description, it is clear that it actually comes from log filtered convergent F-isocrystals.

Let $\mathbb{G}^{\log}_{m,\mathbb{Z}_p}$ be $\mathbb{P}^1_{\mathbb{Z}_p}$ endowed with the fine log structure define by $\{0, \infty\}$. We denote by $K(r)$ the object of $MF_{\mathbb{Q}_p}(\varphi)$ i.e. the filtered φ-module over $\mathbb{Q}_p$ corresponding to the crystalline p-adic representation $\mathbb{Q}_p(r)$ $(r \in \mathbb{Z})$ and by $\mathcal{O}(r)$ its inverse image in $MF^{\nabla}_{\mathbb{G}^{\log}_{m,\mathbb{Z}_p}/\mathbb{Q}_p}(\Phi)$. Then we have the following exact sequences. See (2.3) for the syntomic case.

(4.1)
$$\begin{aligned} 0 \longrightarrow H^1_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{(*)} H^1_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathbb{Q}_p(1)) \longrightarrow H^0(\mathbb{Q}_p, H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathbb{Q}_p(1))) \\ \longrightarrow H^2_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{(*)} H^2_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathbb{Q}_p(1)), \end{aligned}$$

(4.2)
$$0 \to H^1_{\mathrm{syn}}(\mathbb{Z}_p, K(1)) \xrightarrow{(*)} H^1_{\mathrm{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O}(1)) \to H^0_{\mathrm{syn}}(\mathbb{Z}_p, H^1_{\mathrm{crys}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O}(1))) \to 0.$$

Here $H^*_{\text{ét}}$ denotes the continuous étale cohomology ([Jan88]). The pull-backs by the point 1 give the splittings of the homomorphisms $(*)$. On the other hand the third terms of the above exact sequences are both canonically isomorphic to $\mathbb{Q}_p$. Thus we obtain isomorphisms:

$$H^1_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathbb{Q}_p(1)) \cong H^1_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \oplus \mathbb{Q}_p, \tag{4.3}$$

$$H^1_{\text{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O}(1)) \cong H^1_{\text{syn}}(\mathbb{Z}_p, K(1)) \oplus \mathbb{Q}_p. \tag{4.4}$$

We define the smooth $\mathbb{Q}_p$-sheaf $\mathcal{L}og^{(1)}_{\text{ét}}$ on $(\mathbb{G}_{m,\mathbb{Q}_p})_{\text{ét}}$ and the filtered convergent F-isocrystal $\mathcal{L}og^{(1)}_{\text{crys}}$ on $\mathbb{G}^{\log}_{m,\mathbb{Z}_p}$ to be the extensions:

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathcal{L}og^{(1)}_{\text{ét}} \longrightarrow \mathbb{Q}_p \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{L}og^{(1)}_{\text{crys}} \longrightarrow \mathcal{O} \longrightarrow 0$$

corresponding to the elements $(0, 1)$ via the above isomorphisms. See Proposition 2.2 for the interpretation of the first syntomic cohomology as Ext^1. The specializations of these extensions at the point 1 split.

We define $\mathcal{L}og^{(n)}_{\text{ét}}$ and $\mathcal{L}og^{(n)}_{\text{crys}}$ to be the n symmetric tensors of $\mathcal{L}og^{(1)}_{\text{ét}}$ and $\mathcal{L}og^{(1)}_{\text{crys}}$ respectively. Let pr denote the projections from $\mathcal{L}og^{(1)}_{\text{ét}}$ to $\mathbb{Q}_p$ and $\mathcal{L}og^{(1)}_{\text{crys}}$ to $\mathcal{O}$ and let $\mathrm{pr}_i\colon (\mathcal{L}og^{(1)}_{\bullet})^{\otimes n} \to (\mathcal{L}og^{(1)}_{\bullet})^{\otimes (n-1)}$ $(1 \le i \le n)$ be the tensor product of pr on the i-th term and id on the other terms. Then the sum $\sum_{i=1}^n \mathrm{pr}_i$ induces $\mathcal{L}og^{(n)}_{\bullet} \to \mathcal{L}og^{(n-1)}_{\bullet}$. We have short exact sequences:

$$0 \longrightarrow \mathbb{Q}_p(n) \longrightarrow \mathcal{L}og^{(n)}_{\text{ét}} \longrightarrow \mathcal{L}og^{(n-1)}_{\text{ét}} \longrightarrow 0, \tag{4.5}$$

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{L}og^{(n)}_{\text{crys}} \longrightarrow \mathcal{L}og^{(n-1)}_{\text{crys}} \longrightarrow 0. \tag{4.6}$$

Proposition 4.7. *Let d be a positive integer. Then the pull-back of $\mathcal{L}og^{(n)}_{ét}$ by $Spec(\mathbb{Q}_p(\mu_d)) = Spec(\mathbb{Q}_p[t]/\Phi_d(t)) \hookrightarrow \mathbb{G}_{m,\mathbb{Q}_p}$ is canonically isomorphic to $\oplus_{0\le r\le n}\mathbb{Q}_p(r)$, and the pull-back of $\mathcal{L}og^{(n)}_{crys}$ by $Spec(\mathbb{Z}_p[\mu_d]) = Spec(\mathbb{Z}_p[t]/\Phi_d(t)) \hookrightarrow \mathbb{G}^{\log}_{m,\mathbb{Z}_p}$ is canonically isomorphic to $\oplus_{0\le r\le n}\mathbb{Q}_p(\mu_d)(r)$.*

Proof. It suffices to prove the claim in the case $n = 1$. Let $[d]$ denote the multiplication by d on $\mathbb{G}_{m,\mathbb{Q}_p}$. Then the pull-back by $[d]$ preserves the decomposition (4.3) and it induces a multiplication by d on the factor $\mathbb{Q}_p$. Hence $[d]^*(\mathcal{L}og^{(1)}_{\text{ét}})$ is isomorphic to the fiber product of $\mathcal{L}og^{(1)}_{\text{ét}} \to \mathbb{Q}_p \overset{d}{\underset{\sim}{\leftarrow}} \mathbb{Q}_p$. Since $[d]^*(\mathcal{L}og^{(1)}_{\text{ét}})$ splits at $\mathrm{Spec}(\mathbb{Q}_p(\mu_d)) \hookrightarrow \mathbb{G}_{m,\mathbb{Q}_p}$, so does $\mathcal{L}og^{(1)}_{\text{ét}}$. If d is prime to p, $[d]$ induces a morphism of log schemes $\mathbb{G}^{\log}_{m,\mathbb{Z}_p} \to \mathbb{G}^{\log}_{m,\mathbb{Z}_p}$ and we can prove the second claim similarly. □

Proposition 4.8. *$\mathcal{L}og^{(n)}_{ét}$ is crystalline and $D_{crys}(\mathcal{L}og^{(n)}_{ét})$ is canonically isomorphic to $\mathcal{L}og^{(n)}_{crys}$.*

Proof. It suffices to prove in the case $n = 1$. We have a commutative diagram:

$$\begin{tikzcd}
\Gamma(\mathbb{G}_{m,\mathbb{Z}_p}, \mathcal{O}^*) \arrow[r] \arrow[d, equal] & H^1_{\text{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O}(1)) \arrow[d, "\cap"] \\
\Gamma(\mathbb{G}_{m,\mathbb{Z}_p}, \mathcal{O}^*) \arrow[r] & H^1_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathbb{Q}_p(1))
\end{tikzcd}$$

and the images of the canonical coordinate t under the horizontal maps are the extensions $\mathcal{L}og^{(1)}_{\text{ét}}$ and $\mathcal{L}og^{(1)}_{\text{crys}}$. Locally on $\mathbb{G}^{\log}_{m,\mathbb{Z}_p}$, the upper horizontal map is defined by $f \mapsto (df, p^{-1}\log(f^p\varphi(f)^{-1}))$ □

If we define the lifting of Frobenius on the log scheme $\mathbb{G}_{m,\mathbb{Z}_p}^{\log}$ by $t \mapsto t^p$, then using the construction of $\mathcal{L}og^{(1)}_{\mathrm{crys}}$ in the proof of Proposition 4.8, we obtain the following explicit description of $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ ([Ban00a] Definition 5.1):

$$
\begin{aligned}
\mathcal{L}og^{(n)}_{\mathrm{crys}} &= \bigoplus_{0\le r\le n} \mathcal{O}\cdot e_r,\\
Fil^i \mathcal{L}og^{(n)}_{\mathrm{crys}} &= \bigoplus_{i\le -r, 0\le r\le n} \mathcal{O}\cdot e_r,\\
\nabla(e_r) &= \begin{cases} e_{r+1}\cdot \frac{dt}{t} & (0\le r\le n-1),\\ 0 & (r=n),\end{cases}\\
\Phi(1\otimes e_r) &= p^{-r} e_r.
\end{aligned}
\tag{4.9}
$$

5. $\mathcal{P}ol$

In this section, we review the realizations of the motivic polylogarithmic pro-sheaf $\mathcal{P}ol = (\mathcal{P}ol^{(n)})_{n\ge 1}$ on $\mathbb{P}^1_{\mathbb{Q}}\backslash\{0,1,\infty\}$ in the category of smooth $\mathbb{Q}_p$-sheaves on $(\mathbb{P}_{\mathbb{Q}_p}\backslash\{0,1,\infty\})_{\text{ét}}$ and the category of filtered convergent F-isocrystals on $\mathbb{P}^1_{\mathbb{Z}_p}$ with the log structure defined by $\{0,1,\infty\}$. The realization in the category of filtered overconvergent F-isocrystals on $\mathbb{P}^1_{\mathbb{Z}_p}\backslash\{0,1,\infty\}$ is constructed by K. Bannai in [Ban00a] and as we will see below, his construction works also for filtered convergent F-isocrystals and syntomic cohomology with log poles. As in [Ban00a], we follow the construction by A. Huber and J. Wildeshaus in [HW98]§2.

Let U denote the scheme $\mathbb{P}^1_{\mathbb{Q}_p}\backslash\{0,1,\infty\}$ and let $\mathcal{U}^{\log}$ be the scheme $\mathbb{P}^1_{\mathbb{Z}_p}$ endowed with the log structure defined by the divisor $\{0,1,\infty\}$. The pro-sheaf $\mathcal{P}ol_{\text{ét}}$ (resp. $\mathcal{P}ol_{\mathrm{crys}}$) is a compatible system of an extension of $\mathbb{Q}_p$ (resp. $\mathcal{O}$) by the pull-back of $\mathcal{L}og^{(n)}_{\text{ét}}$ (resp. $\mathcal{L}og^{(n)}_{\mathrm{crys}}$) on U (resp. $\mathcal{U}^{\log}$) or equivalently a compatible system of a class in $H^1_{\text{ét}}(U, \mathcal{L}og^{(n)}_{\text{ét}}|_U)$ (resp. $H^1_{\mathrm{syn}}(\mathcal{U}^{\log}, \mathcal{L}og^{(n)}_{\mathrm{crys}}|_{\mathcal{U}^{\log}})$). See Proposition 2.2 for the second case. To simplify the notation, we will denote the pull-back of $\mathcal{L}og^{(n)}_{\text{ét}}$ and $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ on U and $\mathcal{U}^{\log}$ by the same symbols.

Let us consider the Gysin exact sequences:

$$
\begin{aligned}
0 \to H^1_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathcal{L}og^{(n)}_{\text{ét}}) &\to H^1_{\text{ét}}(U, \mathcal{L}og^{(n)}_{\text{ét}})\\
&\to H^0_{\text{ét}}(\mathbb{Q}_p, i_1^*(\mathcal{L}og^{(n)}_{\text{ét}}(-1))) \to H^2_{\text{ét}}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathcal{L}og^{(n)}_{\mathrm{crys}})
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
0 \to H^1_{\mathrm{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{\mathrm{crys}}) &\to H^1_{\mathrm{syn}}(\mathcal{U}^{\log}, \mathcal{L}og^{(n)}_{\mathrm{crys}})\\
&\to H^0_{\mathrm{syn}}(\mathbb{Z}_p, i_1^*(\mathcal{L}og^{(n)}_{\mathrm{crys}}(-1))) \to H^2_{\mathrm{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{\mathrm{crys}})
\end{aligned}
\tag{5.2}
$$

for the unit sections $i_1\colon \mathrm{Spec}(\mathbb{Q}_p) \hookrightarrow \mathbb{G}_{m,\mathbb{Q}_p}$ and $i_1\colon \mathrm{Spec}(\mathbb{Z}_p) \hookrightarrow \mathbb{G}^{\log}_{m,\mathbb{Z}_p}$. Since $i_1^*(\mathcal{L}og^{(n)}_{\text{ét}})$ and $i_1^*(\mathcal{L}og^{(n)}_{\mathrm{crys}})$ are canonically isomorphic to $\bigoplus_{0\le r\le n}\mathbb{Q}_p(r)$ and $\bigoplus_{0\le r\le n} K(r)$ respectively by definition, the third terms of the above exact sequences are canonically isomorphic to $\mathbb{Q}_p$. On the other hand, the first and the fourth terms have the following properties.

Proposition 5.3. *The homomorphisms induced by the projections:*

$$H^1_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathcal{L}og^{(n+1)}_{\acute{e}t}) \to H^1_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathcal{L}og^{(n)}_{\acute{e}t}),$$
$$H^1_{syn}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n+1)}_{crys}) \to H^1_{syn}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys})$$

are 0 *for all* $n \geq 1$ *and we have*

$$H^2_{\acute{e}t}(\mathbb{G}_{m,\mathbb{Q}_p}, \mathcal{L}og^{(n)}_{\acute{e}t}) = 0 \quad (n \geq 2),$$
$$H^2_{syn}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys}) = 0 \quad (n \geq 1).$$

To prove this proposition, we first calculate the geometric étale cohomology of $\mathcal{L}og^{(n)}_{\text{ét}}$ and the crystalline cohomology of $\mathcal{L}og^{(n)}_{\text{crys}}$.

Proposition 5.4. (1) *The natural homomorphisms*

$$\mathbb{Q}_p(n) = H^0_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathbb{Q}_p(n)) \to H^0_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathcal{L}og^{(n)}_{\acute{e}t})$$
$$H^1_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathcal{L}og^{(n)}_{\acute{e}t}) \to H^1_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \cong \mathbb{Q}_p(-1)$$

are isomorphisms and

$$H^i_{\acute{e}t}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathcal{L}og^{(n)}_{\acute{e}t}) = 0 \quad (i \geq 2)$$

for all $n \geq 1$.

(2) *The Hodge spectral sequence for* $H^*_{crys}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys})$ *degenerates at* E_1 *and these cohomology groups can be naturally regard as filtered* φ*-modules over* $\mathbb{Q}_p$ *in the sense of Fontaine. Furthermore, the natural homomorphisms of filtered* φ*-modules:*

$$K(n) = H^0_{crys}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, K(n)) \to H^0_{crys}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys})$$
$$H^1_{crys}(\mathbb{G}^{log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys}) \to H^1_{crys}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, K) \cong K(-1)$$

are isomorphisms and

$$H^i_{syn}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{crys}) = 0 \quad (i \geq 2)$$

for all $n \geq 1$.

Proof. We will give a proof of (2). The proof of (1) is similar. It is well-known that the Hodge spectral sequence for the de Rham cohomology of $\mathbb{P}^1_{\mathbb{Q}_p}$ with log poles along $\{0, 1, \infty\}$ degenerates at E_1. The crystalline cohomology $H^i_{\text{crys}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O})$ is isomorphic to K if $i = 0$, $K(-1)$ if $i = 1$, and 0 if $i \geq 2$ as filtered φ-modules. By definition, the class of $\mathcal{L}og^{(1)}_{\text{crys}}$ in $H^1_{\text{crys}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{O})$ is non-trivial. Hence, by looking at the long exact sequence of crystalline cohomology associated to the extension, we obtain the claim for $n = 1$. One can prove the claim for the general n by induction using the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(n) & \longrightarrow & \mathcal{L}og^{(n)}_{\text{crys}} & \longrightarrow & \mathcal{L}og^{(n-1)}_{\text{crys}} & \longrightarrow & 0 \\
 & & \| & & \uparrow\cup & & \uparrow\cup & & \\
0 & \longrightarrow & \mathcal{O}(n) & \longrightarrow & \mathcal{L}og^{(1)}_{\text{crys}}(n-1) & \longrightarrow & K(n-1) & \longrightarrow & 0
\end{array}$$

and the non-triviality of the lower extension. $\square$

Proof of Proposition 5.3. By Propositions 5.4 (2) and (2.3), we have a short exact sequence

$$0 \to H^1_{\mathrm{syn}}(\mathbb{Z}_p, K(n)) \to H^1_{\mathrm{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{\mathrm{crys}}) \to H^0_{\mathrm{syn}}(\mathbb{Z}_p, K(-1)) \to 0$$

and the last term vanishes. Furthermore the projection $\mathcal{L}og^{(n+1)}_{\mathrm{crys}} \to \mathcal{L}og^{(n)}_{\mathrm{crys}}$ sends $K(n+1)$ to 0. This implies the claim for the first syntomic cohomology. By Propositions 5.4 (2) and (2.3) again, we see $H^2_{\mathrm{syn}}(\mathbb{G}^{\log}_{m,\mathbb{Z}_p}, \mathcal{L}og^{(n)}_{\mathrm{crys}}) \cong H^1_{\mathrm{syn}}(\mathbb{Z}_p, K(-1)) = 0$. One can verify the claim for étale cohomology similarly using Proposition 5.4 (1) and the Leray spectral sequence. The difference from the syntomic case arises from the non-vanishing of $H^2_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$. Note $H^2_{\text{ét}}(\mathbb{Q}_p, \mathbb{Q}_p(r)) = 0$ for $r \geq 2$. □

By taking the projective limit of the Gysin exact sequences (5.1), (5.2) with respect to n and using Proposition 5.3, we obtain isomorphisms:

$$\varprojlim{}_n H^1_{\text{ét}}(U, \mathcal{L}og^{(n)}_{\text{ét}}) \xrightarrow{\sim} \mathbb{Q}_p, \tag{5.5}$$

$$\varprojlim{}_n H^1_{\mathrm{syn}}(\mathcal{U}^{\log}, \mathcal{L}og^{(n)}_{\mathrm{crys}}) \xrightarrow{\sim} \mathbb{Q}_p. \tag{5.6}$$

Definition 5.7. We define $\mathcal{P}ol_{\text{ét}} = (\mathcal{P}ol^{(n)}_{\text{ét}})_{n\geq 1}$ and $\mathcal{P}ol_{\mathrm{crys}} = (\mathcal{P}ol^{(n)}_{\mathrm{crys}})_{n\geq 1}$ to be the projective systems of extensions corresponding to 1 by the above isomorphisms (5.5) and (5.6) respectively.

Let $\mathcal{U}_1^{\log}$ be the p-adic formal completion of the scheme $\mathbb{P}^1_{\mathbb{Z}_p}\backslash\{1\}$ endowed with the log structure defined by the divisor $\{0, \infty\}$ and let $F_{\mathcal{U}_1^{\log}}$ be the lifting of Frobenius defined by $t \mapsto t^p$, where t denotes the canonical coordinate. Note that one can also define a lifting of Frobenius on $\mathbb{P}^1_{\mathbb{Z}_p}$ in the same way, but it does not give a lifting of Frobenius on $\mathcal{U}^{\log}$ because $1 - t^p \neq (1-t)^p\cdot$unit. K. Bannai gave the following explicit description of $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$. Strictly speaking, he proved this theorem for the realization in the category of filtered overconvergent F-isocrystals but his proof still works in our settings with a slight modification.

Theorem 5.8 (K. Bannai [Ban00a] Theorem 2)**.** *The restriction of $\mathcal{P}ol^{(n)}_{crys}$ on $\mathcal{U}_1^{\log}$ with $F_{\mathcal{U}_1^{\log}}$ is explicitly described as follows. (See* (4.9) *for the explicit description of $\mathcal{L}og^{(n)}_{crys}$.)*

$$\begin{aligned}
\mathcal{P}ol^{(n)}_{crys} &= \mathcal{O}\cdot e \oplus \mathcal{L}og^{(n)}_{crys},\\
Fil^i(\mathcal{P}ol^{(n)}_{crys}) &= \begin{cases} \mathcal{O}\cdot e \oplus Fil^i(\mathcal{L}og^{(n)}_{crys}) & (i \leq 0),\\ 0 & (i > 0),\end{cases}\\
\nabla(e) &= \frac{dt}{t-1}e_1,\\
\Phi(1\otimes e) &= e + \sum_{r=1}^{n}(-1)^{r-1} l_r^{(p)}(t) e_r,
\end{aligned}$$

where $l_r^{(p)}(t)$ denote the p-adic polylogarithmic functions defined inductively by the following differential equations:

$$\begin{aligned}
l_1^{(p)}(0) &= 0, \quad dl_1^{(p)}(t) = \Big(1 - \frac{F^*_{\mathcal{U}_1^{\log}}}{p}\Big)\Big(\frac{dt}{1-t}\Big),\\
l_{r+1}^{(p)}(0) &= 0, \quad dl_{r+1}^{(p)}(0) = l_r^{(p)}(t)\otimes\frac{dt}{t} \quad (r \geq 1).
\end{aligned}$$

6. p-ADIC PERIODS AND COMPARISON

Since $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ (resp. $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$) is a successive extension of $\mathcal{O}(r)$ $(0 \leq r \leq n)$, it comes from a crystalline sheaf on $(\mathbb{G}_{m,\mathbb{Q}_p})_{\mathrm{\acute{e}t}}$ (resp. $(\mathbb{P}^1_{\mathbb{Q}_p}\backslash\{0,1,\infty\})_{\mathrm{\acute{e}t}}$) by Theorem 3.3. In this section, we will calculate the p-adic periods of $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ and $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$ and show that the crystalline sheaf associated to $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$ coincides with $\mathcal{P}ol^{(n)}_{\mathrm{\acute{e}t}}$.

Let $\mathcal{U}^{\log}_{1,\infty}$ be the p-adic formal completion of $\mathbb{P}^1_{\mathbb{Z}_p}\backslash\{1,\infty\}$ endowed with the log structure defined by 0 and let A be its ring of coordinates. We define the lifting of Frobenius F on $\mathcal{U}^{\log}_{1,\infty}$ by $t \mapsto t^p$. Choose an algebraic closure of $\mathrm{Frac}(A)$ and define $\overline{A}$ as in §3. We set $G = \mathrm{Gal}(\mathrm{Frac}(\overline{A})/\mathrm{Frac}(A))$ as in §3 and simply write $\mathcal{B}_{\mathrm{crys}}$ for $\mathcal{B}_{\mathrm{crys}}(\overline{A})$ in the following. By the proof of Theorem 3.3, the p-adic crystalline representation of G corresponding to $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ is

$$Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p\otimes A} \mathcal{L}og^{(n)}_{\mathrm{crys}})^{\nabla=0,\varphi=1}. \tag{6.1}$$

If we use the explicit description (4.9) of $\mathcal{L}og^{(n)}_{\mathrm{crys}}$, we have

$$Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes \mathcal{L}og^{(n)}_{\mathrm{crys}}) = \bigoplus_{0\leq r\leq n} Fil^r\mathcal{B}_{\mathrm{crys}} \cdot e_r$$

and for its element $f = \sum_{r=0}^n f_r \cdot e_r$ $(f_r \in Fil^r\mathcal{B}_{\mathrm{crys}})$, we have:

$$\nabla(f) = 0 \iff \nabla(f_0) = 0, \quad \nabla(f_r) = -f_{r-1} \cdot \frac{dt}{t} \quad (1 \leq r \leq n). \tag{6.2}$$

$$\varphi(f) = f \iff \varphi(f_r) = p^r f_r. \quad (0 \leq r \leq n) \tag{6.3}$$

Choose a compatible system $\{t_n\}_{n\geq 1}$, $t^p_{n+1} = t_n$, $t_0 = t$ of p^n-th roots of t in $\overline{A}$ and define the element $\underline{t}$ of $R_{\overline{A}}$ to be $(t_n \mod p)_{n\geq 0}$. Then the element $[\underline{t}]$ of $W(R_{\overline{A}}) \subset A_{\mathrm{crys}} = \mathcal{A}^{\nabla=0}_{\mathrm{crys}}$ satisfies $\varphi([\underline{t}]) = [\underline{t}]^p$ and its image in $A_{\mathrm{crys}}/Fil^1A_{\mathrm{crys}} \subset \mathbb{Q}_p \otimes \widehat{\overline{A}}$ is t. Set $u := [\underline{t}]t^{-1}$. Then, thanks to the log structure at the point 0, u is contained in $1 + Fil^1\mathcal{A}_{\mathrm{crys}}$ and $\log(u)$ converges in $\mathcal{A}_{\mathrm{crys}}$ p-adically. We have $\log(u) \in Fil^1\mathcal{A}_{\mathrm{crys}}$ and

$$\nabla(\log(u)) = -\frac{dt}{t}, \quad \varphi(\log(u)) = p\log(u).$$

Using the fact that the dimension of (6.1) is $n+1$, we obtain the following proposition:

Proposition 6.4. *Choose a non-zero element ε of $\mathbb{Q}_p(1)$. Then*

$$\left\{\varepsilon^{\otimes r}\left(e_r + \log(u)e_{r+1} + \cdots + \frac{1}{(n-r)!}\log(u)^{n-r}e_n\right)\Big| 0 \leq r \leq n\right\}$$

is a basis of the $\mathbb{Q}_p$-vector space (6.1).

Next we will calculate a lifting of $1 \in \mathbb{Q}_p = Fil^0(\mathcal{B}_{\mathrm{crys}})^{\nabla=0,\varphi=1}$ in the p-adic crystalline representation of G associated to $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$:

$$Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p\otimes A} \mathcal{P}ol^{(n)}_{\mathrm{crys}})^{\nabla=0,\varphi=1}. \tag{6.5}$$

By the explicit description of K. Bannai (Theorem 5.8), we have

$$Fil^0(\mathcal{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p\otimes A} \mathcal{P}ol^{(n)}_{\mathrm{crys}}) = Fil^0\mathcal{B}_{\mathrm{crys}} \cdot e \oplus \bigoplus_{0\leq r\leq n} Fil^r\mathcal{B}_{\mathrm{crys}} \cdot e_r$$

and for its element $e+\sum_{0\leq r\leq n} f_r e_r$, we have:

$$(6.6)\quad \nabla(f)=0 \iff \nabla(f_0)=0,\quad \nabla(f_1)=-\frac{dt}{t-1}-f_0\frac{dt}{t},$$
$$\nabla(f_r)=-f_{r-1}\frac{dt}{t}\quad (2\leq r\leq n),$$
$$(6.7)\quad \varphi(f)=f \iff \varphi(f_0)=f_0,\quad (1-p^{-r}\varphi)(f_r)=(-1)^{r-1}l_r^{(p)}\ (1\leq r\leq n).$$

The equations for f_0 imply $f_0\in\mathbb{Q}_p$. Hence by subtracting an element of (6.1), we may assume $f_0=0$.

Proposition 6.8. *There exist $l_r\in Fil^r\mathcal{B}_{crys}$ $(r\geq 1)$ satisfying the following equations:*

$$\nabla(l_1)=\frac{dt}{1-t},\quad \nabla(l_r)=l_{r-1}\frac{dt}{t}\quad (r\geq 2),\quad (1-p^{-r}\varphi)(l_r)=l_r^{(p)}\quad (r\geq 1),$$

where $l_r^{(p)}$ are as in Theorem 5.8. Choose one solution $\{l_r\in Fil^r\mathcal{B}_{crys}|r\geq 1\}$. Then

$$e+\sum_{r=1}^{n}(-1)^{r-1}l_r\cdot e_r$$

is an element of (6.5) *lifting* 1 *in* $\mathbb{Q}_p=Fil^0\mathcal{B}_{crys}^{\nabla=0,\varphi=1}$.

Proof. Choose a non-zero element ε of $\mathbb{Q}_p(1)\subset Fil^1\mathcal{B}_{\rm crys}$. Then ε is invertible in $\mathcal{B}_{\rm crys}$, $\nabla(\varepsilon)=0$, $\varphi(\varepsilon)=p\varepsilon$ and the multiplication by ε^r induces an isomorphism $Fil^0\mathcal{B}_{\rm crys}\xrightarrow{\sim} Fil^r\mathcal{B}_{\rm crys}$. Hence by Proposition 3.1 (2), we have the following exact sequence:

$$0\longrightarrow\mathbb{Q}_p(r)\longrightarrow Fil^r\mathcal{B}_{\rm crys}$$
$$\xrightarrow{\alpha}(Fil^{r-1}\mathcal{B}_{\rm crys}\otimes_A\Omega_A^1(\log))\oplus\mathcal{B}_{\rm crys}\xrightarrow{\beta}\mathcal{B}_{\rm crys}\otimes_A\Omega_A^1(\log)\longrightarrow 0,$$

where $\alpha(x)=(\nabla(x),(1-p^{-r}\varphi)(x))$ and $\beta((y,z))=(1-p^{-r}\varphi)(y)-\nabla(z)$. Since $(1-p^{-1}\varphi)(\frac{dt}{1-t})=\nabla(l_1^{(p)})$, there exists a solution l_1 unique up to addition by an element of $\mathbb{Q}_p(1)$. For $r\geq 2$, if we are given a solution l_s for $1\leq s\leq r-1$, then $(1-p^{-r}\varphi)(l_{r-1}\frac{dt}{t})=(1-p^{-(r-1)}\varphi)(l_{r-1})\frac{dt}{t}=l_{r-1}^{(p)}\frac{dt}{t}=\nabla(l_r^{(p)})$. Hence there exists a solution l_r unique up to addition by an element of $\mathbb{Q}_p(r)$. The second claim follows from (6.6) and (6.7). □

As the proof shows, the solution $\{l_r\}$ is not unique. If we choose a point $\overline{0}\colon \mathrm{Spec}(\overline{\mathbb{Z}_p})\to\mathrm{Spec}(\overline{A})$ over the point $0\colon\mathrm{Spec}(\mathbb{Z}_p)\to\mathrm{Spec}(A)$, then there exists a unique solution such that each l_r is "regular at $\overline{0}$" and its "value at $\overline{0}$" is 0.

If we choose a compatible system of p^n-th roots of $1-t$ in $\overline{A}$ and define the element $[\underline{1-t}]$ in $\mathcal{B}_{\rm crys}(\overline{A})$ similarly as $[\underline{t}]$, then $\log([\underline{1-t}](1-t)^{-1})\in Fil^1\mathcal{B}_{\rm crys}$ is well defined and satisfies the equations for l_1.

Here it is worthwhile to mention that the equations (6.2), (6.3), (6.6) and (6.7) depend on the special choice F of liftings of Frobenius, but their solutions do not. Indeed, if we choose another lifting of Frobenius, then the Frobenius endomorphism on $\mathcal{B}_{\rm crys}\otimes\mathcal{E}$ is twisted by the canonical isomorphism:

$$A_{\varphi'}\otimes_A(\mathcal{B}_{\rm crys}\otimes\mathcal{E})\cong A_\varphi\otimes_A(\mathcal{B}_{\rm crys}\otimes\mathcal{E})$$

induced by the connection. However, for any horizontal element $x\in(\mathcal{B}_{\rm crys}\otimes\mathcal{E})^{\nabla=0}$, the image of $1\otimes x$ is again $1\otimes x$ (cf. (1.1)).

In order to calculate the residue at 1 of the crystalline sheaf associated to $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$, we need to work on the p-adic formal completion of an open log subscheme of $\mathcal{U}^{\log}$ containing 1. Let $\mathcal{U}^{\log}_{0,\infty}$ be the p-adic completion of $\mathbb{P}^1_{\mathbb{Z}_p}\backslash\{0,\infty\}$ endowed with the log structure defined by $\{1\}$. Then $(t-1)\mapsto(t-1)^p$ defines a lifting of Frobenius on $\mathcal{U}^{\log}_{0,\infty}$, which we denote by F'. By the construction of $\mathcal{L}og^{(1)}_{\mathrm{crys}}$ in the proof of Proposition 4.8 (or by calculating the twist of the Frobenius endomorphism directly), we see that the Frobenius of $\mathcal{L}og^{(1)}_{\mathrm{crys}}$ with respect to F' is given by

$$\varphi(e_0)=e_0+p^{-1}\log(F'^{*}(t)t^{-p})e_1.$$

By taking its symmetric powers, we see that the Frobenius of $\mathcal{L}og^{(n)}_{\mathrm{crys}}$ is given by

$$\varphi(e_r)=p^{-r}\sum_{r\leq s\leq n}\frac{1}{(s-r)!}(p^{-1}\log(F'^{*}(t)t^{-p}))^{s-r}\cdot e_s\quad(0\leq r\leq n).$$

The solution of (6.1) for $\mathcal{U}^{\log}_{0,\infty}$ is again given by the same formula as Proposition 6.4 using $\log([\underline{t}]t^{-1})$ in $\mathcal{B}_{\mathrm{crys}}$ associated to $\mathcal{U}^{\log}_{0,\infty}$.

The Frobenius endomorphism of $\mathcal{P}ol^{(2)}_{\mathrm{crys}}$ with respect to F' is simply given by $\varphi(e)=e$ and we see that $e+\log([\underline{1-t}](1-t)^{-1})e_1$ is an element of (6.5) for $\mathcal{U}^{\log}_{0,\infty}$ and $n=2$. Note that $\log([\underline{1-t}](1-t)^{-1})$ is well defined thanks to the log structure at 1.

Theorem 6.9. *The sheaf $\mathcal{P}ol^{(n)}_{\acute{e}t}$ is crystalline and we have a canonical isomorphism $D_{crys}(\mathcal{P}ol^{(n)}_{\acute{e}t})\cong\mathcal{P}ol^{(n)}_{crys}$.*

Proof. Let $\mathcal{F}^{(n)}$ be the crystalline sheaf on $\mathbb{P}^1_{\mathbb{Q}_p}\backslash\{0,1,\infty\}$ associated to $\mathcal{P}ol^{(n)}_{\mathrm{crys}}$. Then by Proposition 4.8, $\mathcal{F}^{(n)}$ is an extension of $\mathbb{Q}_p$ by $\mathcal{L}og^{(n)}_{\mathrm{\acute{e}t}}$ and is compatible with n. Hence by the definition of $\mathcal{P}ol^{(n)}_{\mathrm{\acute{e}t}}$, it suffices to prove that the residue of $\mathcal{F}^{(2)}$ at 1 is 1. The pull-back of $\mathcal{F}^{(2)}$ on $\mathrm{Spec}(A_{0,\infty}[(p(t-1))^{-1}])$ is given by the p-adic representation (6.5) for $\mathcal{U}^{\log}_{0,\infty}$ and $n=2$. Here $A_{0,\infty}$ denotes the ring of coordinates of $\mathcal{U}^{\log}_{0,\infty}$. Since the natural injection $\mathbb{Z}_p[t,t^{-1},(t-1)^{-1}]\to\mathbb{Q}_p[[t-1]][(t-1)^{-1}]$ factors through $A_{0,\infty}[(p(t-1))^{-1}]$, the inertia group acts through this representation. For the element g of the inertia at 1 corresponding to an element ε of $\widehat{\mathbb{Z}}(1)$, we have $g(\log([\underline{1-t}](1-t)^{-1}))=\varepsilon+\log([\underline{1-t}](1-t)^{-1})$. Hence, by the remark before Theorem 6.9 the residue at 1 of $\mathcal{F}^{(2)}$ is 1. □

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