# EXPLICIT RECIPROCITY LAW FOR LUBIN-TATE GROUPS

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Let $K$ be an imaginary quadratic field of class number 1 and let $E$ be an elliptic curve over $K$ with complex multiplication in the ring of integers $O_K$ of $K$. Let $\psi$ be the Grössencharacter of $K$ associated to $E$ and let $p$ be an odd prime over which $E$ has good reduction. In [Kato93], K. Kato defined the zeta element which is, roughly speaking, a compatible system in the Galois cohomology of abelian extensions of $K$ with coefficients in $\mathbb{Z}_p(1)$ defined using the elliptic units. Then he proved that, for any positive integer $k$ and a finite Hecke character $\lambda$, the zeta element gives exactly the value at $s = 0$ of the Hecke $L$-function $L(\psi^{-k}\lambda, s)$ through the dual exponential map (see §3) for $H^1(K \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathcal{F}_k(\lambda)^*(1))$, where $\mathcal{F}_k(\lambda)$ denotes the $p$-adic abelian representation of $\mathrm{Gal}(\overline{K}/K)$ corresponding to $\psi^{-k}\lambda$ (loc. cit. Chap. III, Theorem 1.2.6). A key ingredient of the proof of this theorem is the generalized explicit reciprocity law for Lubin-Tate formal groups proven in loc. cit. Chap. II, §2, which, in the above special case, describes the dual exponential maps for the Galois cohomology of abelian extensions of $K$ with coefficients in $\mathcal{F}_k(\mathbf{1})^*(1)$ explicitly. Here $\mathbf{1}$ denotes the trivial representation.

The generalization of the above Kato's result on the zeta element to the Hecke character of $K$ of the form $\psi^{-k}\overline{\psi}^j\lambda$ $(0 \le j < k)$ was proven by L. Guo [Guo99], B. Han [Han97] and K. Kimura [Kim93] in the case $p$ splits in $K$. L. Guo and K. Kimura proved it by generalizing the Kato's explicit reciprocity law. In the case $p$ is inert, B. Han [Han97] proved a partial result: a result up to $p$-adic units for trivial $\lambda$. However a generalization of the Kato's explicit reciprocity law was not known in the inert case, and it is the main theme of these notes. As a consequence of our explicit reciprocity law, we obtain the expected relation between the zeta element and $L(\psi^{-k}\overline{\psi}^j\lambda, 0)$ (under some condition on $\lambda$ in the case $k - j = 2$.) Our explicit reciprocity law holds for any Lubin-Tate formal group for a finite unramified extension of $\mathbb{Q}_p$. However, in these notes, we will deal with only the Lubin-Tate formal group of height 2 coming from $E$ in order to explain how the formulation of our explicit reciprocity law is inspired by the work of N. Katz [Katz78], [Katz80] on the special values of the Hecke $L$-functions of $K$.

## 1. Special values of Hecke $L$-functions of $K$

In this section, we will review the description of the Hecke $L$-functions of $K$ as values of some algebraic or $p$-adic derivatives of the functions $\theta_a$ $(0 \neq a \in \mathbb{Z}, (6, a) = 1)$ on the elliptic curve $E$ or on the universal elliptic curve. In the case $j = 0$, the special values $L(\psi^{-k}\lambda, 0)$ are described as values of holomorphic Eisenstein series of weight $k$, which is actually algebraic. However, in the case $j > 0$, the special values $L(\psi^{-k}\overline{\psi}^j\lambda, 0)$ are related to real analytic Eisenstein series of weight $k + j$; a $p$-adic description of these values was obtained by N. Katz ([Katz78], [Katz80]) by giving an interpretation of the real analytic differential operator

$$\frac{-\pi}{\mathrm{Im}(\overline{\omega_1}\omega_2)}\Big(\overline{\omega_1}\frac{\partial}{\partial\omega_1}+\overline{\omega_2}\frac{\partial}{\partial\omega_2}\Big) \tag{1.1}$$

on the space of $C^\infty$ elliptic modular forms in terms of the Gauss-Manin connection and the Hodge decomposition of the relative de Rham cohomology of the universal elliptic curve.

First let us review the case $j=0$.

**Proposition 1.2** ([Kato93] Chap. III, Proposition 1.1.5). *For any non-zero integer $a$ prime to 6, there exists a unique rational function $\theta_a$ on $E$ characterized by*

(1) *$div(\theta_a)=-\sum_{P\in{}_aE}P+a^2\cdot O$.*

(2) *$Norm_b(\theta_a)=\theta_a$ for any non-zero integer $b$ prime to $a$, where $Norm_b$ denotes the norm map with respect to the morphism $b\colon E\to E; P\mapsto b\cdot P$.*

By Abel's theorem, there exists a function satisfying (1), which is obviously unique up to $K^*$, and it is uniquely determined by the property (2).

Choose a basis $\omega$ of $\Gamma(E,\Omega^1_E)=\mathrm{coLie}(E)$ and an $O_K$-basis $\gamma\in H_1(E(\mathbb{C}),\mathbb{Z})$. Then we have

$$\mathbb{C}/L\xrightarrow{\cong}E(\mathbb{C}),\quad L=O_K\cdot\int_\gamma\omega \tag{1.3}$$

such that $\omega$ corresponds to $dz$ on the LHS.

Let $\partial$ denote the differential operators

$$\Omega_E^{\otimes r}\longrightarrow\Omega_E^{\otimes(r+1)}\quad(r\in\mathbb{N})$$

defined by

$$\partial(f\cdot\eta)=df\otimes\eta\quad(f\in\mathcal{O}_E,\eta\in\mathrm{coLie}(E)^{\otimes r}).$$

If we identify $\Omega_E^{\otimes r}$ with $\mathcal{O}_E$ using the basis $\omega^{\otimes r}$, then these differential operators become the operator $\frac{d}{\omega}$, which corresponds to $\frac{d}{dz}$ on $\mathbb{C}/L$ via the isomorphism (1.3).

If we apply $\partial$ to $\log(\theta_a)$ repeatedly, we obtain the following differential forms on $E$ ([Wei76], [dS87] Chap. II, §3):

$$\partial^r(\log(\theta_a))=(-1)^{r-1}(r-1)!\left(a^2\sum_{l\in L}\frac{1}{(z+l)^r}-a^r\sum_{l\in L}\frac{1}{(az+l)^r}\right)\otimes(dz)^{\otimes r}\quad(r\geq 3).$$

Let $\mathfrak{f}$ be the conductor of $\psi$. For an integer $N>0$ such that $\mathfrak{f}|N$, if we take the values at primitive $N$-torsion points of the power series in the RHS, then they become the Dirichlet power series at $s=0$ of the partial $L$-functions of $\psi^{-r}$ for the abelian extension $K({}_NE)$ of $K$ generated by $N$-torsion points of $E$ (up to some powers of a period of $\omega$). Thus we obtain a description of the special values of the partial $L$-functions of $\psi^{-r}$ at $s=0$ in terms of the values of $\partial^r(\log(\theta_a))$ at torsion points. In fact, this also holds for $r=1$ and $r=2$.

In order to generalize this to $\psi^{-k}\overline{\psi}^j$ $(0\leq j<k)$, we need to work over the moduli spaces and our description involves some differential operators on them.

Let $N$ be an integer$\geq 4$, let $Y_1(N)$ be the moduli scheme over $\mathbb{Q}$ of elliptic curves with $N$-torsion points, let $E^{\mathrm{univ}}$ be the universal elliptic curve over $Y_1(N)$ and let $\xi_N: Y_1(N)\to E^{\mathrm{univ}}$ be the universal $N$-torsion point.

First one can easily generalize Proposition 1.2 to $E^{\mathrm{univ}}/Y_1(N)$ and obtains the function $\theta_a$ on $E^{\mathrm{univ}}$ for a non-zero integer $a$ prime to 6. One can also define the differential operators

$$\partial\colon \Omega^{\otimes r}_{E^{\mathrm{univ}}/Y_1(N)} \longrightarrow \Omega^{\otimes(r+1)}_{E^{\mathrm{univ}}/Y_1(N)} \quad (r \in \mathbb{N})$$

in the same way.

Set $\underline{\omega} := \mathrm{coLie}(E^{\mathrm{univ}})$ and $\underline{H}^1_{\mathrm{dR}} := H^1_{\mathrm{dR}}(E^{\mathrm{univ}}/Y_1(N))$. We define the *real analytic* differential operators

$$D\colon \underline{\omega}^{\otimes r} \longrightarrow \underline{\omega}^{\otimes(r+2)} \quad (r \in \mathbb{N})$$

to be the composite of

$$\underline{\omega}^{\otimes r} \hookrightarrow \mathrm{Sym}^r \underline{H}^1_{\mathrm{dR}} \xrightarrow{\nabla} \mathrm{Sym}^r \underline{H}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{Y_1(N)}} \Omega^1_{Y_1(N)} \xleftarrow{\simeq} \mathrm{Sym}^r \underline{H}^1_{\mathrm{dR}} \otimes_{\mathcal{O}_{Y_1(N)}} \underline{\omega}^{\otimes 2} \longrightarrow \underline{\omega}^{\otimes(r+2)}$$

Here the second map is the Gauss-Manin connection, the third one is the isomorphism induced by the Kodaira-Spencer isomorphism and the last one is given by the Hodge decomposition of $\underline{H}^1_{\mathrm{dR}}$. Note that the last map is real analytic and the remaining ones are all algebraic.

In [Katz78], N. Katz showed that for a $C^\infty$-modular form $f \in \Gamma(Y_1(N)_{\mathbb{C}}, \underline{\omega}^{\otimes r}(C^\infty))$ of weight $r$, if we regard it as a $C^\infty$ function $f(\omega_1,\omega_2)$ on $\{(\omega_1,\omega_2) \in \mathbb{C}^2 \,|\mathrm{Im}((\omega_1)^{-1}\omega_2) > 0\}$ such that $f((\omega_1,\omega_2)\gamma) = f(\omega_1,\omega_2)$ $(\gamma \in \Gamma_1(N))$ and $f(a^{-1}\omega_1, a^{-1}\omega_2) = a^r f(\omega_1,\omega_2)$ $(a \in \mathbb{C}^*)$, then the modular form $D(f)$ of weight $r+2$ corresponds to the derivative of $f(\omega_1,\omega_2)$ by the differential operator (1.1). This implies ([Wei76], [dS87] Chap. II, §3) that, for $0 \le j < k$, $k-j \ge 3$, the values of

$$D^j \circ \xi_N^* \circ \partial^{k-j}(\log(\theta_a)) \in \underline{\omega}^{\otimes(k+j)}$$

at $(E,\alpha)$, $\alpha \in {}_NE(\mathbb{C})$ is

$$(-1)^{k-j-1}(k-1)!A(L)^{-j}\left(a^2 \sum_{l\in L} \frac{(\overline{z(\alpha)+l})^j}{(z(\alpha)+l)^k} - a^{k-j}\sum_{l\in L}\frac{(\overline{az(\alpha)+l})^j}{(az(\alpha)+l)^k}\right) \otimes (dz)^{\otimes(k+j)}.$$

Here we define $A(L)$ to be $\pi^{-1}\mathrm{Im}(\overline{l_1}l_2)$ for a basis $l_1, l_2$ of $L$ such that $\mathrm{Im}(l_2/l_1) > 0$. Note that we have

$$\left(\overline{\omega_1}\frac{\partial}{\partial\omega_1} + \overline{\omega_2}\frac{\partial}{\partial\omega_2}\right)\frac{(\overline{N^{-1}\omega_1 + n\omega_1 + m\omega_2})^j}{(N^{-1}\omega_1 + n\omega_1 + m\omega_2)^k} = -k\cdot\frac{(\overline{N^{-1}\omega_1 + n\omega_1 + m\omega_2})^{j+1}}{(N^{-1}\omega_1 + n\omega_1 + m\omega_2)^{k+1}}$$

for integers $n, m$.

Thus we obtain the following real analytic description of the special values at 0 of the partial $L$-functions of $\psi^{-k}\overline{\psi}^j$. (For $k-j = 1, 2$, we need a little more delicate argument.)

**Theorem 1.4.** *Let $L$ be the abelian extension of $K$ contained in $\mathbb{C}$ and generated by $N$-torsion points of $E$. Let $\gamma$ be a generator of the free $O_K$-module $N^{-1}H_1(E(\mathbb{C}),\mathbb{Z})$ of rank 1, and set $\alpha := \exp(\gamma) \in {}_NE(\mathbb{C}) = {}_NE(L)$. Let $\mathfrak{a}$ be a non-zero ideal of $O_K$ prime to $6N$ and satisfying $\psi(\mathfrak{a}) \in \mathbb{Z}$. Set $\sigma_{\mathfrak{a}} := (\mathfrak{a}, L/K) \in Gal(L/K)$. For $g \in Gal(L/K)$, we denote by $x_{E,g\alpha}$ the point of $Y_1(N)(\mathbb{C})$ corresponding to the elliptic curve $E \otimes_K \mathbb{C}$ endowed with the $N$-torsion point $g\alpha \in {}_NE(\mathbb{C}) = {}_NE(L)$. Then for any $g \in Gal(L/K)$ and any $k, j \in \mathbb{Z}$ such that $0 \le j < k$, the value of*

$$D^j \circ \xi_N^* \circ \partial^{k-j}(\log(\theta_{\psi(\mathfrak{a})})) \in \underline{\omega}^{\otimes(k+j)} \tag{1.5}$$

*at the point $x_{E,g\alpha}$ is*

$$(-1)^{k-j-1}(k-1)!(\pi^{-1}A(\mathbb{C}/O_K)N^2)^{-j}\Big(\int_\gamma \omega\Big)^{-k-j}\cdot$$
$$\{N(\mathfrak{a})L(\psi^{-k}\overline{\psi}^j, g\text{-}part, 0) - \psi^k\overline{\psi}^{-j}(\mathfrak{a})L(\psi^{-k}\overline{\psi}^j, g\sigma_{\mathfrak{a}}\text{-}part, 0)\}\cdot\omega^{\otimes(k+j)}.$$

*Here $A(\mathbb{C}/O_K)$ denotes the area of a fundamental parallelogram of the lattice $O_K$ in $\mathbb{C}$.*

*Remark* 1.6. Since the Hodge decomposition of $H^1_{\mathrm{dR}}(E_{\mathbb{C}}/\mathbb{C}) = H^1_{\mathrm{dR}}(E/K)\otimes_K \mathbb{C}$ is defined over $K$ (by the eigen-spaces of the action of $K\cong \mathbb{Q}\otimes \mathrm{End}_K(E)$), we see that the value of (1.5) at $x_{E,g\alpha}$ is contained in $\mathrm{coLie}(E)^{\otimes(k+j)}\otimes_K L$ (Damerell's theorem).

In order to study these values $p$-adically, we will replace the Hodge decomposition in the definition of the operator $D$ by a $p$-adically analytic decomposition on the formal moduli space.

Let $\mathfrak{p}$ be an inert prime of $K$ at which $E$ has good reduction, let $\mathcal{K}$ be the completion of $K$ at $\mathfrak{p}$, let $\mathcal{E}/\mathcal{O}_{\mathcal{K}}$ be the proper smooth model of the base change of $E/K$ to $\mathcal{K}$, and let $\mathcal{E}^{\mathrm{univ}}/\mathcal{M}$ be the formal moduli of $\mathcal{E}/\mathcal{O}_{\mathcal{K}}$. Then on the PD-envelope $\mathcal{M}^{\mathrm{PD}}$ of $\mathrm{Spec}(O_{\mathcal{K}})\hookrightarrow\mathcal{M}$, we have an isomorphism

$$\underline{H}^1_{\mathrm{dR}}\cong \mathcal{O}_{\mathcal{M}^{\mathrm{PD}}}\otimes_{\mathcal{O}_{\mathcal{K}}} H^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{O}_{\mathcal{K}})$$

(the Gauss-Manin connection has enough solutions on $\mathcal{M}^{\mathrm{PD}}$), and hence the Hodge decomposition of $H^1_{\mathrm{dR}}(E/K)$ (which is, in fact, defined over $\mathcal{O}_{\mathcal{K}}$ because $K$ is unramified at $\mathfrak{p}$ over $\mathbb{Q}$) gives rise to a splitting of the short exact sequence:

$$0\longrightarrow \mathrm{coLie}(\mathcal{E}^{\mathrm{univ}})\longrightarrow \underline{H}^1_{\mathrm{dR}}\longrightarrow \mathrm{Lie}(\mathcal{E}^{\mathrm{univ}})\longrightarrow 0.$$

Note that the composite of

$$\mathcal{O}_{\mathcal{M}^{\mathrm{PD}}}\otimes_{\mathcal{O}_{\mathcal{K}}}\mathrm{Lie}(\mathcal{E})\hookrightarrow \underline{H}^1_{\mathrm{dR}}\to \mathrm{Lie}(\mathcal{E}^{\mathrm{univ}})$$

is an isomorphism. By replacing the Hodge decomposition with this splitting in the definition of the operator $D$, we define the differential operators on $\mathcal{M}^{\mathrm{PD}}$:

$$D_p\colon \underline{\omega}^{\otimes r}\longrightarrow \underline{\omega}^{\otimes(r+2)}\quad (r\in\mathbb{N}).$$

Let $\mathcal{M}_N$ be the kernel of the multiplication by $N$ on $\mathcal{E}^{\mathrm{univ}}$ and let $\xi_N\colon\mathcal{M}_N\to\mathcal{E}^{\mathrm{univ}}\times_{\mathcal{M}}\mathcal{M}_N$ be the $N$-torsion point induced by the canonical inclusion $\mathcal{M}_N\hookrightarrow\mathcal{E}^{\mathrm{univ}}$. Since $\mathcal{M}_N$ is étale over $\mathcal{M}$ after inverting $p$, the differential operators $D_p$ are naturally extended to $\mathcal{K}\otimes_{O_{\mathcal{K}}}\underline{\omega}^{\otimes r}\to\mathcal{K}\otimes_{O_{\mathcal{K}}}\underline{\omega}^{\otimes(r+2)}$ on $\mathcal{M}_N\times_{\mathcal{M}}\mathcal{M}^{\mathrm{PD}}$, which we again denote by $D_p$.

For any primitive $N$-torsion point $\alpha\in E(K({}_NE))$, the morphism $\mathrm{Spec}(O_{\mathcal{K}})\hookrightarrow\mathcal{M}$ is uniquely extended to a morphism $\varepsilon_\alpha\colon\mathrm{Spec}(K({}_NE)\otimes_K\mathcal{K})\to\mathcal{M}_N$ such that the inverse image of $\xi_N$ is $\alpha$.

**Proposition 1.7.** *For any integers $0\le j<k$, the value of $D^j\circ\xi_N^*\circ\partial^{k-j}(\log(\theta_a))$ on $Y_1(N)$ at $(E,\alpha)$ considered in Theorem 1.4 coincides with $\varepsilon_\alpha^*\circ(D_p)^j\circ\xi_N^*\circ\partial^{k-j}(\log(\theta_a))$.*

By combining Theorem 1.4 with Proposition 1.7, we obtain a $p$-adic description of the values at $s=0$ of the partial Hecke $L$-functions of $\psi^{-k}\overline{\psi}^j$ $(0\le j<k)$.

## 2. Lifting of Coleman power series

The $p$-adic description of the special values of Hecke $L$-functions in §1 involves differential operators on the formal moduli space or on the universal elliptic curve over it. Hence such differential operators should appear in an explicit reciprocity law related to such special values, and we are naturally led to the question of lifting Coleman power series to the universal elliptic curve on the formal moduli space.

First let us recall the main theorem in the theory of Coleman power series in our special setting. We keep the notation in the previous section and set $\pi := \psi(\mathfrak{p})$. First note that the completion $\widehat{\mathcal{E}}$ is the Lubin-Tate formal group associated to $(\mathcal{K}, \pi)$. Let $H$ be a finite unramified extension of $\mathcal{K}$, let $\sigma$ be the Frobenius of $H$ over $\mathcal{K}$ and let $\varphi$ denote the morphism $\sigma \otimes [\pi] \colon O_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}} \to O_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}}$, where $[\pi]$ denotes the multiplication by $\pi \in O_\mathcal{K}$ on $\widehat{\mathcal{E}}$. $\varphi$ is a lifting of the square of the absolute Frobenius. Let $H_n$ be the extension of $H$ generated by $\pi^n$-torsion points of $\widehat{\mathcal{E}}$ (or equivalently $\mathcal{E}$).

**Theorem 2.1** (R. Coleman [Col79]). *Let $\eta = (\eta_m)_{m \geq 1}$ be a basis of the Tate module $T\widehat{\mathcal{E}} = \varprojlim_n ({}_{\pi^n}\widehat{\mathcal{E}})(\overline{\mathcal{K}})$ of $\widehat{\mathcal{E}}$ as an $O_\mathcal{K}$-module. Then we have an isomorphism*

$$\Gamma(O_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}}, \mathcal{O}^*_{O_H \widehat{\otimes} \widehat{\mathcal{E}}})^{Norm_\varphi = 1} \xrightarrow{\sim} \varprojlim_m O^*_{H_m}$$

*which sends $f$ to $((\sigma^{-m}(f))(\eta_m))_{m \geq 1}$. In the RHS, we take the projective limit with respect to the norm maps.*

For a norm compatible system $u = (u_m)_{m \geq 1}$ of a unit $u_m$ of $H_m$, we call the corresponding function on $\mathcal{O}_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}}$ the Coleman power series of $u$. Note that we have a non-canonical isomorphism $\Gamma(O_H \widehat{\otimes}_{O_\mathcal{K}} \mathcal{E}, \mathcal{O}) \cong O_H[[X]]$.

By the universal property, $\varphi$ is naturally extended to $O_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}^{\mathrm{univ}}}$ and Coleman power series can be canonically lifted to $\widehat{\mathcal{E}^{\mathrm{univ}}}$ as follows:

**Proposition 2.2** ([Tsu00] Proposition 5.2). *The specialization map*

$$\Gamma(O_H \widehat{\otimes}_{O_\mathcal{K}} \widehat{\mathcal{E}^{univ}}, \mathcal{O}^*)^{Norm_\varphi = 1} \xrightarrow{\sim} \Gamma(O_H \widehat{\otimes}_{O_\mathcal{K}} \mathcal{E}, \mathcal{O}^*)^{Norm_\varphi = 1}$$

*is an isomorphism.*

## 3. Exponential maps and dual exponential maps

In this section, $K$ denotes a finite extension of $\mathbb{Q}_p$ and $K_0$ denotes the maximal unramified extension of $\mathbb{Q}_p$ contained in $K$. In [BK90], S. Bloch and K. Kato generalize the exponential maps

$$K \xrightarrow{\exp} \mathbb{Q}_p \otimes \widehat{K^*} \cong H^1(K, \mathbb{Q}_p(1)),$$

$$Lie(A_K) \xrightarrow{\exp} \mathbb{Q}_p \otimes \widehat{A(K)} \hookrightarrow H^1(K, \mathbb{Q}_p \otimes T_p A)$$

for $\mathbb{G}_m$ and an abelian scheme $A$ over $O_K$ to any crystalline representation of $Gal(\overline{K}/K)$.

By the theory of J.-M. Fontaine ([Fon82], [Fon94a], [Fon94b]), we have the notion "crystalline" for $p$-adic representations of $\mathrm{Gal}(\overline{K}/K)$ and a fully faithful functor $D_{\mathrm{crys}}$ from the category of crystalline $p$-adic representations of $\mathrm{Gal}(\overline{K}/K)$ to the category $MF_K(\varphi)$ of finite dimensional filtered $\varphi$-modules over $K$. Here a $p$-adic representation of $\mathrm{Gal}(\overline{K}/K)$ means a finite dimensional $\mathbb{Q}_p$-vector space $V$ endowed with a continuous and linear action of $\mathrm{Gal}(\overline{K}/K)$ and a finite dimensional filtered

$\varphi$-module over $K$ is a finite dimensional $K$-vector space $D$ endowed with a separated and exhaustive decreasing filtration $Fil^iD$ ($i \in \mathbb{Z}$) by $K$-subspaces, a $K_0$-structure $D_0$ and a semi-linear automorphism $\varphi$ on $D_0$.

The crystalline conjecture, which is now a theorem, asserts that for any proper smooth scheme over $O_K$, the $p$-adic representation $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline and the corresponding filtered $\varphi$-module is canonically isomorphic to the de Rham cohomology $H^m_{\text{dR}}(X_K/K)$ endowed with the Hodge filtration and the $K_0$-structure with $\varphi$ given by the crystalline cohomology of the special fiber of $X$.

The essential image of the functor $D_{\text{crys}}$ is stable under extensions and hence $D_{\text{crys}}$ defines a natural injective homomorphism:

$$\mathrm{Ext}^1_{MF_K(\varphi)}(K, D) \hookrightarrow H^1(K, V)$$

for any crystalline representation $V$ of $\mathrm{Gal}(\overline{K}/K)$ and $D = D_{\text{crys}}(V)$. On the other hand, we have a natural map:

$$D/Fil^0D \to \mathrm{Ext}^1_{MF_K(\varphi)}(K, D)$$

sending the class of $a \in D$ to $K \oplus D$ whose filtration is twisted by $a$:

$$Fil^i(K \oplus D) = \begin{cases} Fil^iD & (i > 0), \\ Fil^iD \oplus K \cdot (1, a) & (i \leq 0). \end{cases}$$

The $K_0$-structure and the Frobenius are the direct sums. We define the exponential map:

$$\exp\colon D/Fil^0D \to H^1(K, V)$$

to be the composite of these two maps. Using Tate duality, we obtain the dual exponential map:

$$\exp^*\colon H^1(K, V) \to Fil^0D.$$

Note that in the case $V = \mathbb{Q}_p \otimes T_pA$ for an abelian scheme $A$ over $O_K$, $D$ is canonically isomorphic to the dual of $H^1_{\text{dR}}(A_K/K)$ and $D/Fil^0D$ is isomorphic to $Lie(A_K)$.

## 4. Explicit reciprocity law

Before giving our generalized explicit reciprocity law, we shall review how the classical explicit reciprocity law for $\mathbb{Q}_p(\mu_{p^n})$ ($n \geq 1$) by Artin-Hasse and K. Iwasawa can be reformulated in terms of the dual exponential maps for the trivial $p$-adic representation $\mathbb{Q}_p$.

By the definition of the dual exponential map, we have

$$u \cup c = \mathrm{Trace}_{\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p}(\log(u)\exp^*(c))$$

for $u \in \mathbb{Q}_p \otimes \mathbb{Z}_p[\mu_{p^n}]^* \subset H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbb{Q}_p(1))$ and $c \in H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbb{Q}_p)$. Hence the classical explicit reciprocity law says that the image of $f \in (\mathbb{Z}_p[[X-1]]^*)^{\mathrm{Norm}_\varphi = 1}$ under the composite of

$$\begin{aligned} (\mathbb{Z}_p[[X-1]]^*)^{\mathrm{Norm}_\varphi=1} &\xrightarrow[\text{Coleman}]{\sim} \varprojlim{}_m \mathbb{Z}_p[\mu_{p^m}]^* \to \varprojlim{}_m H^1(\mathbb{Q}_p(\mu_{p^m}), \mathbb{Z}_p(1)) \\ &\xrightarrow{\sim} \varprojlim{}_m H^1(\mathbb{Q}_p(\mu_{p^m}), \mathbb{Z}/p^m\mathbb{Z}(1)) \xrightarrow[t^{\otimes(-1)}]{\sim} \varprojlim{}_m H^1(\mathbb{Q}_p(\mu_{p^m}), \mathbb{Z}_p/p^m\mathbb{Z}) \\ &\xleftarrow{\sim} \varprojlim{}_m H^1(\mathbb{Q}_p(\mu_{p^m}), \mathbb{Z}_p) \xrightarrow{\text{proj}} H^1(\mathbb{Q}_p(\mu_{p^n}), \mathbb{Z}_p) \xrightarrow{\exp^*} \mathbb{Q}_p(\mu_{p^n}) \end{aligned}$$

is

$$p^{-n}\Big(X\frac{d}{dX}\Big)\log(f)\Big|_{X=\zeta_n}$$

for $n \geq 1$. Here we choose a generator $t = (\zeta_m)_{m\geq 1}$ of $\mathbb{Z}_p(1)$ and the first map is defined by taking the values at $X = \zeta_m$ $(m \geq 1)$.

The generalized explicit reciprocity law of Bloch-Kato in [BK90] §2 and K. Kato in [Kato93] Chap. II, §2 asserts that, if we replace $t^{\otimes(-1)}$ by $t^{\otimes(-r)}$ $(r \geq 2)$ and $\mathbb{Z}_p$ by $\mathbb{Z}_p(1-r)$, then we obtain

$$\frac{1}{(r-1)!}p^{-nr}\Big(\Big(X\frac{d}{dX}\Big)^r\log(f)\Big)\Big|_{X=\zeta_n}.$$

The classical explicit reciprocity law of Artin-Hasse and K. Iwasawa is generalized by A. Wiles [Wil78] to a finite extension $K$ of $\mathbb{Q}_p$ and the abelian extensions $K$ generated by torsion points of the Lubin-Tate formal group associated to $K$ and a uniformizer $\pi$ of $K$. In [Kato93], K. Kato also showed that the above generalized explicit reciprocity law still holds if we replace $\mathbb{Q}_p(\mu_{p^n})$ by the extension of $K$ generated by $\pi^n$-torsion points of the Lubin-Tate group, $t$ by a basis $\eta$ of the Tate module $T$ of the Lubin-Tate group as an $O_K$-module and the derivation $X\frac{d}{dX}$ by $\partial$ defined in the same way as in §1. If we apply this to $\mathcal{K}$ and $\mathcal{E}$, then by §1, we see that the translations of $\theta_a$ by torsion points of order prime to $p$ are related to the values at $s = 0$ of the partial Hecke $L$-functions of $\psi^{-r}$ $(r \geq 1)$ through the dual exponential maps for $T^{\otimes(-r)}(1)$.

Now we will give the precise statement of our generalization of the above Kato's explicit reciprocity law. We keep the notation of §1 and §2. Let $T$ be the Tate module of $\widehat{\mathcal{E}}$, let $\overline{T}$ be the same $\mathbb{Z}_p$-representation of $\mathrm{Gal}(\overline{\mathcal{K}}/\mathcal{K})$ as $T$ with the action of $O_{\mathcal{K}}$ twisted by the unique non-trivial automorphism of $\mathcal{K}/\mathbb{Q}_p$, and set $T_{k,j} := T^{\otimes(-k)} \otimes_{O_{\mathcal{K}}} \overline{T}^{\otimes j}$ for integers $0 \leq j < k$. Choose and fix a basis $\eta = (\eta_n)_{n\geq 1}$ of $T = \varprojlim_n ({}_{\pi^n}\widehat{\mathcal{E}})(\overline{\mathcal{K}})$ and let $\overline{\eta}$ be the same element as $\eta$ regarded as an element of $\overline{T}$.

**Theorem 4.1** ([Tsu00] Theorem 6.3). *For any integers $0 \leq j < k$, the image of an element $f$ of $\Gamma(O_H\widehat{\otimes}_{O_{\mathcal{K}}}\widehat{\mathcal{E}^{univ}}, \mathcal{O}^*)^{Norm_\varphi=1}$ under the composite of the maps*

$$\begin{aligned}\Gamma(O_H\widehat{\otimes}_{O_{\mathcal{K}}}\widehat{\mathcal{E}^{univ}}, \mathcal{O}^*)^{Norm_\varphi=1} &\xrightarrow{\sim} \Gamma(O_H\widehat{\otimes}_{O_{\mathcal{K}}}\widehat{\mathcal{E}}, \mathcal{O}^*)^{Norm_\varphi=1} \underset{Coleman}{\xrightarrow{\sim}} \varprojlim_m O_{H_m}^*\\ &\longrightarrow \varprojlim_m H^1(H_m, \mathbb{Z}_p(1)) \xrightarrow{\eta^{\otimes(-k)}\otimes\overline{\eta}^{\otimes j}} \varprojlim_m H^1(H_m, T_{k,j}(1))\\ &\xrightarrow{proj} H^1(H_n, T_{k,j}(1)) \xrightarrow{\exp^*} (coLie(E))^{\otimes(k+j)} \otimes_K H_n\end{aligned}$$

*is*

$$(-1)^j\frac{1}{(k-1)!}\pi^{-nk}\overline{\pi}^{nj}\cdot\varepsilon_{\eta_n}^*\circ D_p^j\circ\xi_{p^n}^*\circ\partial^{k-j}(\log(\sigma^{-n}(f))).$$

*See §1 for the definition of $\partial$, $D_p$, $\xi_{p^n}$ and $\varepsilon_{\eta_n}$. Here $\overline{\pi}$ denotes the image of $\pi \in \mathcal{K}$ under the unique non-trivial automorphism of $\mathcal{K}/\mathbb{Q}_p$.*

If we apply this theorem to the translation of $\theta_a$ by torsion points of $\mathcal{E}$ of order prime to $p$, then, by §1, the images are related to the special values at $s = 0$ of the partial $L$-functions of $\psi^{-k}\overline{\psi}^j$. By the similar argument as in [Kato93] Chap. III, we see that the Kato's zeta element gives $(-1)^{k-j-1}$ times the special value at 0 of the Hecke $L$-function of $\psi^{-k}\overline{\psi}^j\lambda$ (for integers $0 < j \leq k$ and a finite Hecke character

$\lambda$) through the dual exponential map for $H^1(\mathcal{K}, \mathcal{F}_{k,j}(\lambda)^*(1))$ under some condition on $\lambda$ in the case $k-j=2$ ([Tsu00] Theorem 12.8). Here $\mathcal{F}_{k,j}(\lambda)$ denotes the $p$-adic representation of $\mathrm{Gal}(\overline{K}/K)$ such that, for any prime ideal $\mathfrak{p}$ of $O_K$ prime to the conductor of $\psi^{-k}\overline{\psi}^j\lambda$, $\mathcal{F}_{k,j}(\lambda)$ is unramified at $\mathfrak{p}$ and the action of the *geometric* Frobenius at $\mathfrak{p}$ is given by the multiplication by $(\psi^{-k}\overline{\psi}^j\lambda)(\mathfrak{p})$.

The above explicit reciprocity law can be generalized to any Lubin-Tate formal group for a finite unramified extension of $\mathbb{Q}_p$ using the formal moduli space of the Lubin-Tate group. See [Tsu00] for details.

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