Model Theory and the Mordell-Lang conjecture

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1 Introduction

These notes are based on the lectures about Model Theory and Diophantine Geometry given by Anand Pillay at the University of Arizona in May, 2000. The goal is to present the basic ideas and concepts of (Geometric) Model Theory and Diophantine Geometry and to give a quick overview of how do they fit together in Hrushovski's model theoretic proof of the famous Mordell-Lang conjecture. For details that are missed in this article we refer the interested reader to the following three main sources: [19], [5], [3], [4].

This article comprises the following parts:

- Section 1: Introduction.
- Section 2: Basic definitions and facts from Algebraic Geometry and Number Theory.
- Section 3: Highlights of the basic notions and facts of Model Theory and of Geometric Model Theory.
- Section 4: Intermission, a short summary.
- Section 5: A list of important propositions and definitions that will be used in the proof of Mordell-Lang conjecture in Section 6.
- Section 6: Sketch of the proof of the Mordell-Lang conjecture.
- Section 7: Historical overview of the Mordell-Lang Conjecture and other related theorems.

All definitions, propositions, facts, theorems and etc. are labeled as Z X.Y; where Z is a definition, proposition, etc., X is a section number and Y is the number of the object Z inside the section X.

Sections 2 and 3 consist of a list of definitions and facts that the reader will most likely encounter in reading any text about this or a related topic. But we tried to restrict ourselves, most of the time, for presenting the minimal necessary concepts, ideas and facts that will be needed in section 6.

So what is this paper all about?

One one hand we have Diophantine Geometry. In general Diophantine or Arithmetic Geometry is concerned about solution sets and their interactions in some field K of a system of polynomial equations over K. In particular the main topic of this paper the Mordell-Lang conjecture describes the properties of the intersection of a subvariety X of a semiabelian variety A, both defined by polynomial equations over a function field K, with special subgroups Γ of A.

On the other hand we have Model Theory which is a branch of mathematical logic in which one studies mathematical structures by considering the first order sentences true of those structures and sets definable in those structures by first order formulas. In Geometric Model theory one studies the behavior and interaction of these structures inside bigger structures.

Now Hrushovski used model theoretical tools to prove the Mordell-Lang conjecture and pooled together seemingly very different areas of mathematics. Our goal then will be to describe the salient features of his proof.

The general idea in applying model theory to problems in Diophantine geometry is to find a suitable model theoretic structure (e.g. theory of algebraically closed fields, $\aleph_1$-saturated models, etc.) in which the objects from the original question are replaced by appropriate model theoretic objects (e.g. definable sets, complete types, etc.). Then one can apply the theorems and results of model and geometric model theory (e.g. Zilber's conjecture) and use these results to infer something about the original claim.

2 Basic definitions and facts from Algebraic Geometry and Number Theory

Definition 2.1 *A field K is said to be* **algebraically closed** *if any non zero polynomial with coefficients from K in a single indeterminant has a zero in K.*

We will work with fields of infinite cardinality.

Definition 2.2 *Let R be a characteristic zero ring. A* **derivation** *on R is an additive map δ on R such that for all x and y in R, $\delta(xy) = x\delta(y) + y\delta(x)$.*
A **differential** *ring is a ring equipped with a derivation δ. A* **differential field** *is a differential ring which is also a field (i.e. R is a field with a derivation).*
Given a differential field R, the **differential polynomial ring** *$R\{y\}$ in one differential indeterminant y is the ring $R\{y\} = R[y, \delta y, \delta^2 y, \dots]$, i.e. the (usual algebraic) polynomial ring over R in the infinite set of indeterminants $y, \delta y, \delta^2 y, \dots$.*
By extending δ to all of $R\{y\}$ in the obvious manner, we get that $R\{y\}$ is a differential ring. For nonzero $f \in R\{y\}$ we define **ord(f)***, the order of f, to be the largest n, if any, such that $\delta^n y$ appears in f with non zero coefficient. For $x \in R$ we take $ord(x) = -1$.*
A differential field K is **differentially closed** *provided for all $f, g \in K\{y\}$ such that $ord(f) > ord(g)$, K contains an element a such that $f(a) = 0$ and $g(a) \neq 0$. Roughly speaking the field K is differentially closed when it contains*

enough solutions of ordinary differential equations.
Given a differential field K, the **constant field** *of K is the field $C_K = \{x \in K : \delta(x) = 0\}$.*

Fact 2.1 *Differentially closed fields exist and every differential field has a differential closure which is unique up to isomorphism.*

Fact 2.2 *Any differentially closed field is algebraically closed.*

Definition 2.3 *Let K be a field of characteristic $p > 0$. We say that a polynomial over K is separable if all of its roots are distinct. Let x be algebraic over K, f its minimal polynomial. x is sad to be* **separable** *over K if f is separable. The set of separably algebraic elements over K form a subfield K^s of the algebraic closure K^a of K. K^s is called the* **separable closure** *of K, and K is said to be* **separably closed** *if $K = K^s$.*

Definition 2.4 *Let K be a field of characteristic $p > 0$. If K has dimension p^ν over its subfield of p-th powers, K^p, i.e. $p^\nu = [K : K^p]$ we say that K has a* **finite degree of imperfection** *equal to ν.*

Fact 2.3 *Algebraic extensions never increase the degree of imperfection.*

Fact 2.4 *Let k be an algebraically closed field. Let K be a separable extension of k, that is every $x \in K$ is separable over k. Then $k = K^{p^\infty} = \cap_{n\in\mathbb{N}} K^{p^n}$ where K^{p^n} is the subfield of p^n-th powers.*

Fact 2.5 *Separably closed fields are not algebraically closed.*

Definition 2.5 *A* **number field** *is a finite extension of the field of rational numbers $\mathbb{Q}$.*

Definition 2.6 *By a* **function field** *we will mean a finitely generated extension of an algebraically closed field.*

Definition 2.7 *Let K be a field. By a* **Zariski closed subset** *of K^n we mean a set Y which is a finite intersection of zero sets of polynomials over K (in n indeterminants). By* **Zariski topology** *we mean a topology generated by the Zariski closed subsets. A* **Zariski open set** *is a complement of a Zariski closed set.*

<u>Note</u>: In this presentation any topological concept (e.g. closed, dense) will be meant in the sense of Zariski topology.

Definition 2.8 *Let K be an algebraically closed field. We say that X is an* **(affine) variety** *in K^n if it is the solution set of finite many polynomial equations from $K[X_1, \ldots, X_n]$. We say that X is defined over k, a subfield of K, if the defining polynomials of X can be chosen with coefficients from k. An*

abstract (affine) variety *is something modeled locally on affine varieties i.e. it is pieced together in a "suitable way" from finite many affine varieties (an idea similar to that of abstract manifolds). If $L \subset K$ is a field, by $X(L)$, the* L **rational points** *of X, we mean the set of all points in X whose coordinates are in L. A* **subvariety** *of a variety X is a variety which is a subset of X. A variety is* **irreducible** *(i.e. connected) if it cannot be written as a finite union of subvarieties. Note: varieties and subvarieties are closed sets in the Zariski topology.*

Remark: Abstractly one should think about an affine variety X as a set of polynomial equations over some field. If the field is not mentioned explicitly one can take any algebraically closed field K which contains the coefficients of the defining polynomial equations and think of X as $X(K)$. But it should be realized that some statements about the variety X are true regardless of the underlying field. In this case the field is not mentioned explicitly. $\triangleright$

Fact 2.6 *Let K be an algebraically closed field and k a subfield of K. Let W be any subset of k^n. Let V be the Zariski closure of W in K^n. Then V is defined over k.*

Definition 2.9 *Let K be an algebraically closed field. We define* **regular functions** *on K^n as follows:*

- *constant functions are regular functions*
- *coordinate functions i.e. projection functions* $\pi_i : K^n \to K$ *where* $\pi_i(x_1, \ldots, x_n) = x_i$ *are regular functions*
- *the sum and the product of regular functions is a regular function*

Thus, regular functions are expressed polynomially in terms of the coordinate functions π_i.

For infinite fields K we can identify the regular functions on K^n with polynomials over K in n variables.

Definition 2.10 *Let $W \subset K^n$ and $V \subset K^m$ be two affine varieties. A mapping $f : W \to V$ is said to be a* **morphism** *if it can be extended to a mapping of the ambient spaces $\tilde{f} : K^n \to K^m$ which is given by m regular functions $K^n \to K$.*

Since abstract varieties are obtained by gluing together affine varieties one defines a morphism between two abstract varieties as a family of morphisms as defined above that behave "properly" under gluing.

Definition 2.11 *Let K be an algebraically closed field. A* **rational function** *on K^n is a ratio f/g of two regular functions on K^n, f and g, where $g \neq 0$ and f and g have the same degree.*

Then one proceeds as above in the case of a regular function to obtain the definition of a rational mapping (morphism) between two (abstract affine) varieties.

One can similarly define projective (algebraic) varieties, abstract projective (algebraic) varieties, regular functions, morphisms and rational mappings in an analogous manner using the projective space $\mathbb{P}^n(K)$. For more details on the complete definitions of the above objects the reader should consult the reference [11].

Definition 2.12 *An* **algebraic group** *G is by definition a variety X together with a pair of morphisms $\mu : X \times X \to X$ and $\rho : X \to X$, such that μ yields a group operation on X, ρ is the map $x \mapsto x^{-1}$. G is said to be defined over k, if X, μ, ρ are all defined over k.*

Example Let K be a field. The group $GL_n(K)$, the group of $n \times n$ invertible matrices over K is an algebraic group whose underlying variety is isomorphic to an irreducible affine subvariety of an affine $n^2 + 1$space. $\diamond$

Fact 2.7 *In a commutative algebraic group the Zariski closure of a finite boolean combination of cosets (translates of subgroups) is just a finite union of cosets of algebraic subgroups.*

Definition 2.13 *We will say that an algebraic group is* **connected** *(i.e. irreducible) if it has no proper closed subgroups of finite index.*

Definition 2.14 *A variety X is said to be* **complete** *if for any variety Y the projection map $\pi : X \times Y \to Y$ is a closed map in the Zariski topology, namely takes closed sets to closed sets.*

Definition 2.15 *An* **abelian variety** *is a connected (i.e. irreducible) complete algebraic group. An* **abelian subvariety** *of A is a closed connected subgroup of A.*

Example An abelian variety over the field of complex numbers, $\mathbb{C}$, is a compact complex Lie group (a torus), that is, a quotient $\mathbb{C}^r/L$, where L is a lattice in $\mathbb{C}^r$. $\diamond$

Definition 2.16 *Over an algebraically closed field K, a* **semiabelian** *variety S is a (commutative) connected algebraic group such that some algebraic subgroup T of S is an algebraic torus (i.e. a product of a finite number of copies of the multiplicative group $(K\backslash\{0\}, \cdot)$) and A/T is an abelian variety.*

We quote the next few facts to elucidate the structure of algebraic groups.

Fact 2.8 *Let G be an algebraic group. If H is a subgroup of G then the Zariski closure of H in G is also a subgroup of G.*

Fact 2.9 *If G is a subgroup of an abelian variety A, then $\overline{G}$ is an abelian subvariety of A.*

Fact 2.10 *Let G be an algebraic group. Then the irreducible component of G which contains the identity e coincides with the smallest closed subgroup of G of finite index. We call this subgroup G^0, the* **connected component** *of G. The other irreducible components of G are just cosets of G^0 in G. G^0 is normal in G.*

Definition 2.17 *An algebraic group is said to be* **linear** *if it is isomorphic (as an algebraic group) to a closed subgroup of some $GL_n(K)$.*

Fact 2.11 *Let G be a connected algebraic group. Then G has a maximal normal linear closed subgroup N, and G/N is an abelian variety.*

Fact 2.12 *Abelian varieties are commutative.*

Fact 2.13 *Abelian varieties are smooth.*

Example Abelian varieties of *dimension one* are known as *elliptic curves.* $\diamond$

Fact 2.14 *Let G be a subgroup of an abelian variety A. Then the Zariski closure of G is an abelian subvariety of A.*

Fact 2.15 *Let K be an algebraically closed field and A an abelian variety over K. If A is defined over $k \subset K$, and G is a closed subgroup of A, then G is defined over the algebraic closure of k, in characteristic zero, and over the separable closure of k in characteristic $p > 0$.*

Fact 2.16 *If B is an abelian subvariety of an abelian variety A, there exists another abelian subvariety C such that $A = B + C$ and $B \cap C$ is finite. In other words the map from $B \times C$ to A defined by $(b, c) \mapsto b + c$ is an isogeny (i.e. a surjective homomorphism with finite kernel).*

Definition 2.18 *If we have the situation as in the above* **Fact 2.16** *we say that A is an* **almost direct sum** *of B and C.*

We give two definitions for a *finite rank group.* One in characteristic zero and one in positive characteristic. Let A be an abelian variety defined over a field K.

Definition 2.19 *(char $p = 0$) Let $\Gamma \subset A(K)$. We say that the group Γ is of finite rank if there is a finitely generated subgroup $\Gamma' \subset \Gamma$ such that for every $x \in \Gamma$ there is some positive integer n such that $nx \in \Gamma'$*

Definition 2.20 *(char $p > 0$) Let $\Gamma \subset A(K)$. We say that the group Γ is of finite rank if there is a finitely generated subgroup $\Gamma' \subset \Gamma$ such that for every $x \in \Gamma$ there is some positive integer n coprime with p such that $nx \in \Gamma'$*

3 Highlights of the basic notions and facts of Model and Geometric Model Theory

As we vaguely described in the introduction model theory is a branch of mathematical logic in which one studies mathematical structures by considering the first order sentences true of those structures and sets definable in those structures by first order formulas.
In this section, among other things, we will try to make explicit the meaning of the above sentence.

Definition 3.1 *A* **structure** $\mathcal{M}$ *consist of*

1. *A non empty set* M.
2. *A family* $(R_i^{\mathcal{M}})_{i\in I}$ *of relations or subsets of* $\cup_{n\in\mathbb{N}} M^n$, *that is, for each* i, $R_i^{\mathcal{M}}$ *is a subset of* M^{n_i} *for some* $n_i \geq 1$. *We add the extra condition that the diagonal of* M^2 *is one of the* $R_i^{\mathcal{M}}$*'s. (We note that there are two different notations in use:* $\bar{m} \in R^{\mathcal{M}}$ *and* $R^{\mathcal{M}}(\bar{m})$ *both denote that the* n*-tuple* $\bar{m} \in M^n$ *is the element of the* n*-ary relation* $R^{\mathcal{M}}$*.)*
3. *A family of maps* $(f_j^{\mathcal{M}})_{j\in J}$, *where* $f_j^{\mathcal{M}}$ *is an* n_j*-ary map,* $f_j^{\mathcal{M}} : M^{n_j} \to M$.
4. *A set of constants* $(c_k^{\mathcal{M}})_{k\in K}$, *where* $c_k^{\mathcal{M}} \in M$.

Remarks:

1. We can treat all functions and constants as relations by referring to their graphs as relation in the structure $\mathcal{M}$. Then we can simply talk about $\mathcal{M}$ as a **relational structure**.
2. One can easily generalize this to the concept of **many sorted structure** where instead of a single set M we have a family of underlying sets $(M_s)_{s\in S}$ (S is indexing the "sorts") and the constants, relations, and the graphs of the functions are subsets of cartesian products of finite many various M_s's. ▹

Definition 3.2 *The* **signature** *or* **language** $\mathcal{L}$ *associated to a structure* $\mathcal{M}$ *consist of*

1. *For each relation* $R_i^{\mathcal{M}}$, *a relational symbol,* R_i *of arity* n_i *(arity is the number* n *such that* $R_i^{\mathcal{M}} \subset M^n$*). As the diagonal of* M^2 *is by assumption always one of the basic relations, there always is a particular binary relation in* $\mathcal{L}$, *which corresponds to equality in the structure, and which we allow ourselves not to mention when describing a specific language.*
2. *For each map* $f_j^{\mathcal{M}}$ *a function symbol,* f_j *of arity* n_j.
3. *For each constant,* $c_k^{\mathcal{M}}$ *a constant symbol* c_k.

We will use the following notation for an arbitrary language

$$\mathcal{L} = \{(R_i)_{i\in I}; (f_j)_{j\in J}; (c_k)_{k\in K}\}.$$

If $\mathcal{L}$ is the language (or signature) associated to the structure $\mathcal{M}$, we say that $\mathcal{M}$ is an **$\mathcal{L}$-structure**. *(The language $\mathcal{L}$ need not be countable).*
Note: *Sometimes for convenience we will abuse the notation and write M (the underlying set of the structure $\mathcal{M}$) instead of $\mathcal{M}$.*

Example Algebraically closed fields and separably closed fields as $\mathcal{L}_{ring} = \{+, \cdot, -, 0, 1\}$ structures. The binary maps $+$ and $\cdot$ stand for addition and multiplication in the field. The unary map $-$ for the inverse in the additive group and two constants 0 and 1 for the neutral elements. The symbol for equality is implicitly assumed as mentioned above. $\diamond$

Example Differential fields as $\mathcal{L}_{diff} = \mathcal{L}_{ring} \cup \{\delta\}$-structures. We add a new symbol, δ, for the derivation map. $\diamond$

Example Let K be a field of characteristic $p > 0$ and of finite degree of imperfection ν. Then K as a vector space over K^p has a basis $b_1, \dots, b_{p^\nu}$ and there are p^ν component functions $\lambda_1, \dots, \lambda_{p^\nu}$ so that every $x \in K$ can be written as $x = \sum_{i=1}^{p^\nu} \lambda_i(x)^p b_i$. Let $\mathcal{L}_{p,\nu} = \mathcal{L}_{ring} \cup \{b_1, \dots, b_{p^\nu}, \lambda_1, \dots, \lambda_{p^\nu}\}$. Then a separably closed field K of characteristic $p > 0$ and finite degree of imperfection is an example of a $\mathcal{L}_{p,\nu}$ structure ν. $\diamond$

Example A vector space V over a fixed field F is an example of a two sorted structure. The sets V and F are the sorts. We have a language consisting of the language for the group V of the language for the field F and a binary function symbol *lambda*, which should be interpreted as multiplication by scalars, $\lambda : F \times V \to V$. $\diamond$

Definition 3.3 *Let $\mathcal{L} = \{(R_i)_{i\in I}; (f_j)_{j\in J}; (c_k)_{k\in K}\}$ be a language and $\mathcal{M}$ and $\mathcal{N}$ two $\mathcal{L}$-structures.*

1. *A map h from M to N is a* **homomorphism** *from $\mathcal{M}$ to $\mathcal{N}$ if the following holds*

   - *for every constant $c \in \mathcal{L}$, $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$*
   - *for every n-ary function $f \in \mathcal{L}$, for every $\bar{m} \in M^n$, $h(f^{\mathcal{M}}(\bar{m})) = f^{\mathcal{N}}(h(\bar{m}))$*
   - *for every n-ary relation $R \in \mathcal{L}$, for every $\bar{m} \in M^n$, if $\bar{m} \in R^{\mathcal{M}}$, then $h(\bar{m}) \in R^{\mathcal{N}}$.*

2. *A homomorphism h from $\mathcal{M}$ to $\mathcal{N}$ is an* **embedding** *if, for every n-ary relation $R \in \mathcal{L}$, for every $\bar{m} \in M^n$, $\bar{m} \in R^{\mathcal{M}}$ if and only if $h(\bar{m}) \in R^{\mathcal{N}}$. Note that, as the diagonal is always in $\mathcal{L}$, an embedding is injective.*

3. *An* **isomorphism** *from $\mathcal{M}$ to $\mathcal{N}$ is a surjective embedding. An* **automorphism** *is an isomorphism from $\mathcal{M}$ onto itself.*

4. *$\mathcal{M}$ is an $\mathcal{L}$-**substructure** of $\mathcal{N}$, denoted $\mathcal{M} \subset_{\mathcal{L}} \mathcal{N}$, if $M \subset N$ and the inclusion map from M into N is an $\mathcal{L}$ embedding.*
   It is equivalent to the following conditions:
   - *for every constant $c \in \mathcal{L}$, $c^{\mathcal{M}} = c^{\mathcal{N}}$*
   - *for every n-ary function $f \in \mathcal{L}$, for every $\bar{m} \in M^n$, $f^{\mathcal{M}}(\bar{m}) = f^{\mathcal{N}}(\bar{m}) \in M$*
   - *for every n-ary relation $R \in \mathcal{L}$, $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^n$.*

Now, we would like to define, $Form(\mathcal{L})$, the set of formulas in a given language $\mathcal{L}$. We will do this inductively. So, let $\mathcal{L}$ be a language and let $v, w, x, y, x_i, \ldots$ denote an infinite set of variables.

Definition 3.4 *The definition of a **term** in the language $\mathcal{L}$:*

1. *every variable is a term*
2. *every constant is a term*
3. *if $f \in \mathcal{L}$ is any n-ary function, if $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.*

*An **atomic formula** of $\mathcal{L}$ is an expression of the form: $R(t_1, \ldots, t_n)$ where R is an n-ary relation and $t_1, \ldots, t_n$ are n terms in $\mathcal{L}$.*
Note that as equality is always in $\mathcal{L}$, for any two terms s and t, $''s = t''$ is an atomic formula.
*Definition of a **formula** in language $\mathcal{L}$:*

1. *all atomic formulas are formulas*
2. *if $\phi_1(x_1, \ldots, x_n)$ and $\phi_2(x_1, \ldots, x_n)$ are formulas, then $(\phi_1 \wedge \phi_2)(x_1, \ldots, x_n)$ is a formula*
3. *if $\phi_1(x_1, \ldots, x_n)$ is a formula, then $\neg\phi_1(x_1, \ldots, x_n)$ is a formula*
4. *if $\phi_1(x_1, \ldots, x_n)$ is a formula, then $\exists x_n \phi_1(x_1, \ldots, x_n)$ is a formula*
5. *similar clauses for $\vee$ and $\forall$.*

The symbols $\vee, \wedge, \neg, \exists, \forall$ stand for the standard logical connectives and quantifiers.
If $\phi \in Form(\mathcal{L})$ we write $\phi(x_1, \ldots, x_n)$ to denote that the set of all free variables (i.e. the ones which are not under the "effect" of any quantifier in ϕ) are among the $x_1, \ldots, x_n$.
The cardinality of $Form(\mathcal{L})$ is equal to the maximum of the cardinality of the language $\mathcal{L}$ and $\aleph_0$.

Example In the language $\mathcal{L}_{ring}$ any polynomial $P(x_1, \ldots, x_n)$ in $\mathbb{Z}[x_1, \ldots, x_n]$ is a term. If $P, Q \in \mathbb{Z}[x_1, \ldots, x_n]$ then $P(x_1, \ldots, x_n) = Q(x_1, \ldots, x_n)$ is an example of an atomic formula, and $\exists x_1(P(x_1, \ldots, x_n) = 0 \wedge \neg(Q(x_1, \ldots, x_n) = 0))$ is an example of a formula. $\diamond$

Definition 3.5 *A* **sentence** *is a formula with no free variables.*

Example Let P and Q be as above. Then $\exists x_1 \ldots \exists x_n (P(x_1, \ldots, x_n) = 0 \wedge \neg(Q(x_1, \ldots, x_n) = 0))$ is an example of a sentence in $\mathcal{L}_{ring}$. $\diamond$

If ϕ is a sentence in the language $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-structure, ϕ is either true or false in $\mathcal{M}$. We denote this respectively by $\mathcal{M} \models \phi$ and $\mathcal{M} \not\models \phi$. But suppose now that the formula $\phi(x_1, \ldots, x_n)$ has fee variables. Let $\bar{m} \in M^n$ be an n-tuple $\bar{m} = (m_1, \ldots, m_n)$, we say that $\bar{m}$ **satisfies** $\phi(x_1, \ldots, x_n)$ in $\mathcal{M}$, written as $\mathcal{M} \models \phi(m_1, \ldots, m_n)$, if $\phi(m_1, \ldots, m_n)$ is true in $\mathcal{M}$.

Remark: All this can be generalized to many sorted structures. A many sorted language is a language which contains *sorts*, *relation symbols, function symbols, constant symbols*, and for each sort a supply of *variables of that sort.* To each n-ary relation symbol R we associate an n-tuple $(S_1, \ldots, S_n)$ of sorts, and to each n-ary function symbol f we associate an n-tuple of sorts for the domain and one more sort for the target. Well-formed formulas are built up as usual, except we require that variables in the formulas be of the right sort. For example for the n-ary relation R with the associated n-tuple $(S_1, \ldots, S_n)$ of sorts we require that in $R(x_1, \ldots, x_n)$ the variable x_i be of the sort S_i or $\exists x_i \ldots$ means that there exists x_i in a sort S_i $\ldots$. A structure $\mathcal{M}$ for such a language will consist of a collection of disjoint domains $S^{\mathcal{M}}$, corresponding to the various sorts S (so $S^{\mathcal{M}}$ will be the interpretation for S) and relations, functions and constants all interpreted as subsets of cartesian products of interpretations of appropriate sorts.
All of model theory then goes through for this situation. $\triangleright$

So far we have defined structures, languages and formulas. Now we give the definition of definable sets.

Definition 3.6 *Let $\mathcal{M} = \langle M, (R_i^{\mathcal{M}})_{i \in I} \rangle$ be a relational structure. We define the family $Def^0(\mathcal{M})$, the family of* **zero-definable** *subsets of the structure $\mathcal{M}$ (i.e. definable over the $\emptyset$ set), to be the smallest family of subsets of M^n for various n which contains $\cup_{i \in I} \{R_i^{\mathcal{M}}\}$ and is closed under finite intersections, finite unions, complementation, cartesian products and (coordinate) projections. The sets $R_i^{\mathcal{M}}$ are called* **atomic sets***.*
The family $Def(\mathcal{M})$ of **definable** *sets in $\mathcal{M}$ (we say definable with parameters) is obtained from the family $Def^0(\mathcal{M})$ by further closing under taking fibres (i.e. if $A \in Def(\mathcal{M})$, $A \subset M^{n+k}$ and if $\bar{m} \in M^n$ then $A(\bar{m}) = \{\bar{x} \in M^k : (\bar{m}, \bar{x}) \in A\} \in Def(\mathcal{M})$).*

Example Let $\mathcal{K}$ be an arbitrary algebraically closed field considered as a relational structure with the atomic sets taken to be the solutions of single polynomial equations with coefficients from K. Closing under finite intersections we get the Zariski closed subsets of K^n, for all $n \geq 1$. If we close under finite boolean combinations we get the **constructible sets**. *Chevalley's* theorem states that a projection of a constructible set is constructible. Hence in $\mathcal{K}$constructible sets are exactly the definable sets.

In general for an arbitrary field which is not algebraically closed there will be definable set which are not constructible $\diamond$

There is a natural correspondence between definable sets of an $\mathcal{L}$-structure $\mathcal{M}$ and formulas in the language $\mathcal{L}$. It is based on the natural interpretation of the boolean symbols $\wedge$ as intersection, $\vee$ as union, $\neg$ as complementation and $\exists$ as projection.

So we have the following definition.

Definition 3.7 *We say that $D \subset M^n$ is a* **definable subset** *in the $\mathcal{L}$-structure $\mathcal{M}$ if we can find $b_1, \dots, b_m \in M$ and a formula $\phi(x_1, \dots, x_n, x_{n+1} \dots, x_m)$ such that*

$$D = \{(a_1, \dots, a_n) \in M^n : \mathcal{M} \models \phi(a_1, \dots, a_n, b_1, \dots, b_m)\}.$$

*We say that D is B-***definable** *or definable over B, where $B \subset M$ if we can choose $b_1, \dots, b_m \in B$.*
Note that the set of formulas with parameters in B, $\{\phi(\bar{x}, \bar{b}) : \bar{b} \subset B, \phi(\bar{x}, \bar{y}) \in Form(\mathcal{L})\}$ is exactly the set of formulas of the language $\mathcal{L}(B) = \mathcal{L} \cup \{c_b : b \in B\}$, where one adds to the original language $\mathcal{L}$ a new constant for each element of B.

Example Consider the relational structure $\mathcal{K}$ of an algebraically closed field K as in the previous example. It is an $\mathcal{L}_{ring}$-structure $\mathcal{K}$. The definable sets in the language $\mathcal{L}_{ring}$ according to **Definition 3.6** are exactly the definable set according to the **Definition 3.7**. In this context Chevalley's theorem states that any formula is equivalent to a formula without quantifiers i.e. it is equivalent to a boolean combination of atomic formulas. This is known as *quantifier elimination* property (more about this later). $\diamond$

Example Let K be a differential field with a derivation δ. Then $C_K = \{x \in K : \delta(x) = 0\}$, the constant field of K is a δ- definable subset of K in the language $\mathcal{L}_{diff}$. It is sometimes useful to introduce the following convention: we will say that the set Y is δ-definable if it is definable in the language $\mathcal{L}_{diff}$ and reserve the word definable if it is definable in $\mathcal{L}_{ring}$. $\diamond$

The definable sets in the above sense are exactly the definable sets of the relational structure $\langle \mathcal{M}, (R_i^{\mathcal{M}})_{i \in I} \rangle$.

Definition 3.8 *In a structure $\mathcal{M}$ for $A \subset M$ by $dcl(A)$, the* **definable closure** *of A, we mean the set of elements (considered as singleton sets) which are definable over A.*

Remark: The set $dcl(A)$ is the set of elements which are fixed under all automorphisms of $\mathcal{M}$ which fixes A pointwise.
In general a definable set X in $\mathcal{M}$ is definable over A if X is invariant under all automorphisms of $\mathcal{M}$ which fixes A pointwise. $\triangleright$

Definition 3.9 *Let $\mathcal{M}$ be an $\mathcal{L}$ structure and $A \subset M$. An element $b \in M$ contained in a finite A-definable set is said to be algebraic over A. We call $acl(A)$, the set of elements algebraic over A, the algebraic closure of A.*

Example Let K be a field generated by the set A in the $\mathcal{L}_{ring}$-structure $\mathcal{M}$. Then $acl(A)$ is the usual field theoretic algebraic closure of K. $\diamond$

Remark: There is another way to define $acl(\_)$. The set $acl(A)$ in $\mathcal{M}$ is the set of elements which have only finitely many images under automorphisms of $\mathcal{M}$ which fixes A pointwise. In general a definable set X in $\mathcal{M}$ is definable over A if X is invariant under all automorphisms of $\mathcal{M}$ which fixes A pointwise. $\triangleright$

Definition 3.10 *In a structure $\mathcal{M}$ an* **infinitely-definable** *subset of M^n over A is an infinite intersection of A-definable subsets of M^n.*

Example In a separably closed field of characteristic $p > 0$, the field $K^{p^\infty} = \cap_{n\in\mathbb{N}} K^{p^n}$ is not definable but infinitely definable. $\diamond$

Definition 3.11 *By a* **definable group** *in an $\mathcal{L}$- structure $\mathcal{M}$ we mean a group G such that both the underlying set of G and the group operation are definable sets.*

Fact 3.1 *Let G be an algebraic group. If H is a definable subgroup of G then H is closed in G.*

By the above fact we can restate **Definition 2.13** for a definable group in an $\mathcal{L}$-structure as follows.

Definition 3.12 *A group G is* **connected** *if it has no proper definable subgroups of finite index.*

In order to compare structures we need to define the concept of a theory and of an elementary equivalence.

Definition 3.13 *A set Σ of sentences in $\mathcal{L}$ is* **consistent** *(or satisfiable) if there is an $\mathcal{L}$-structure $\mathcal{M}$ such that for every sentence $\sigma \in \Sigma$, $\mathcal{M} \models \sigma$. In this case we say that $\mathcal{M}$ is a* **model** *of Σ, denoted $\mathcal{M} \models \Sigma$.*
A **theory** *in $\mathcal{L}$ is a consistent set of sentences in $\mathcal{L}$. A theory $\mathcal{T}$ in $\mathcal{L}$ is said to be* **complete** *if it is maximal, that is for any sentence σ in $\mathcal{L}$, $\sigma \in \mathcal{T}$ or $\neg\sigma \in \mathcal{T}$ (of course at most one of σ and $\neg\sigma$ can be a member of a consistent set of sentences).*

Example Consider the theory $\mathcal{T}$ in $\mathcal{L}_{diff}$ of all differentially closed fields of characteristic zero, that is the theory consisting of all sentences which are true in all differentially closed fields of characteristic zero. It is an example of a complete theory. In particular this implies that all differentially closed fields of characteristic zero satisfy exactly the same sentences. $\diamond$

Example Consider a single $\mathcal{L}$-structure $\mathcal{M}$. Then the associated theory $\mathcal{T}(\mathcal{M}) = \{\sigma \in Form(\mathcal{L}) : \sigma \text{ is a sentence}, \mathcal{M} \models \sigma\}$ is a complete theory. $\diamond$

Notation :

- Let $\mathbf{ACF}_p$ stand for the theory of algebraically closed fields of characteristic p in the language $\mathcal{L}_{ring}$ i.e. the theory consisting of all sentences which are true in all algebraically closed fields of characteristic p.

- Let **DCF** stand for the theory of differentially closed fields of characteristic zero in the language $\mathcal{L}_{diff}$ i.e. the theory consisting of all sentences which are true in all differentially closed fields of characteristic zero.
- Let $\mathbf{SCF}_{p,\nu}$ stand for the theory of separably closed fields of characteristic p and of finite degree of imperfection ν in the language $\mathcal{L}_{p,\nu}$ i.e. the theory consisting of all sentences which are true in all separably closed fields of characteristic p and fixed degree of imperfection ν.

Fact 3.2 *The above three theories are complete.*

The above fact implies that all algebraically closed fields of a fixed characteristic p satisfy exactly the same sentences. The same conclusion applies for differentially closed fields of characteristic zero and for separably closed fields of fixed characteristic p.

Definition 3.14 *Given two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ we say that $\mathcal{M}$ and $\mathcal{N}$ are* **elementary equivalent**, *denoted $\mathcal{M} \equiv \mathcal{N}$, if for all sentences $\sigma \in Form(\mathcal{L})$, $\mathcal{M} \models \sigma$ if and only if $\mathcal{N} \models \sigma$. That is $\mathcal{T}(\mathcal{M}) = \mathcal{T}(\mathcal{N})$.*

Definition 3.15 *Let $\mathcal{M} \subset_{\mathcal{L}} \mathcal{N}$ i.e. $\mathcal{M}$ is an $\mathcal{L}$-substructure of $\mathcal{N}$. We say that $\mathcal{M}$ is an* **elementary substructure** *of $\mathcal{N}$ (or that $\mathcal{N}$ is an elementary extension of $\mathcal{N}$), denoted $\mathcal{M} \preceq \mathcal{N}$, if for all $\phi \in Form(\mathcal{L})$, for all $\bar{a} = (a_1, \ldots, a_n) \in M^n$, $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(a_1, \ldots, a_n)$.*

Note that $\mathcal{M} \preceq \mathcal{N}$ implies that they are elementary equivalent. If $\mathcal{M} \preceq \mathcal{N}$, then any definable subset D of M^n (defined with parameters from M) has a canonical extension to a definable set D' in N^n (defined with parameters from M), such that $D' \cap M^n = D$. If $D = \{\bar{x} \in M^n : \mathcal{M} \models \phi(\bar{x}, \bar{m}), \bar{m} \in M^m\}$ then $D' = \{\bar{x} \in N^n : \mathcal{N} \models \phi(\bar{x}, \bar{m}), \bar{m} \in M^m\}$.

Fact 3.3 *(Tarski-Vaught test) Let $\mathcal{M} \subset_{\mathcal{L}} \mathcal{N}$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if for all $\mathcal{L}$-formula $\phi(x, y_1, \ldots, y_n)$, for all $m_1, \ldots, m_n \in M$, such that $\mathcal{N} \models \exists x\, \phi(x, m_1, \ldots, m_n)$, there is some $m_0 \in M$ such that $\mathcal{N} \models \phi(m_0, m_1, \ldots, m_n)$.*

Proposition 3.1 *(Corollary to Tarski-Vaught test) Let $\mathcal{M} \subset_{\mathcal{L}} \mathcal{N}$. Then $\mathcal{M} \preceq \mathcal{N}$ if and only if for any nonempty definable subset E of N^n, defined with parameters from M, $E \cap M^n$ is nonempty.*

Definition 3.16 *A map f from $\mathcal{M}$ to $\mathcal{N}$ is an* **elementary embedding** *from $\mathcal{M}$ into $\mathcal{N}$ if for all formula $\phi(x_1, \ldots, x_n) \in Form(\mathcal{L})$, for all $\bar{a} \in M^n$, $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(f(a_1), \ldots, f(a_n))$.*

This means exactly that $\mathcal{M}$ is isomorphic to an elementary substructure of $\mathcal{N}$. Not all embeddings are elementary. But we have the following simple fact.

Fact 3.4 *Any isomorphism is an elementary embedding.*

Definition 3.17 *We say that the theory $\mathcal{T}$ in the language $\mathcal{L}$ has* **quantifier elimination** *if any formula in $\mathcal{L}$ is equivalent to a formula in $\mathcal{L}$ without quantifiers.*

Fact 3.5 *The following theories in the appropriate languages admit quantifier elimination.* $\mathbf{ACF}_p$ *in the language* $\mathcal{L}_{ring}$*,* **DCF** *in the language* $\mathcal{L}_{diff}$ *and* $\mathbf{SCF}_{p,\nu}$ *in the language* $\mathcal{L}_{p,\nu}$*.*

Quantifier elimination allows us to avoid technical difficulties in proving statements about formulas but more importantly if a theory in a language $\mathcal{L}$ admits quantifier elimination then every embedding of $\mathcal{L}$-structures is an elementary embedding. In particular then the inclusion of an $\mathcal{L}$-structure $\mathcal{M}$ into a larger structure $\mathcal{N}$ is an elementary extension. If this property holds i.e. every inclusion is an elementary extension we say that the theory is **Model complete**. Hence we see that a theory with quantifier elimination is model complete. Also, understanding definable sets in a structure is a difficult task. If the theory has quantifier elimination this task is made easier by reducing all definable sets to sets definable with formulas without quantifiers.

Definition 3.18 *Let $\mathcal{F} = \{D_i : i \in I\}$ be an infinite family of sets (in some "universe"). We say that the family $\mathcal{F}$ has the* **finite intersection property** *if for any finite $J \subset I$ the set $\cap_{i \in J} D_i$ is non empty.*

Definition 3.19 *Let κ be an infinite cardinal. The $\mathcal{L}$-structure $\mathcal{M}$ is κ-***saturated** *if for any family $\mathcal{F} = \{D_i : i < \kappa\}$ of definable subsets of M^n with the finite intersection property, there is some $\bar{a} \in M^n$, $\bar{a} \in \cap_{i<\kappa} D_i$.*
We say $\mathcal{M}$ is **saturated** *if $\mathcal{M}$ is $\mid M \mid$-saturated.*

The two most important cases that we are going to need is when κ is either $\aleph_0$ or $\aleph_1$.

Example The field of complex numbers $\mathbb{C}$ as an $\mathcal{L}_{ring}$ structure is $\aleph_1$ saturated. $\diamond$

We list now several classical theorems of model theory about existence of models and elementary extensions.

Fact 3.6 *(Compactness theorem)*

1. *(Version I) Let Σ be a set of sentences in $\mathcal{L}$. Then Σ is consistent (i.e. has a model) if and only if every finite subset of Σ is consistent (has a model).*

2. *(Version II) Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\Sigma(x_1, \ldots, x_n) = \{\phi_i(x_1, \ldots, x_n) : i \in I\}$ be a set of formulas in $\mathcal{L}$ with n variables (allowing parameters from $\mathcal{M}$). Suppose that for each finite subset $\{i_1, \ldots, i_k\}$ of I,*

$$\mathcal{M} \models \exists x_1 \ldots \exists x_n (\phi_{i_1}(x_1, \ldots, x_n) \wedge \cdots \wedge \phi_{i_k}(x_1, \ldots, x_n)).$$

Then there is some $\mathcal{N} \succeq \mathcal{M}$ and some $\bar{a} \in N^n$ such that for every $i \in I$ $\mathcal{N} \models \phi_i(\bar{a})$.

In terms of definable sets, this means that if we have an infinite family $\mathcal{F} = \{D_i : i \in I\}$ of definable subsets of M^n which has the finite intersection property, then in some elementary extension $\mathcal{N}$ of $\mathcal{M}$, the family $\mathcal{F}$ has a non empty intersection ($\cap_{i \in I} D_i \neq \emptyset$).

The following two theorems are consequences of the compactness theorem. The first one says in particular that a theory in a countable language with an infinite model has a model of any cardinality.

Fact 3.7 *(Löwenheim-Skolem theorem)*

1. *Let $\mathcal{L}$ be a language, $\mathcal{M}$ an $\mathcal{L}$-structure and $X \subset M$. Then there exists $\mathcal{M}_0 \preceq \mathcal{M}$ such that $X \subset M_0$ and $\mid M_0 \mid \leq \mid X \mid + \mid \mathcal{L} \mid$.*

2. *Let $\mathcal{L}$ be a language, let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Then for any cardinal $\kappa > \mid M \mid$, $\mathcal{M}$ has an elementary extension of cardinality κ.*

The second fact is about the existence of saturated models.

Fact 3.8

1. *Let κ be an infinite cardinal, let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then there is a κ-saturated $\mathcal{L}$-structure $\mathcal{N}$ such that $\mathcal{M} \preceq \mathcal{N}$.*

2. *Let κ be an infinite cardinal, let $\mathcal{M}$ be a κ-saturated $\mathcal{L}$-structure. If $\mathcal{N}$ is an $\mathcal{L}$-structure of cardinality less then κ and $\mathcal{N} \equiv \mathcal{M}$, there is an elementary embedding from $\mathcal{N}$ into $\mathcal{M}$.*

These existence theorems are very general in nature. Later on we will be working only with the theories **DCF** and $\mathbf{SCF}_{p,\nu}$ which are known to be *stable theories* (see **Definition 3.24**) so for us it is enough to point out the following fact.

Fact 3.9 *Stable theories have arbitrary large saturated models.*

But in general it is important that the above theorems guarantee that one can assume the existence of a saturated *monster* model of big enough cardinality such that all substructures of interest are of strictly smaller cardinality then the monster model.

In addition saturated models have the property of *homogeneity*. We say that an $\mathcal{L}$-structure $\mathcal{M}$ of cardinality κ is **homogeneous** if for any $A, B \subset M$, of cardinality less then κ, and any partial elementary embedding f from A onto B, f extends to an automorphism of $\mathcal{M}$.

In the rest of this section we will make the following assumptions: $\mathcal{T}$ is a complete theory in a language $\mathcal{L}$ and $\mathcal{C}$ is a saturated monster model for $\mathcal{T}$. The cardinality of the language $\mathcal{L}$ is strictly smaller then $|C|$, the cardinality of C. All models $\mathcal{M}$ under consideration will be elementary substructures of $\mathcal{C}$ of cardinality strictly less than $|C|$ and all subsets (of parameters) $A, B, \ldots$ will be subsets of C of cardinality less than $|C|$.

We will need the following important notions.

Definition 3.20 *Let $A \subset C$ and let $\Sigma(x_1, \ldots, x_n) = \{\phi_i(x_1, \ldots, x_n), i \in I\}$ be a set of $\mathcal{L}$-formulas with n variables and parameters from A (i.e. it is a set of $\mathcal{L}(A)$ formulas in n variables). We say that $\Sigma(x_1, \ldots, x_n)$ is* **finitely satisfiable** *if for any finite subset F of $\Sigma(x_1, \ldots, x_n)$, there is some $(c_1 \ldots, c_n) \in C^n$ such that $\mathcal{C} \models \phi(c_1, \ldots, c_n)$ for every $\phi(x_1, \ldots, x_n) \in F$.*

This is the same as taking a family of A definable subsets of C^n with the finite intersection property.

Definition 3.21 *A* **complete n-type** *over A is a finitely satisfiable set $\Sigma(x_1, \ldots, x_n)$ of formulas in n variables with parameters from A which is maximal, i.e. such that for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}(A)$, $\phi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$ or $\neg\phi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$.*

By our assumptions $\mathcal{C}$ is saturated and $|A|, |\mathcal{L}| < |C|$ hence there exist $(c_1, \ldots, c_n) \in C^n$ such that $\mathcal{C} \models \phi(c_1, \ldots, c_n)$ for every $\phi(x_1, \ldots, x_n) \in \Sigma(x_1, \ldots, x_n)$. We say that $(c_1, \ldots, c_n)$ **realizes** the complete n-type Σ over A and write $\mathcal{C} \models \Sigma(c_1, \ldots, c_n)$.

A κ-saturated model realizes every type over every subset A where $|A| < \kappa$.

In a model $\mathcal{M}$ for an n-type q over $A \subset M$ by $q(M)$ we mean the set $\{\bar{m} \in M^n : \bar{m} \text{ realizes } q\}$.

Definition 3.22 *Conversely, if we consider any $(c_1, \ldots, c_n) \in C^n$, then let the* **type of** $(c_1, \ldots, c_n)$ **over** *A, be the following set of formulas: $t(c_1, \ldots, c_n/A) = \{\phi(x_1, \ldots, x_n) \in \mathcal{L}(A) : \mathcal{C} \models \phi(c_1, \ldots, c_n)\}$. Then $t(c_1, \ldots, c_n/A)$ is a complete n-type over A.*

This corresponds to considering all A-definable subsets of C^n containing $(c_1, \ldots, c_n)$.

Remark: If f is an automorphism of $\mathcal{C}$ fixing the set A pointwise, in notation $f \in Aut_A(C)$, then f must fix setwise all A-definable sets and hence must fix setwise all complete types over A. By homogeneity of $\mathcal{C}$, the converse is true too. Hence we have: let $\bar{c}, \bar{d} \in C^n$, $t(\bar{c}/A) = t(\bar{d}/A)$ if and only if there is some $f \in Aut_A(C)$ such that $f(\bar{c}) = \bar{d}$. That is in a saturated model complete n-types over A correspond exactly to orbits of n-tuples under the action of $Aut_A(C)$. ▹

For a fixed $n \in \mathbb{N}$ let Def_A^n denote the collection of all A-definable subsets of C^n. Def_A^n is a boolean algebra for the usual operations of intersection, union and complementation. Since each definable set can be identified with an equivalence class of formulas used to define it a complete n type over A can be seen as an *ultrafilter* (see [2], [1]) in this boolean algebra.

Definition 3.23 *Let $A \subset C$. Let $S_n(A)$ denote the space of complete n-types over A with the topology where a basis of open sets is given by $\langle\phi(\bar{x}, \bar{a})\rangle = \{q \in S_n(A) : \phi(\bar{x}, \bar{a}) \in q\}$ for all formulas $\phi(\bar{x}, \bar{y})$ in $\mathcal{L}$ and all finite $\bar{a} \subset A$. This space is the "Stone space" (see [2], [1]) associated to the boolean algebra Def_A^n.*

The structure of the spaces $S_n(A)$ (e.g. its topology) contain valuable information about the properties of the models of a given theory (e.g. about the richness and interaction of definable sets in a model). In particular the cardinalities of these spaces will be important for us to single out theories which have a well behaved structure of definable sets. It is interesting to remark that the cosequence of Compactness theorem, **Fact 3.6**, is that the spaces $S_n(A)$ are compact. Also note that $\langle\phi(\bar{x},\bar{a})\rangle \neq \emptyset$ if and only if $\mathcal{C} \models \exists\bar{x}\phi(\bar{x},\bar{a})$.

Definition 3.24 *We say that the theory $\mathcal{T}$ is* **stable** *if for any model $\mathcal{M}$ of $\mathcal{T}$ the set of complete types over M, $S(M) = \cup_{n\in\mathbb{N}} S_n(M)$, has cardinality at most $|M|^{|T|}$.*

Fact 3.10 *Stable theories have saturated models of arbitrary large cardinality.*

Definition 3.25 *A theory $\mathcal{T}$ is ω-***stable** *if $S_1(A)$ is countable for every countable A (in every model $\mathcal{M}$ of $\mathcal{T}$).*

Definition 3.26 *An $\mathcal{L}$-structure $\mathcal{M}$ is* **stable** *if the theory $\mathcal{T}(\mathcal{M})$ is stable.*

Fact 3.11 *An ω-stable theory is stable.*

Example DCF is an ω-stable theory. $\mathbf{SCF}_{p,\nu}$ is a stable but not an ω stable theory. $\diamond$

Definition 3.27 *We say that a group G with possible extra structure is ω-***stable** *if the theory of G with the extra structure in the appropriate language $\mathcal{L}$ is ω-stable.*

It is also possible to weaken this definition and replace it by the following.

Definition 3.28 *(version II of an ω-stable group) We say that a definable group G in the structure $\mathcal{M}$ is ω-***stable** *if the structure $\mathcal{M}$ is ω-stable.*

Fact 3.12 *In an ω-stable group there is no infinite decreasing chain of definable subgroups.*

From the above it follows that in an ω-stable group every infinitely-definable subgroup is definable. Moreover we have the following.

Fact 3.13 *In an ω-stable theory a group which is infinitely-definable is definable.*

Fact 3.14 *In an ω-stable theory every definable subset of a set D definable over B is definable with parameters from $B \cup D$.*

For definable sets and complete types in $\mathcal{C}$ we can define generalized concepts of dimension. In some cases these dimensions coincide with the geometric dimensions of geometric objects (e.g dimension of a variety). These notions were originally introduced to help counting and classifying models of first order theories. But it will turn out the the following definition of a dimension, called Morley rank, will be very useful for us.

We want to assign to each definable set an ordinal number (or -1 and ∞) as follows.

Definition 3.29 *Let α be an ordinal and X be a definable set in $\mathcal{C}$. We define the* **Morley rank** *of X, $Mr(X)$ recursively as follows:*
$Mr(X) \geq$ if X is not empty
$Mr(X) \geq \alpha$ if $Mr(X) \geq \lambda$ for all $\lambda < \alpha$, (α is a limit ordinal)
$Mr(X) \geq \alpha + 1$ if there is a an infinite family $\{X_i\}$ of pairwise disjoint definable subsets of X , such that $Mr(X_i) \geq \alpha$ for all i. We put $Mr(X) = sup\{\alpha : Mr(X) \geq \alpha\}$ with the convention that $Mr(\emptyset) = -1$ and $Mr(X) = \infty$ if $Mr(X) \geq \alpha$ for all α (here we say that X has no rank, otherwise we say that it is ranked). If $Mr(X) = \alpha$, then there will be a greatest number d such that X can be partitioned into d definable subsets, each of Morley rank α. d is called the **Morley degree** *of X, $Md(X)$.*

Special cases: X has Morley rank 0 if it is finite and non empty, X has Morley rank 1 if it is infinite but does not contain an infinite family of disjoint infinite definable subsets.
If $Mr(X) = 0$, then $Md(X) = |X|$. If X has no rank, then the Morley degree is not defined.

Example If our model $\mathcal{C}$ is an algebraically closed field and X a (possibly reducible) variety, then $Mr(X)$ is equal to the *algebraic-geometric dimension* of the constructible set X and $Md(X)$ is equal to the number of irreducible components of X of maximal dimension. $\diamond$

Fact 3.15 *In an ω-stable theory every definable set has Morley rank.*

As we mentioned earlier there is a correlation between definable sets and formulas. Hence we can state the following definitions.

Definition 3.30 *(Morley rank of a formula) Let ϕ be a formula in the language $\mathcal{L}$. We define the* **Morley rank** *and* **Morley degree of the formula** *ϕ to be the Morley rank and Morley degree of the definable set D of $\mathcal{C}$ defined by the formula ϕ.*

Definition 3.31 *(***Morley rank and degree of a complete type***) Let q be a (complete n-)type over A. If q does not contain a ranked formula (a formula with a defined Morley rank) the Morley rank, $Mr(q)$, of q is ∞ by definition. Otherwise choose a formula ϕ which has minimal rank α and, among the formulas of rank α, has minimal Morley degree d. We define $Mr(q)$ to be α and the Morley degree, $Md(q)$ to be d.*

Definition 3.32 *Let A be a subset of B and p a type over A. A* **nonforking extension** *q of p is a type over B which extends p and has the same rank as p.*

Definition 3.33 *A type $q \in S(A)$ is* **stationary** *if it has exactly one nonforking extension to every set which contains A.*

Definition 3.34 *Let X be a ranked A-definable set in $\mathcal{C}$. Let $a \in X$ and $A \subset B$. We define the* **Morley rank** *of a* **over** *B, $Mr(a/B)$ to be the least α such that there is a B-definable set Y of Morley rank α with $a \in Y$. If $B \subset F$, we say that a is* **independent** *from F over B if $Mr(a/F) = Mr(a/B)$.*

An equivalent definition of independence is given by the following definition.

Definition 3.35 *We say that a and B are independent over A if $t(a/AB)$ does not fork over A.*

Definition 3.36 *Let X and Y be two A-definable sets. We will say that X and Y are* **orthogonal** *if for all $B \supset A$, for all $a \in X$ and for all $b \in Y$, a is independent from b over B i.e. if $Mr(a/B) = Mr(a/bB)$ where bB is a short hand for $\{b\} \cup B$.*
In other words X and Y are orthogonal if any two elements $a \in X$ and $b \in Y$ are independent over any set of parameters over which X and Y can be defined. The notation is $X \perp Y$.
Another way to characterize orthogonality is to say that X and Y are orthogonal if and only if any definable set $Z \subset X \times Y$ is a finite boolean combination of sets of the form $X_i \times Y_i$, where X_i and Y_i are definable subsets of X and Y respectively.

Another important notion is a notion of a *strongly minimal set.* Understanding strongly minimal sets amounts to understanding sets of finite Morley rank.

Definition 3.37 *A definable set is* **strongly minimal** *if it has Morley rank and degree equal to one.*

Fact 3.16 *A definable set X is strongly minimal if and only if for any definable subset Y of X, either Y is finite or $X \backslash Y$ is finite.*

Definition 3.38 *We say that a definable set Y is almost strongly minimal (with respect to X) if it is contained in the algebraic closure of a strongly minimal set X.*

Fact 3.17 *A strongly minimal set X is nonorthogonal to the definable set Y if and only if $X \subset acl(A \cup Y)$ for some finite parameter set A.*

Fact 3.18 Nonorthogonality *is an equivalence relation for strongly minimal sets.*

Definition 3.39 *Let X be a definable set (in $\mathcal{C}$ over $A \subset C$) of finite Morley rank. We will say that X is* **modular** *if for any finite tuple $\bar{a}$ of elements in X and for any $B \supset A$, $\bar{a}$ is independent from $acl(B)$ over $acl(A \cup \bar{a}) \cap acl(B)$.*
We say that the group G is modular if it is modular as a definable set.

In the case when X is a strongly minimal set, modularity has a more geometric description: There should be no definable family, of rank $n \geq 2$, of strongly minimal subsets of $X \times X$ with finite pairwise intersections.

Proposition 3.2 *Let G be a modular group. Then any definable subset of G^n is a finite boolean combination of cosets of definable (connected) subgroups of G^n.*

Note that if the group G is an abelian modular group, then according to **Fact 2.7** we can replace in the above **Proposition 3.2** the phrase "finite boolean combination" with "finite union".

Fact 3.19 *The sum of modular groups is a modular group.*

Definition 3.40 *Let X be a topological space. We say that the topology on X is* **Noetherian** *if there is no infinite descending sequence of closed sets.*

Definition 3.41 *Let X be a topological space with Noetherian topology and D a closed subset of X. We say that D is* **irreducible** *if whenever D_0 and D_1 are closed in X and $D = D_0 \cup D_1$, then $D = D_0$ or $D = D_1$.*

Fact 3.20 *Let X be a topological space with Noetherian topology and D a closed subset of X. There are irreducible closed set $D_1, \ldots, D_n$ such that $D = D_1 \cup \cdots \cup D_n$. Moreover, if we choose $D_1, \ldots, D_n$ such that $D_i \not\subset D_j$ for $i \neq j$, then $D_1, \ldots, D_n$ are unique up to permutation.*

Definition 3.42 *We call the sets $D_1, \ldots, D_n$, from the above statement,* **irreducible components** *of D.*

Definition 3.43 *Let $\mathcal{M}$ be a structure and X a definable set in $\mathcal{M}$ of Morley rank and Morley degree 1. X will be called a* **Zariski geometry** *if for each $n \in \mathbb{N}$, X^n is equipped with a Noetherian topology, satisfying the following conditions:*

1. *every closed set in X^n is definable in $\mathcal{M}$ and any subset of X^n definable in $\mathcal{M}$ is a finite boolean combination of closed sets.*
2. *the diagonal is a closed subset of X^2, and if Y is a closed subset of X^{n+m}, then for each $a \in X^n$ the fibre $Y_a = \{b \in X^m : (a, b) \in X^{n+m}\}$ is a closed subset of X^m.*
3. *if Y and Z are irreducible closed subsets of X^n, then every irreducible component of $Y \cap Z$ has Morley rank at least $Mr(Y) + Mr(Z) - n$.*

Proposition 3.3 *Let K be a model for* **DCF**. *Let $Y \subset K^n$ be strongly minimal. There is $F \subset Y$ finite such that $D \backslash F$ is a Zariski geometry (with a Noetherian topology generated by the set of solutions of differential polynomial equations over K). Reference [29].*

Zariski geometries play an important role in Diophantine problems in model theory because of the consequences of the celebrated Zilber (Zilber-Hrushovski) conjecture.

Proposition 3.4 *Let $\mathcal{M}$ be a structure, and let X be a definable set in $\mathcal{M}$ which is a Zariski geometry. Then either X is modular, or there is some algebraically closed field k definable in $\mathcal{M}$ such that k has Morley rank one and is nonorthogonal to X.*

Fact 3.21 *In the case of a differentiably closed field K, as a model for* **DCF**, *any field k of finite Morley rank definable in K is definably isomorphic to C_K, to the constant field of K. In particular they all have Morley rank one. Reference [28].*

In the next few paragraphs we will introduce a useful technical device which will enable us to view definable sets as objects of the same kind as elements in $\mathcal{C}$. Under the assumption of stability this will also permit types to be viewed as elements. The way to do this is to introduce a many sorted structure $\mathcal{C}^{eq}$ with a many sorted language $\mathcal{L}^{eq}$ and theory $\mathcal{T}^{eq}$ constructed from the one sorted $\mathcal{L}$-structure $\mathcal{C}$ which will naturally imbedd in $\mathcal{C}^{eq}$.

We do this as follows. First we define the language $\mathcal{L}^{eq}$ (which depends on $\mathcal{T}$). Let $\mathcal{ER}(\mathcal{T})$ be the collection of $\mathcal{L}$-formulas $E(\bar{x},\bar{y})$, where $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$, which define in $\mathcal{C}$ an equivalence relation on C^n. For each formula $E(\bar{x},\bar{y}) \in \mathcal{ER}(\mathcal{T})$, $\mathcal{L}^{eq}$ will contain a sort S_E. In particular there will be a sort $S_=$. For E as above $\mathcal{L}^{eq}$ will also contain a new n-ary function symbol f_E, whose domain sort is $S_=^n$ and whose target sort is S_E. All the relation and function symbols of $\mathcal{L}$ will also be in $\mathcal{L}^{eq}$ but the associated tuples of sorts will be of the form $(S_=, \ldots, S_=)$. So we get a natural "embedding" of $\mathcal{L}$ in $\mathcal{L}^{eq}$.

We now enlarge $\mathcal{C}$ to a canonical $\mathcal{L}^{eq}$-structure $\mathcal{C}^{eq}$. The interpretation of the sort $S_=$ in $\mathcal{C}^{eq}$ will be the set C itself, and those relation and function symbols of $]calL^{eq}$ which are in $\mathcal{L}$ will be interpreted in C in the usual way. The interpretation of the sort S_E in $\mathcal{C}^{eq}$ will be the set $\{\bar{c}/E : \bar{c} \in C^n\}$, and the interpretation of the function symbol f_E will be the function which takes $\bar{c}$ to $\bar{c}/E$. The new elements of $\mathcal{C}^{eq}$ are called **imaginary** elements. Similarly for any model $\mathcal{M}$ of $\mathcal{T}$ we produce an $\mathcal{L}^{eq}$-structure $\mathcal{M}^{eq}$ such that the interpretation in $\mathcal{M}^{eq}$ of the sort $S_=$ is exactly the structure M. By virtue of the natural embedding of $\mathcal{L}$ in $\mathcal{L}^{eq}$, we can treat any formula $\phi(\bar{x})$ as an $\mathcal{L}^{eq}$-formula, by considering all the variables in ϕ (bound as well as free) as being of sort $S_=$. That is we have the following fact.

Fact 3.22 *For all models $\mathcal{M}$ of $\mathcal{T}$, $\mathcal{L}$-formulas $\phi(\bar{x})$, and tuples $\bar{c}$ from M*

$$\mathcal{M} \models \phi(\bar{c}) \text{ if and only if } \mathcal{M}^{eq} \models \phi(\bar{c}).$$

The theory $\mathcal{T}^{eq}$ will be the $\mathcal{L}^{eq}$-theory consisting of all the sentences of $\mathcal{T}$ together with axioms which assert that for each $E \in \mathcal{ER}(\mathcal{T})$, f_E is a map from $S_=^n$ (n = arity of f_E) onto S_E, and $f_E(\bar{x}) = f_E(\bar{y})$ if and only if $E(\bar{x},\bar{y})$. $\mathcal{M}^{eq}$ is a model of $\mathcal{T}^{eq}$ whenever $\mathcal{M}$ is a model of $\mathcal{T}$.

Fact 3.23
1. *Every model $\mathcal{M}^*$ of $\mathcal{T}^{eq}$ is of the form $\mathcal{M}^{eq}$ where $\mathcal{M}$ is a model of $\mathcal{T}$, and $(S_=)^{\mathcal{M}^*} = M$.*
2. *$\mathcal{T}^{eq}$ is complete (under the assumption that $\mathcal{T}$ is complete).*
3. *Whenever $E_1, \ldots, E_k \in \mathcal{ER}(\mathcal{T})$, and $\phi(x_1, \ldots, x_k)$ is an $\mathcal{L}^{eq}$-formula with x_i a variable of sort S_{E_i}, there is an $\mathcal{L}$-formula $\psi(y_1, \ldots, y_n)$ such that for any model $\mathcal{M}^*$ of $\mathcal{T}^{eq}$ $\mathcal{M}^* \models (\forall y_1, \ldots, y_k$ from $S_=)(\psi(y_1, \ldots, y_k) \Leftrightarrow \phi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k))$.*

In particular if $\phi(x_1, \ldots, x_k)$ is a formula of $\mathcal{L}^{eq}$ and each x_i is a variable of sort $S_=$ then there is an $\mathcal{L}$-formula $\psi(x_1, \ldots, x_k)$ which is equivalent to ϕ in $\mathcal{T}^{eq}$.

Fact 3.24 *1. For every elementary substructure $\mathcal{M}$ of $\mathcal{C}$, $\mathcal{M}^{eq}$ is an elementary substructure of $\mathcal{C}^{eq}$.*

2. $\mathcal{C}^{eq}$ is in the definable closure (in the sense of $\mathcal{L}^{eq}$) of $\mathcal{C}$. Similarly $\mathcal{M}^{eq} = dcl^{eq}(\mathcal{M})$.

3. Every subset of C^n which is definable (with parameters) in the structure $\mathcal{C}^{eq}$ is definable in $\mathcal{C}$.

4. $\mathcal{C}^{eq}$ is saturated (we assumed that $\mathcal{C}$ is saturated).

One nice property that we have in $\mathcal{C}^{eq}$ is that for any $E \in \mathcal{ER}(\mathcal{T})$, and E-equivalence class $X \subset C^n$ there is an element c in $\mathcal{C}^{eq}$, and an $\mathcal{L}^{eq}$-formula $\phi(\bar{x}, y)$ such that

1. the formula $\phi(\bar{x}, c)$ defines X, and
2. if $\phi(\bar{x}, c')$ also defines X, then $c = c'$ ($=$ is in $\mathcal{C}^{eq}$).

The element c is just $\bar{c}/E$ where $\bar{c} \in X$, and the formula $\phi(\bar{x}, y)$ is simply $f_E(\bar{x}) = y$. So equivalence classes from $\mathcal{C}$ are definable sets in $\mathcal{C}^{eq}$.

We could generalize the above idea in the form of the following definition.

Definition 3.44 *We will say that a (possibly many sorted) theory $\mathcal{T}$ has* **elimination of imaginaries** *if for any (sufficiently saturated) model $\mathcal{M}$ of $\mathcal{T}$, zero-definable equivalence relation E on M^n, and E-class X, there is some tuple $\bar{c}$ from $\mathcal{M}$ such that that for some $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y})$, (i) X is defined by $\phi(\bar{x}, \bar{c})$ and (ii) whenever $\phi(\bar{x}, \bar{c}')$ is equivalent to $\phi(\bar{x}, \bar{c})$ (i.e. also defines X), then $\bar{c} = \bar{c}'$. We call such $\bar{c}$ a* **code** *for X.*

If a tuple $\bar{c}$ is a code for the equivalence class X then $\bar{c}$ is unique up to **interdefinability**, in the sense that if $\bar{c}'$ is another code for X, then $\bar{c} \in dcl(\bar{c}')$ and $\bar{c}' \in dcl(\bar{c})$.

Fact 3.25 *$\mathcal{T}^{eq}$ has elimination of imaginaries.*

Fact 3.26 *The theories* $\mathbf{ACF}_p$, **DCF**, $\mathbf{SCF}_{p,\nu}$ *in the languages $\mathcal{L}_{ring}$, $\mathcal{L}_{diff}$, and $\mathcal{L}_{p,\nu}$ respectively eliminate imaginaries.*

Remark: Let X be a definable set in $\mathcal{C}$, defined by the formula $\phi(\bar{x}, \bar{a})$ where ϕ is an $\mathcal{L}$-formula. Let $E(\bar{y_1}, \bar{y_2})$ be the $\mathcal{L}$-formula $(\forall \bar{x})(\phi(\bar{x}, \bar{y_1}) \Leftrightarrow \phi(\bar{x}, \bar{y_2}))$. So $E \in \mathcal{ER}(\mathcal{T})$. Let Y be the E-class of $\bar{a}$ defined by the formula $E(\bar{y}, \bar{a})$. Then X and Y are "interdefinable" in the sense that for any tuple $\bar{b}$, X is $\bar{b}$-definable if and only if Y is $\bar{b}$-definable. In particular if some tuple $\bar{b}$ is a code for Y, then we can say that $\bar{b}$ is a *code* for X, meaning that $\bar{b}$ satisfies (i) and (ii) from

Definition . Thus if the theory $\mathcal{T}$ has elimination of imaginaries then every definable set in $\mathcal{C}$ has a code which is again unique up to interdefinability.
Then one can say that a theory $\mathcal{T}$ has elimination of imaginaries if every definable set in $\mathcal{C}$ (or equivalently in any model of $\mathcal{T}$) has a code. $\triangleright$

In general for the sake of convenience one passes from $\mathcal{C}$ to $\mathcal{C}^{eq}$. In $\mathcal{C}^{eq}$ one can build quotient structures. For example if G is a definable group in $\mathcal{C}$ and H is a zero-definable normal subgroup of G, the quotient group G/H can be considered as a group definable in $\mathcal{C}^{eq}$. Or if for example $f : G \to H$ is a zero-definable homomorphism between two definable groups G and H in $\mathcal{C}$ then for any $g \in G$ the element $h = f(g)$ is definable from g, but also from any $g' \in g \cdot ker(f)$. It is more canonical to consider that h is definable from the imaginary element $g \cdot ker(f)$. Also while working with $\mathcal{C}^{eq}$ the distinction between elements and tuples becomes somewhat blurred. For example an n-tuple from C is an element of the sort C^n. So in $\mathcal{C}^{eq}$ one can simplify the notation and to make no difference between tuples and elements unless one has a specific reason to do it.
It is important to stress that the above facts and remarks point out that if the theory eliminates imaginaries then the objects of $\mathcal{C}^{eq}$ already exist in $\mathcal{C}$ and one can deal with them in the usual model theoretic way.

Here is a very useful fact.

Proposition 3.5 *(In $\mathcal{T}^{eq}$) An infinite almost strongly minimal abelian group G is nonorthogonal to the definable set Y if and only if there is a definable subgroup $H \subset dcl^{eq}(Y)$ and a definable surjective homomorphism $f : G \to H$ with finite kernel.*

If the theory $\mathcal{T}$ has elimination of imaginaries the above statement applies in $\mathcal{T}$ itself.

4 Intermission, a short summary

As was mentioned earlier the general idea of model theoretic proofs of Diophantine problems is based on choosing a right structure in which one can formulate the original question and replace certain objects with sets which are definable in that structure since model theory deals only with definable sets. If the underlying theory of the structure has nice technical properties (e.g. it has elimination of imaginaries, has elimination of quantifiers, is ω-stable or stable, etc.), then one has much more control over the definable sets one uses in proving the theorem. The original structure can also be enlarged (e.g. by taking elementary extensions) if it is needed to obtain a better handle on the problem (e.g. saturated elementary extensions). Proving theorems in this larger structure will yield results about the original structure.

We will see that for instance to prove the Mordell-Lang conjecture in characteristic $p = 0$ one makes use of **DCF**. The idea is to replace the small "finite rank" group Γ by a bigger <u>definable</u> object which is still "small". One cannot do this in the context of algebraically closed fields and constructible sets since the smallest definable group containing Γ would be its Zariski closure. And for or example if Γ is the group of torsion elements of A then the Zariski closure of Γ is A itself. Therefore one needs to extend the class of definable subsets to a richer class and it turns out that **DCF** is just perfect for this role. In addition we saw that **DCF** is a complete ω-stable theory, hence has saturated models of any cardinality, it admits quantifier and imaginary elimination and hence has all sorts of other nice properties. In a $\aleph_0$-saturated ω-stable theory one can use Morley rank and Morley degree to specify certain definable sets (e.g strongly minimal sets) for which powerful theorems with "geometric content" hold (e.g. Zariski geometries, Zilber's conjecture, modularity, orthogonality) and apply them toward the solution of the problem.

One can follow a similar idea in the case of characteristic $p > 0$ but now using the theory $\mathbf{SCF}_{p,\nu}$. Unfortunately this theory is not ω-stable, just stable and Morley rank is then not defined for all definable sets. Hence one will have to work a little bit harder to get some handle on definable sets which have geometric content. But the major theorems and concepts can be slightly redefined so that they work in this setting too.

5 A list of important propositions and definitions that will be used in the proof of Mordell-Lang conjecture in Section 6

Here is a list of propositions and definitions to which we directly refer in the proof of the Mordell-Lang conjecture (theorem).

First we consider characteristic $p = 0$.

Proposition 5.1 *Let K be a differentially closed field and A be an abelian variety over K and let Γ be a subgroup of $A(K)$ of finite rank. Then there exists a δ-definable homomorphism f from A into K^n for some n. Also, there exists a δ-definable subgroup H of A such that $H = f^{-1}(W)$ where W is a subgroup of K^n, and H has the properties that $\Gamma \subset H$, and that it has a finite Morley rank.*

Proposition 5.2 *Let H be a finite Morley rank group in a model for the theory* **DCF**. *There exist a δ-definable subgroup G of H which has the following properties:*

1. *G is connected, there is some finite F such that $G \subset acl(F \cup Y_1 \cup \cdots \cup Y_n)$, with each Y_i strongly minimal δ-definable, and G is maximal such.*
2. *$G = G_1 + \cdots + G_r$, where each G_i is an almost strongly minimal connected δ-definable subgroup of G, and the G_i's are pairwise orthogonal.*

Proposition 5.3 *Let K be a differentially closed field and let $D \subset K^n$ be strongly minimal. There is $F \subset D$ finite such that $D \backslash F$ is a Zariski geometry.*

Definition 5.1 *Let H be a definable group of finite Morley rank and Y a definable subset of H. We define, $Stab_H(Y)$, the* **model theoretic stabilizer** *of Y in H to be $Stab_H(Y) = \{h \in H : Mr((h+Y) \cap Y) = Mr(Y)\}$.*

Fact 5.1 *Let K be an $\aleph_0$-saturated model for the theory of differentially closed fields in the language $\mathcal{L}_{diff}$. Let A be an abelian variety over K, X an irreducible subvariety of A, and H a δ-definable subgroup of $A(K)$ with finite Morley rank, such that $X \cap H$ is Zariski dense in X. Then there is a complete stationary type (see* **Definition 3.33***) q in $X \cap H$ (over some finite set E) of minimal Morley rank, such that $Y = q(K)$ is dense in X, and the model theoretic stabilizer of Y in H is finite (***Definition 5.1***).*

Definition 5.2 *We say that a definable group G, defined over some set E, is* **rigid** *if every definable connected subgroup of G is definable over the algebraic closure of E.*

Proposition 5.4 *Let K be a model for* **DCF***. Let H be a definable commutative finite Morley rank group defined over K, let G be a maximal connected definable subgroup of H such that $G \subset acl(F \cup X_1 \cup \cdots \cup X_n)$, where F is finite and the X_i's are strongly minimal.*
*Let $E \supset F$ be a finite set such that H,G and the X_i's are defined over E. Suppose that G is rigid (***Definition 5.2***). Let q be a complete stationary type in H, over E such that $Stab_H(q(K))$ is finite.*
Then $q(K)$ is contained in a unique coset of G.

Proposition 5.5 *Let K be an $\aleph_0$-saturated differentially closed field and $k \subset K$, the constant field of K. Let A be an abelian variety defined over K and let G_1 be an almost strongly minimal connected δ-definable subgroup of A. Suppose that G_1 is nonmodular.*

1. *Then there exists an abelian variety V defined over k, and a bijective rational morphism h from $\overline{G_1}$ into V, such that $h(G_1) = V(k)$.*
2. *Let X be a subvariety of A, defined over K such that $X \cap G_1$ is dense in X. Then there is a subvariety X_V of V, defined over k, such that $X = h^{-1}(X_V)$.*

Proof: The main steps in this proof are a nice illustration of how all the major theorems come into play and how do they interact.
Since G_1 is almost strongly minimal by **Definition 3.38** we have that there exists a strongly minimal δ-definable set B such that $G_1 \subset acl(B)$. By **Proposition 3.3** B is a Zariski geometry up to finite many points. Nonmodularity of G_1 implies nonmodularity of B. As Zilber's conjecture, **Proposition 3.4**, implies that B is nonorthogonal to k we have that G_1 has to be nonorthogonal to k too. Now as K is a model for **DCF** and **DCF** has elimination of imaginaries we can apply **Proposition 3.5** to get that there is H a subgroup of G_1 defined with parameters from k and a δ-definable homomorphism f with finite kernel which maps G_1 onto H. A little bit of tinkering with the homomorphism f and

the group H one can come up with a definable isomorphism h and an abelian variety V defined over k such that 1. holds.
As for 2. let $X_V = h(X)$ where h is a map from 1. such that $h(G_1) \subset k^n$. As $\overline{X \cap G_1} = X$, X is irreducible (and closed) it follows that $h(X)$ is irreducible (and closed), hence X_V is the Zariski closure of $h(X \cap G_1)$. Since $h(X \cap G_1) \subset h(G_1) \subset k^n$ we have that $h(X \cap G_1) \subset k^n$ and by **Fact 2.6** it Zariski closure X_V is also defined over k. □

Below is the list of propositions used in the proof of the characteristic $p > 0$ case. Here the underlying structure is a model for $\mathbf{SCF}_{p,\nu}$ which is a stable theory and not ω-stable as **DCF**. But the major ideas with some modifications go through in this case too.
The concepts U-rank, minimal types and semiminimal sets in $\mathbf{SCF}_{p,\nu}$ are the analogues of Morley rank, strongly minimal sets and almost strongly minimal sets from **DCF**. For details we refer to [5], [19].

Proposition 5.6 *Let K be an $\aleph_1$-saturated separably closed field of finite degree of imperfection and $k \subset K$ such that $k = K^{p^\infty}$. Let $\widetilde{K}$ denote the algebraic closure of K. Let A be an abelian variety defined over K and let H be a semiminimal infinitely-definable connected subgroup of A. Suppose that H is nonmodular.*

1. *Then there exists an abelian variety V defined over k, and a bijective morphism h from $\overline{H}$ into $V(\widetilde{K})$, such that $h(H) = V(k)$.*
2. *Let X be a subvariety of A, defined over K such that $X \cap H$ is dense in X. Then there is a subvariety X_V of V, defined over k, such that $X = h^{-1}(X_V)$.*

Fact 5.2 *Let K be a model for $\mathbf{SCF}_{p,\nu}$ and G a commutative algebraic group defined over K. Let $G^* = \cap_{n \in \mathbb{N}} p^n G(K)$. Then G^* has a finite U-rank.*

Fact 5.3 *Let K be an $\aleph_1$-saturated separably closed field, ($\widetilde{K}$ the algebraic closure of K), and X an irreducible closed set defined over K. Let F be any finite subset of K over which X is defined. If $Y = \cap_{i \in \mathbb{N}} Y_i$, where for each i, Y_i is a definable subset of K over F and Y_i is Zariski dense in X, and if the Y_i's form a descending chain, then Y is Zariski dense in X.*

Fact 5.4 *In a separably closed field K of characteristic $p > 0$ and finite degree of imperfection ν any field k of finite U-rank definable in K is definably isomorphic to K^{p^∞}.*

Remark: K^{p^∞} has U-rank one.
Also the statement of the Zilber's conjecture (**Proposition 3.4**) holds in models of $\mathbf{SCF}_{p,\nu}$ when Morley rank is replaced by U-rank. ▹

Proposition 5.7 *Let K be a separably closed field, $k = K^{p^\infty}$ and A a simple abelian variety defined over K. Either there is a bijective rational homomorphism between A and a simple abelian variety defined over k or A^* is modular and has U-rank one.*

Definition 5.3 *In* $\mathbf{SCF}_{p,\nu}$ *an infinitely-definable set of U-rank one is called a* **minimal type**.

Fact 5.5 *In* $\mathbf{SCF}_{p,\nu}$ *a minimal type (see* **Definition 5.3***) is a Zariski geometry up to finite many points.*

Proposition 5.8 *Suppose G is an infinitely-definable (abelian) group of finite U-rank in the $\mathcal{L}_{p,\nu}$-structure $\mathcal{M}$ which is a model for* $\mathbf{SCF}_{p,\nu}$*. Suppose moreover that G is modular. Then every definable subset of G is a finite boolean combination (finite union) of translates of definable subgroups of G. In a structure $\mathcal{M}$ by a definable subset of an infinitely-definable set X we mean a set $Y \subset X$ such that there is a definable set Z in M and $Y = X \cap Z$.*

6 Sketch of the proof of the Mordell-Lang conjecture

Theorem 6.1 *Mordell-Lang conjecture for all characteristic. Let $k \subset K$ be two distinct algebraically closed fields (of any characteristic $p \geq 0$), A an abelian variety defined over K, X an infinite irreducible subvariety of A and Γ a finite rank subgroup of $A(K)$. Suppose that $X \cap \Gamma$ is Zariski dense in X, i.e. $\overline{X \cap \Gamma} = X$, and that the stabilizer, $Stab_X = \{a \in A : a + X = X\}$, of X in A is finite. Then there is a abelian subvariety A' of A and there are V, an abelian variety defined over k, X_V a subvariety of V defined over k and a bijective (rational) homomorphism h from A' onto V, such that $X = a + h^{-1}(X_V)$ for some a in A.*

Remarks:

1. We would like to remind the reader that the definition of a finite rank subgroup is slightly different for $cahr\, p = 0$ then for $char\, p > 0$ (see **Definitions 2.19** and **2.20**).

2. If we drop the assumption that $X \cap \Gamma$ is dense in X we would need to work with the irreducible components of the Zariski closure of $X \cap \Gamma$ separately and get the conclusion that there exist $A'_1, \ldots, A'_n$ abelian subvarieties of A, $V_1, \ldots, V_n$ abelian varieties defined over k and for each i, $1 \leq i \leq n$, X_{V_i} a subvariety of V_i, also defined over k and h_i, a (rational) homomorphism from A'_i onto V_i, such that $X \cap \Gamma \subset \bigcup_{i=1,\ldots,n}(a_i + h_i^{-1}(X_{V_i})) \subset X$.

3. The stabilizer $Stab_X$ is a closed subgroup of A and there is no harm in quotienting A by the connected component of the stabilizer, so we can assume that it is finite.

4. Since the conclusion is given as a translate of $h^{-1}(X_V)$ we can replace X by a translate of X in A at any time during the proof.

5. As to simplify the discussion we consider A to be an abelian variety. But we note that Hrushovski's proof applies to a more general case where one takes A to be a semiabelian variety (**Definition 2.16**). $\triangleright$

Proof: First we deal with the **characteristic 0** case. Our goal is to set up the stage for applying **Proposition 5.5**.
So in order to replace Γ by a "small" definable group, we enrich the structure and pass to a differentially closed field. There is no problem in enlarging K so we may assume that K is differentially closed with a derivation δ such that k is the field of constants of K (then K is also algebraically closed and k is definable in K). But in order to be able to utilize properly the results of model theory and the theorems from section 5 we have to have an $\aleph_0$-saturated model. So let K' be an $\aleph_0$-saturated *elementary extension* of K and k' the field of constants of K'. K' exists because **DCF** is a stable theory. Obviously $k \subset k'$. Now if we prove **Theorem 6.1** with k' and K' replacing k and K we need to be able to recover the original statement (with k and K). This can be achieved because the statement of the theorem is a first order sentence, ϕ, in the language of differential fields with parameters from K needed to define A, Γ, X and parameters from k' (which is definable in K' by writing $\delta(x) = 0$ for $x \in k'$) for V and X_V. Since K is algebraically closed by **Fact 2.15** A' is defined over K so the parameters for A' can be taken from K directly.
So if we prove the theorem in the model K', that is we have that
$K' \models \phi(a_1, \ldots, a_n, b_1, \ldots b_m)$ where a_i's come from K and b_i's come from K', then by the corollary to the Tarski-Vaught test, **Proposition 3.1**, the formula $\phi(a_1, \ldots, a_n, x_1, \ldots x_m)$ must be satisfiable in K since it defines a nonempty set in the structure K' which is an elementary extension of K. Hence the theorem is then valid in the pair (k, K) too. So from now on we can assume that K is an $\aleph_0$-saturated differentially closed field and k is the field of constants of K.

Now we replace the group Γ by a δ-definable group of finite Morley rank. A itself has has an infinite Morley rank as a δ-definable set. This is done as follows: suppose $dim(A) = d$, as an algebraic variety. The theory of differential algebraic groups (see [14], [18]) enables us to find a *definable* homomorphism f from A onto $(K, +)^d$ whose kernel is known as the Manin kernel. The image of Γ under f is a finite dimensional vector space over the rationals $\mathbb{Q}$ and now tensoring with k we get a finite dimensional vector space W over k. Let $H = f^{-1}(W)$. Then H is an infinite definable subgroup of A, with finite Morley rank and contains Γ (**Proposition 5.1**). Since $\Gamma \subset H$ and $\overline{\Gamma \cap X} = X$ $X \cap H$ will be dense in X too.

In order to replace H with a strongly minimal set we need couple of intermediate steps.

First we will replace H by a sum of almost strongly minimal groups (**Definition 3.38**). We do this by applying **Proposition 5.2**. So there exist a δ-definable subgroup G of H which has the following properties:

1. G is connected, there is some finite F such that $G \subset acl(F \cup Y_1 \cup \cdots \cup Y_n)$, with each Y_i strongly minimal δ-definable, and G is maximal such.

2. $G = G_1 + \cdots + G_r$, where each G_i is an almost strongly minimal connected δ-definable subgroup of G, and the G_i's are pairwise orthogonal.

Let $F_0 \supset F$ denote some finite set in K such that A, X, H, G and all the G_i's are defined over F_0. Since our goal is to replace H by G but G is contained in H we will need to replace $X \cap H$ by a smaller set Y which is contained in $X \cap G$ and is dense in X. **Fact 5.1** says that there is a complete stationary type q in $X \cap H$, over some finite $E \supset F_0$, such that $Y = q(K)$ is dense in X, and the stabilizer of Y in H is finite. As the group G is rigid and maximal in H by **Proposition 5.4** we get that $Y = q(K)$ is contained in a unique coset of G. Also $G \subset H$ implies that the stabilizer of Y in G will be finite too. If we need we can replace X by a translate of X and get that Y is a subset of G.

To summarize, so far we can assume that there is a complete stationary type Y over $E \supset F_0$ in $X \cap G$ such that $\overline{Y} = X$ and the stabilizer of Y in G is finite.

We claim that G cannot be modular: if it were, then $X \cap G$ would be a finite union of cosets of connected subgroups of G (**Proposition 3.2** and **Fact 2.7**). But then as $Y \subset G$ and Y is dense in X we have that $X \cap G$ is dense in X too and if we take the Zariski closure of the union of cosets we get that $X = \overline{X \cap G}$ is a finite union of translates of abelian subvarieties of A (**Fact 2.9**). X being irreducible implies that X is a single translate of an abelian subvariety and that contradicts the fact that the stabilizer of X, $Stab_X$, is finite. So G cannot be modular.

Since $G = G_1 + \cdots + G_k$ and the sum of modular groups is a modular group (**Fact 3.19**), it follows that one of the G_i's must be nonmodular. This nonmodular summand has to be unique. This is so because for any two nonmodular almost strongly minimal summands of G the corresponding strongly minimal set would be nonorthogonal to k by Zilber's conjecture, but as nonorthogonality is an equivalence relation for almost strongly minimal sets (**Fact 3.18**) the two strongly minimal sets would have to be nonorthogonal which would also imply that the corresponding almost strongly minimal summands have to be nonorthogonal. But they are not, it was assumed that the different G_i's are pairwise orthogonal. Hence there must be just one nonmodular summand. Let us call it G_1.

The next claim is that $X \cap G_1$ is dense in X. We will prove this by showing that (up to translation) Y is contained in G_1 (the proof of this claim will be used in the characteristic $p > 0$ case too !). So let $G = G_1 + B$ where $B = G_2 + \cdots + G_n$. B is modular and orthogonal to the nonmodular group G_1. Since $Y \subset G = G_1 + B$ and the two groups are orthogonal Y is a finite union of definable sets of the form $U_i + V_i$ where $U_i \subset G_1$ and $V_i \subset B$ (**Definition 3.36**). But since Y is dense in X and X is irreducible that union has to contain only one set, that is $Y = U + V$. V being a subset of a modular group B it is a finite union of cosets of definable subgroups of B, but again because X is irreducible the union consists of a single set and we have that $Y = U + (c + D)$ where D is a subgroup of B. As every element of D stabilizes Y and we know that the stabilizer of Y in G is finite it must be that D is finite. And again since $\overline{Y} = X$ irreducibility of X implies that D is trivial hence $Y = U + c$. Replacing X by its translate we can assume that $Y \subset G_1$ and consequently $X \cap G_1$ is dense in X.

Finally we have an almost strongly minimal connected nonmodular group G_1 such that $\overline{X \cap G_1} = X$. That is the conditions of **Proposition 5.5** are now fulfilled and we can apply it to conclude the proof of the theorem. (A' will be $\overline{G}_1$.) $\diamond$

The **characteristic** $p > 0$ case. The general idea is the same as in characteristic $p = 0$ but the setting is different and hence it will be necessary to modify the above theorems in order to apply them in the positive characteristic case.

The major differences are: one needs to work with the theory of separably closed fields $\mathbf{SCF}_{p,\nu}$ of fixed finite degree of imperfection ν. This theory is not ω-stable, only stable. Hence there is no concept of rank like Morley rank in the case of ω-stable theories that applies to all definable sets. But in an analogous manner to Morley rank one can define a notion of U-rank or Lascar rank which is well defined for certain infinitely-definable sets (see [30]). Many of the above theorems used in the characteristic zero case will be valid for infinitely-definable sets of finite U-rank. Since one has to consider infinitely-definable objects one has to work in $\aleph_1$-saturated model. The role of strongly minimal sets from characteristic $p = 0$ case will be taken over by the so called minimal types in char $p > 0$ (**Definition 5.3**). It turns out that these minimal types are Zariski geometries ("in the sense of separably closed fields", **Fact 5.5**). For minimal types Zilber's conjecture holds and the argument follows the same line as in characteristic zero case (**Proposition 5.6**).

The setting is as follows. By assumption there is a finite rank group Γ and a finitely generated subgroup $\Gamma' \subset \Gamma$ such that for every $x \in \Gamma$ there is a positive integer n not divisible by p and such that $nx \in \Gamma'$. Some subtle *extension* arguments allow one to assume that there exists L an $\aleph_1$-saturated separably closed field of a finite degree of imperfection such that A and X are defined over L and the generators of Γ' are L rational points of A. In addition one can assume that $k = L^{p^\infty} = \bigcap_{n \in \mathbb{N}} L^{p^n}$ and that K is the algebraic closure of L.

In characteristic $p > 0$ one can replace the group Γ by an infinitely-definable group $H = \bigcap_{n \in \mathbb{N}} p^n A(K)$. H has finite U-rank and $\overline{H \cap X} = X$. The rest is similar to the characteristic zero case: one needs to show that H can be eventually replaced by a *semiminimal* group (i.e. by a group that is contained in the algebraic closure of a minimal type) and then since sets of minimal type are Zariski geometries extract geometric information using the properties of Zariski geometries guaranteed by Zilber's conjecture.

Instead going into details of this method we would like to describe the slightly shorter approach which appears in [3]. Unfortunately this one cannot be extended to semi-abelian varieties.

We will use the following notation:
From now on let K denote the field L as defined above ! Then $k = K^{p^\infty}$ and k

is algebraically closed.
For any abelian algebraic group S defined over a field F let S^* denote $\cap_{n\in\mathbb{N}} S^{(n)}$ where $S^{(n)} = p^n S(F) = \{p^n \cdot s : s \in S(F)\}$.
Note that $S^{(n+1)} \subset S^{(n)}$ and by **Fact 5.2** S^* has finite U-rank.
In this notation H, as defined above, becomes A^*.

First we show that $A^* \cap X$ is dense in X. As $p^n\Gamma$ has finite index in Γ, for any n, and $\Gamma^{(n)} \subset A^{(n)}$ we have that Γ meets only finitely many translates of $A^{(n)}$. Thus by irreducibility of X for some translate T_n of $A^{(n)}$, $X \cap T_n$ is Zariski dense in X. By $\aleph_1$-saturation we can assume that they form a descending chain i.e. we can find an $a \in X$ such that, for each n, $T_n = a + A^{(n)}$. This is so because for any fixed $n \in \mathbb{N}$ the set of $a \in X$ such that $T_n = a + A^{(n)}$ is dense in X is a definable set (definable by the formula that expresses this property). So we get a countable family of definable sets. Moreover this family is descending chain of sets because the family $\{A^{(n)} : n \in \mathbb{N}\}$ is descending chain too. Hence it has the finite intersection property and by $\aleph_1$-saturation the intersection of all sets in the families nonempty. Now by **Fact 5.3** $(a + A^*) \cap X$ is dense in X, so by translating X we may assume that $A^* \cap X$ is dense in X.

Next we decompose the abelian variety A as an almost direct sum (**Definition 2.18**) of finitely many simple abelian subvarieties $S_1, \ldots, S_r$ say. By **Proposition 5.7** and Zilber's conjecture for each $i \in \{1, \ldots, r\}$ we have either (1) S_i^* is modular and orthogonal to k or (2) S_i descends to k, that is, there is a bijective rational homomorphism between S and a simple abelian variety defined over k. Let D_1 be the sum of those S_i satisfying (1) and D_2 the sum of those S_i satisfying (2).
Then we have all the following. A^* is an almost direct sum of D_1^* and D_2^*. D_1^* is modular, since it is a sum of modular groups. D_2 is rationally isomorphic to an abelian variety V defined over k, hence D_2^* is also definable over k. D_2^* being definable over k is nonorthogonal to k and orthogonal to D_1^*.

Now, let $Y = X \cap A^*$. Y is dense in X. As $Stab_X$ is finite it follows that $Stab_{A^*}(Y)$, the stabilizer of Y in A^*, is finite too. From orthogonality of D_1^* and D_2^* together with the modularity of D_1^* (see **Proposition 5.8**) it follows, by the same argument as in characteristic $p = 0$, that Y is up to a translation contained in the nonmodular group D_2^*. Thus replacing Y and X by suitable translates, we may assume that Y is contained in D_2^*.

As D_2 descends to k and is isomorphic to V via some rational homomorphism h, all defined over k, we have that $h(D_2^*) \subset V(k)$ and we let $X_V = h(X)$. Since $X_V = \overline{h(Y)}$ and $h(Y) \subset h(D_2^*) \subset V(k)$ by **Fact 2.6** we have that X_V is defined over k too. This concludes the proof of the theorem. $\square$

7 Historical overview of the Mordell-Lang Conjecture and other related theorems

Historically the Mordell-Lang conjecture (version I below) was stated for abelian varieties in characteristic zero where the group Γ was a finitely generated subgroup of the abelian variety. It was proved by Faltings [16].

Mordell-Lang conjecture I (also known as just Lang's conjecture) Assume that K is an algebraically closed field of characteristic $p = 0$. Let A be an abelian variety defined over K, X a subvariety of A, and Γ a finitely generated subgroup of A. Then $X \cap \Gamma$ is a finite union of translates of subgroups of A. Equivalently, the Zariski closure of $X \cap \Gamma$ is a finite union of translates of abelian subvarieties of A. $\diamond$

Later the above conjecture was extended to semiabelian varieties and to finite rank subgroups. This version II was proved by McQuillan [22] following the work of Faltings, Vojta, Raynaud, Buium and Hindry. For references see [6].

Mordell-Lang conjecture II Assume that K is an algebraically closed field of characteristic $p = 0$. Let S be a semiabelian variety defined over K, X a subvariety of S, and Γ a finite rank subgroup of S. Then $X \cap \Gamma$ is a finite union of translates of subgroups of S. $\diamond$

In the background of these theorems was another famous conjecture, the Mordell conjecture.

Mordell conjecture I If C is a *curve* of *genus* greater than or equal to 2 which is defined over a number field K, then the set of K rational points of C, $C(K)$, is finite.

Note: a curve is a one dimensional variety. For the definition of a genus and dimension see [11].

The Mordell conjecture I as stated above was proved by Faltings [17]. It follows from the Mordell-Lang conjecture I using the Mordell-Weil theorem (see [8], [3]).

Remark: The Mordell-Weil theorem states that if A is an abelian variety defined over K, where K is a finitely generated field over the rationals $\mathbb{Q}$ (e.g. a number field) then $A(K)$ is a finitely generated abelian group.

Originally a somewhat different version of the Mordell conjecture (Mordell conjecture II) was proved by Manin [18]. His proof had a mistake that was noticed by Coleman 20 years later. Coleman in [20] gave a different proof of the same theorem while Chai managed to fix the original version of the proof in [21].

Mordell conjecture II (Manin's version) Assume characteristic $p = 0$ and let K be a function field over an algebraically closed field k. Let C be a curve of genus greater than or equal to 2 which is defined over K. Then either
i) $C(K)$ is finite, or
ii) C is isomorphic to some curve C' defined over k, and moreover all but finitely many points of $C(K)$ come from points of $C'(k)$ under this isomorphism. $\diamond$

If in the Mordell-Lang conjecture I one takes Γ to be the group of torsion points of the abelian variety A one would get another famous conjecture which is known as the Manin-Mumford conjecture. The Manin-Mumford conjecture was proved by Raynaud [24] and later generalized by Hindry [26] and others.

Unfortunately Mordell-Lang conjecture I and II or the Mordell conjecture II (Manin's version) do not generalize to characteristic $p > 0$. For a counterexample see [6], [3]. Hence a more general analogue of the Mordell-Lang conjecture in characteristic $p > 0$ was proposed by Abramovich and Voloch in [25]. They have proved several important cases of their conjecture but finally the most general case **Theorem 6.1** was proved by Hrushovski following some ideas of Buium [15]. Buium proved a special case of the conjecture and he was using differential algebraic geometry techniques while Hrushovski used model theory.

Mordell-Lang conjecture III, Theorem 6.1 (function field case in all characteristics) Assume that $k \subset K$ are two distinct algebraically closed field of some characteristic $p \geq 0$. Suppose Γ is a finite rank subgroup of the semiabelian variety S defined over the field K, and X is a subvariety of S. Then there is a finite set $X_1, \ldots, X_n$ of *special* subvarieties of S such that $X \cap \Gamma \subset X_1 \cup \cdots \cup X_n \subset X$.

X is a special subvariety of S if there are a semiabelian subvariety S' of S, a semiabelian variety S_0 defined over k and a subvariety X_0 of S_0 also defined over k, and a surjective homomorphism of algebraic groups $h : S' \to S_0$ such that $X = h^{-1}(X_0) + c$ for some $c \in S$.

The story does not end up here. One can be looking for effective bounds on the number of cosets that is guaranteed by the theorem. The model theoretic approach already yields pretty "good" bounds almost "automatically" by the compactness theorem but other techniques are available too. For more details about these questions and results see [6].

Remarks: One can prove the Manin-Mumford conjecture directly using model theoretical tools, see [3]. The underlying theory in this case is **ACFA**, the theory of fields of characteristic zero with a *generic* automorphism.

For the end we note that the Mordell-Lang conjecture in the case when one allows torsion points of Γ whose order is divisible by the characteristic ($p > 0$) of the field is still open.

For those interested in more details about model theory and geometric model theory I would recommend the following references [31], [30], [7], [9], [10], which I found very helpful. ▹

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