14
Renormalization 1: Basics

14.1 Introduction

Functional integrals over fields have been mentioned briefly in Part I devoted to *path* integrals.† In brief, time ordering properties and Gaussian properties generalize immediately from *paths* to *field* integrals. The singular nature of Green's functions in field theory introduces new problems which have no counterpart in path integrals. This problem is addressed by regularization and is briefly presented in the appendix.

The fundamental difference between quantum mechanics (systems with a finite number of degrees of freedom) and quantum field theory (systems with an infinite number of degrees of freedom) can be labelled "radiative corrections": In quantum mechanics "a particle is a particle" characterized, for instance, by mass, charge, spin. In quantum field theory the concept "particle" is intrinsically associated to the concept "field." The particle is affected by its field. Its mass and charge are modified by the surrounding fields, its own, and other fields interacting with it.

An early example of this situation is Green's calculation of the motion of a pendulum in fluid media. The mass of the pendulum in its equation of motion is modified by the fluid. Nowadays we say the mass is "renormalized." The remarkable fact is that the equation of motion remains valid provided one replaces the "bare" mass by the renormalized mass. Green's example is presented in section 14.2.

Although regularization and renormalization are different concepts,

† Namely in section 1.1 "The beginning", in section 1.3 "The operator formalism", in section 2.3 "Gaussian in Banach spaces", in section 2.5 "Scaling and coarse-graining", and in section 4.3 "The anharmonic oscillator".

they are often linked together. The following handwaving argument could lead to the expectation that radiative corrections (renormalization) regularize the theory: We have noted (2.40) that the Green's functions (covariances) G are the values of the variance W evaluated at pointlike sources. In general, given a source J,

$$W(J) = \langle J, GJ \rangle; \tag{14.1}$$

it is singular only if J is pointlike. On the other hand radiative corrections "surrounding" a point particle can be thought of as extending the particle. Unfortunately calculations do not bear out this expectation in general. In addition to inserting radiative corrections into "bare" diagrams, one needs to insert counterterms regularizing the divergences. In section 16, "Renormalization 3: Scaling", counterterms do not appear as a regularizing artifact, but as a consequence of working with scale dependent effective actions.

14.2 Green's example

Alain Connes made us aware of a delightful paper by Green which is an excellent introduction to modern renormalization in quantum field theory. Green derived the harmonic motion of an ellipsoid in fluid media. This system can be generalized and described by the Lagrangian of an arbitrary solid C in an incompressible fluid F, under the influence of an arbitrary potential $V(x)$, $x \in F$. In vacuum (i.e. when the density ρ_F of the fluid is negligible compared to the density ρ_C of the solid) the Lagrangian of the solid moving with velocity $\vec{v}$ under $V(x)$ is

$$L_C := \int_C d^3x \left(\frac{1}{2} \rho_C |\vec{v}|^2 - \rho_C V(x) \right). \tag{14.2}$$

In the static case, $\vec{v} = 0$, Archimedes' principle can be stated as the renormalization of the potential energy of a solid immersed in a fluid; the potential energy of the system fluid plus solid is

$$E_{\text{pot}} := \int_{\mathbb{R}^3 \setminus C} d^3x \rho_F V(x) + \int_C d^3x \rho_C V(x) \tag{14.3}$$

$$= \int_{\mathbb{R}^3} d^3x \rho_F V(x) + \int_C d^3x \left(\rho_C - \rho_F \right) V(x). \tag{14.4}$$

The potential energy of the immersed solid C is

$$E_{\text{pot}} = \int_C d^3x\,(\rho_C - \rho_F)\,V(x). \tag{14.5}$$

In modern parlance ρ_C is the bare coupling constant, $\rho_C - \rho_F$ is the coupling constant renormalized by the immersion. The infinite term $\int_{\mathbb{R}^3} d^3x\rho_F V(x)$ is irrelevant.

In the dynamic case, $\vec{v} \neq 0$, Green's calculation can be shown to be the mass of a solid body by immersion in a fluid. The kinetic energy of the system fluid plus solid is

$$E_{\text{kin}} := \frac{1}{2}\int_F d^3x\rho_F \left|\vec{\nabla}\Phi\right|^2 + \frac{1}{2}\int_C d^3x\rho_C|\vec{v}|^2 \tag{14.6}$$

where Φ is the velocity potential of the fluid,

$$\vec{v}_F = \vec{\nabla}\Phi. \tag{14.7}$$

If the fluid is incompressible

$$\Delta\Phi = 0.$$

If the fluid is nonviscous, the relative velocity $\vec{v}_F - \vec{v}$ is tangent to the boundary ∂C of the solid. Let $\vec{n}$ be the unit normal to ∂C, and $\partial_n\Phi := \vec{n}\cdot\vec{\nabla}\Phi$, then $\vec{v}_F - \vec{v}$ is tangent to ∂C if

$$\partial_n\Phi = \vec{v}\cdot\vec{n} \quad \text{on} \quad \partial C = \partial F. \tag{14.8}$$

Moreover Φ and $\vec{\nabla}\Phi$ must vanish at infinity sufficiently rapidly for $\int_F d^3x \left|\vec{\nabla}\Phi\right|^2$ to be finite. Altogether Φ satisfies a Neumann problem

$$\Delta\Phi = 0 \text{ on } F, \quad \Phi(\infty) = 0, \quad \partial_n\Phi = \vec{v}\cdot\vec{n} \text{ on } \partial F, \tag{14.9}$$

where $\vec{n}$ is the inward normal to ∂F. If ρ_F is constant, the Green formula

$$\int_F d^3x \left|\vec{\nabla}\Phi\right|^2 = -\int_{\partial F} d^2\sigma\ \Phi\cdot\partial_n\Phi \tag{14.10}$$

together with the boundary condition (14.8) simplifies the calculation of $E_{\text{kin}}(F)$, namely

$$E_{\text{kin}}\,(F) = \frac{1}{2}\rho_F\int_{\partial F} d^2\sigma\,(-\Phi)\,\vec{v}\cdot\vec{n}. \tag{14.11}$$

Example Let C be a ball of radius R centered at the origin. The dipole potential

$$\Phi(\vec{r}) = -\frac{1}{2}(\vec{v}\cdot\vec{r})(R/r)^3 \tag{14.12}$$

satisfies the Neumann conditions (14.9). On the sphere $\partial C = \partial F$,

$$\Phi(R) = -\frac{1}{2}R\ \vec{v}\cdot\vec{n} \tag{14.13}$$

and

$$E_{\text{kin}}(F) = \frac{1}{2}\rho_F \frac{R}{2}\int_C d^2\sigma\,(\vec{v}\cdot\vec{n})^2\,. \tag{14.14}$$

By homogeneity and rotational invariance

$$\int_C d^2\sigma\ \ (\vec{v}\cdot\vec{n})^2 = \frac{1}{3}|\vec{v}|^2\int_C d^2\sigma\ \ = \frac{1}{3}|\vec{v}|^2\ \ 4\pi R^2$$

and

$$E_{\text{kin}}(F) = \frac{1}{2}\rho_F\ \text{vol}(C)\ \ \frac{1}{2}|\vec{v}|^2 \tag{14.15}$$

with $\text{vol}(C) = (4/3)\pi R^3$. Finally the total kinetic energy (14.6) is

$$E_{\text{kin}} = \left(\rho_C + \frac{1}{2}\rho_F\right)\ \text{vol}(C)\ \ \frac{1}{2}|\vec{v}|^2\,. \tag{14.16}$$

The mass density ρ_C is renormalized $\rho_C + \frac{1}{2}\rho_F$ by immersion.

In conclusion, one can say that the bare Lagrangian

$$\int_C d^3x\ \ \left(\frac{1}{2}\rho_C|\vec{v}|^2 - \rho_C V(x)\right) \tag{14.17}$$

is renormalized by immersion to

$$\int_C d^3x\ \ \left(\frac{1}{2}\left(\rho_C + \frac{1}{2}\rho_F\right)\ \ |\vec{v}|^2 - (\rho_C - \rho_F)\ \ V(x)\right). \tag{14.18}$$

Green's calculation for a vibrating ellipsoid in a fluid media leads to a similar conclusion. As in (14.16) the density of the vibrating body is increased by a term proportional to the density of the fluid, obviously more complicated than $\rho_F/2$. From Archimedes' principle, to Green's example, to the Lagrangian formulation, one sees the evolution of the concepts used in classical mechanics: Archimedes gives the *force* felt by an immersed solid, Green gives the harmonic *motion* of an ellipsoid in fluid media, and the Lagrangian gives the kinetic and the potential *energy* of the system. In quantum physics, action functionals defined

in infinite dimensional spaces of functions play a more important role than Lagrangians or Hamiltonians defined on finite dimensional spaces. Renormalization is often done by renormalizing the lagrangian. Renormalization in quantum field theory by scaling (section 16), on the other hand, uses the action functional.

14.3 From path to field integrals

Before applying functional integrals to renormalization in Quantum Field Theory, we review briefly key features of field integrals. Two lessons from path integrals can be applied readily to field integrals:

- Time-ordering by path integrals
- Gaussian properties of path integrals.

In quantum mechanics, one integrates functionals of functions x defined on a line $\mathbb{T} = [t_a, t_b]$; the integrals represent either solutions of Schrödinger's equation (see section 6) or matrix elements of operators on Hilbert spaces. For example:

$$\langle b, t_b | T(\mathbf{x}(t)\mathbf{x}(s)) | a, t_a \rangle_Q = \int_{\mathbb{X}_{a,b}} \mathcal{D}_{s,Q}(x) \exp\left(\frac{2\pi i}{h} Q(x)\right) x(t)x(s) \tag{14.19}$$

where x is a function and $\mathbf{x}(t)$ is an operator operating on the state $|a, t_a\rangle$. The matrix element on the l.h.s. is the probability amplitude that a system (action functional Q) in the state a at time t_a be found in the state b at time t_b. This path integral gives the 2-point function of the Gaussian Wiener integral (see section 3.1 and eq. 2.47) when $s = 1$, $h = 1$, $Q(x) = \frac{1}{2}\int_{\mathbb{T}} dt \dot{x}(t)\dot{x}(s)$:

$$\langle 0, t_b | T(\mathbf{x}(t)\mathbf{x}(s)) | 0, t_a \rangle = \begin{cases} t - t_a & \text{if } t < s \\ s - t_a & \text{if } s < t \end{cases} . \tag{14.20}$$

The path integral on the r.h.s. of (14.19) has time ordered the matrix element on the l.h.s. of (14.19).

In field theory time ordering becomes causal ordering dictated by light cones: if a point x_j is in the future light cone of x_i, written

$$j \succcurlyeq i,$$

then the causal (time, chronological) ordering T of the field operators $\boldsymbol{\phi}(x_j)\boldsymbol{\phi}(x_i)$ is the symmetric function

$$T(\boldsymbol{\phi}(x_j)\boldsymbol{\phi}(x_i)) = T(\boldsymbol{\phi}(x_i)\boldsymbol{\phi}(x_j)) \tag{14.21}$$

which satisfies the equation

$$T(\boldsymbol{\phi}(x_j)\boldsymbol{\phi}(x_i)) = \begin{cases} \boldsymbol{\phi}(x_j)\boldsymbol{\phi}(x_i) & \text{for } j \succcurlyeq i \\ \boldsymbol{\phi}(x_i)\boldsymbol{\phi}(x_j) & \text{for } i \succcurlyeq j \end{cases} .$$

In general

$$\langle \text{out}|T(\mathcal{F}(\boldsymbol{\phi}))|\text{in}\rangle_S = \int_{\Phi_{\text{in,out}}} \mathcal{F}(\phi) v_S(d\phi) \tag{14.22}$$

where $\mathcal{F}$ is a functional of the field ϕ (or a collection of fields) and $v_S(d\phi)$ is a volume element that we shall characterize in the following.

Volume Elements

Recall the Gaussian volume element

$$d\gamma_{s,Q}(x) \stackrel{\int}{=} \mathcal{D}_{s,Q}(x) \exp\left(-\frac{\pi}{s}Q(x)\right) \tag{14.23}$$

where $\mathcal{D}_{s,Q}(x)$ has the following properties

- $\mathcal{D}_{s,Q}$ is translation invariant
- $\mathcal{D}_{s,Q}$ has no physical dimension. If $Q(x) = \sum Q_{ij}x^i x^j$ then

$$\mathcal{D}_{s,Q}(x) = (\det Q_{ij})^{1/2} \prod_{n=1}^{D} dx^n \quad \text{for} \quad x \in I\!R^D. \tag{14.24}$$

The guiding principles for determining the volume element $v_S(d\phi)$ are:

- Schwinger's variational principle (1.11)

$$\delta(A|B) = 2\pi i \langle A|\delta S/h|B\rangle, \quad h = 2\pi\hbar. \tag{14.25}$$

- Dirac's quantum analogue of the classical action function (a.k.a Hamilton's principal function) adds to the real classical action function an imaginary part of order $\hbar$ [I.3]. Correspondingly the quantum action

functional $S_{\rm qu}$ is, up to terms of order $\hbar^2$, the sum of the classical action functional $S_{\rm cl}$ and an imaginary term of order $\hbar$

$$\begin{aligned} \frac{1}{\hbar}S_{\rm qu} &= \frac{1}{\hbar}S_{\rm cl} + i\sigma + O(\hbar) && (14.26)\\ &= \frac{1}{\hbar}S_{\rm cl} - i\log\mu + O(\hbar) \quad \mu := \exp(-\sigma) + O(\hbar). && (14.27)\end{aligned}$$

- A volume element $v(d\phi)$ convenient for integration by parts, namely,

$$v(d\phi) = \exp A(\phi)[d\phi] \tag{14.28}$$

where $[d\phi]$ is translation invariant; i.e. $[d\phi]$ is characterized by

$$\int_{\mathbf{\Phi}} \frac{\delta F(\phi)}{\delta\phi(x)}[d\phi] = 0. \tag{14.29}$$

Henceforth the quantum action functional

$$S\left(\phi\right) := S_{\rm cl}\left(\phi\right) - \imath\hbar\ \log\mu\left(\phi\right) + O\left(\hbar^2\right); \tag{14.30}$$

Guided by the characterization of volume elements in path integrals, we propose the following characterization of $v_{_S}(\phi)$ for a system of quantum action functional S

$$\int_{\Phi}\left(\frac{\delta F\left(\phi\right)}{\delta\phi\left(x\right)} + \frac{2\pi\imath}{h}F\left(\phi\right)\frac{\delta S\left(\phi\right)}{\delta\phi\left(x\right)}\right)\ v_{_S}\left(d\phi\right) = 0. \tag{14.31}$$

where

$$\delta F\left(\phi\right) = \int \frac{\delta F\left(\phi\right)}{\delta\phi\left(x\right)}\ \delta\phi\left(x\right)\ d^Dx,\ x \in I\!R^D. \tag{14.32}$$

This equation gives a characterizations of $v_{_S}\left(d\phi\right)$ in terms of the unknown functional $\mu\left(\phi\right)$ as desired in the guiding principle (14.26):

$$\begin{aligned} v_{_S}\left(d\phi\right) &= \mu\left(\phi\right)\ \exp\left(\frac{2\pi\imath}{h}S_{\rm cl}\left(\phi\right)\right)[d\phi] && (14.33)\\ &= \exp\left(\frac{2\pi\imath}{h}S_{\rm qu}\left(\phi\right)\right)[d\phi]\,. \end{aligned}$$

The characterization (14.33) of the volume element $v_{_S}\left(d\phi\right)$ is similar to the characterization of the volume element $v_g\left(x\right)$ on a $D-$dimensional Riemannian manifold with metric g given in section 7.4.

Application $\mathbf{F = 1}$

Equation (14.31) says

$$\int_{\Phi} \frac{\delta S(\phi)}{\delta \phi(x)} v_S(d\phi) = 0 \tag{14.34}$$

which translates (14.22) in terms of the matrix element of a time ordered operator to

$$0 = \int_{\Phi_{\text{in, out}}} \frac{\delta S(\phi)}{\delta \phi(x)} v_S(d\phi) = \left\langle \text{out} \left| T \frac{\delta S(\phi)}{\delta \phi(x)} \right| \text{in} \right\rangle. \tag{14.35}$$

Eq. (14.35) is a particular case of Schwinger's variational principle (14.25) when the initial and final states are kept fixed. It is the quantum version of the classical equation of motion

$$\frac{\partial S_{\text{cl}}}{\partial \phi(x)} = 0 \tag{14.36}$$

where the classical action S_{cl} is the limit of the quantum action when $\hbar \to 0$

$$S = S_{\text{cl}} - \imath\hbar \log \mu + O\left(\hbar^2\right). \tag{14.37}$$

When $F \neq 1$, the term ∂F can be used to account for the variation of $|A\rangle$ and $|B\rangle$ in a matrix element $\langle B|M|A\rangle$ represented by a functional integral. Most often we choose the domain of integration $\Phi_{A,B}$ of the functional integral to be fixed; but variations of A and B are also interesting. In which case we write

$$\int_{\Phi_{A,B}} = \int_{\Phi} \text{appropriate characteristic functions},$$

and insert appropriate terms for characterizing $|A\rangle$ and $|B\rangle$.

Application $\boldsymbol{F(\phi) = \exp(2\pi\imath \langle J, \phi \rangle)}$

Let

$$Z(J) := \int_{\Phi} v_S(d\phi) \exp(-2\pi\imath \langle J, \phi \rangle) \tag{14.38}$$

with $v_S(d\phi)$ given by (14.16) or equivalently

$$v_S(d\phi) = \exp\left(\frac{2\pi\imath}{\hbar} S_{\text{qu}}(\phi)\right)[d\phi] \tag{14.39}$$

or

$$v_S(d\phi) = \exp\left(\frac{2\pi\imath}{\hbar} S_{\text{cl}} + \log \mu\right)(x)\,[d\phi]. \tag{14.40}$$

It follows that

$$0 = \frac{\partial Z(J)}{\partial \phi(x)} = \int_{\Phi} \left(\frac{2\pi\imath}{\hbar} \frac{\partial S_{\mathrm{cl}}(\phi)}{\partial \phi(x)} + \frac{1}{\mu} \frac{\partial \mu(\phi)}{\partial \phi(x)} - 2\pi\imath J(x) \right) \times \exp\left(-2\pi\imath \langle J, \phi \rangle\right) v_S(d\phi). \tag{14.41}$$

For an action which is the sum of a quadratic form $-\frac{1}{2}\langle D\phi, \phi\rangle$ and a polynomial $\mathcal{P}(\phi)$ of order higher than 2

$$\frac{\partial S_{\mathrm{cl}}(\phi)}{\partial \phi(x)} = -D_x\phi + \mathcal{P}'(\phi).$$

The differential D_x can be taken outside the functional integral, and the property of Fourier transforms

$$\int_{\Phi} v_S(d\phi)\, \phi(x) \exp\left(-2\pi\imath \langle J, \phi\rangle\right) = \int v_S(d\phi) \frac{-1}{2\pi\imath} \frac{\partial}{\partial J(x)} \exp\left(-2\pi\imath \langle J, \phi\rangle\right)$$

used to derive a functional differential equation for $Z(J)$:

$$J(x) Z(J) + \frac{1}{2\pi\imath} D_x \frac{\partial}{\partial J(x)} Z(J) + \mathcal{P}'\left(\frac{1}{2\pi\imath} \frac{\partial}{\partial J(x)}\right) Z(J) = 0.$$

$Z(J)$ can be normalized by dividing it by $Z(0)$. Let

$$Z(J)/Z(0) =: \exp\left(-\frac{2\pi\imath}{h} W(J)\right). \tag{14.42}$$

It has been shown [Ryder] that the diagram expansion of $Z(J)$ contains disconnected diagrams, but that the diagram expansion of $\exp W(J)$ contains only connected diagrams.

TO BE COMPLETED (SEE CECILE'S NOTES AND CARTIER SCH 5, SCH 6, SCH 7.)