Hodge Theory

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Lecture 1

1.1 Some Sample Theorems

We begin with the broad goal of gaining an understanding of the topology and geometry of complex algebraic varieties.

Let $\mathbf{P}^n$ denote complex n-dimensional projective space, which is the space of lines in $\mathbf{C}^{n+1}$ and which can also be thought of as the union of $\mathbf{C}^n$ and an a copy of $\mathbf{P}^{n-1}$. In the case $n = 1$, the latter interpretation gives us the union of $\mathbf{C}$ and a point at infinity ∞, and we obtain the Riemann sphere.

We consider algebraic varieties inside $\mathbf{P}^n$. These may be hypersurfaces defined by $F(x_0, \ldots, x_n) = 0$. For example, we have the smooth Fermat hypersurface given by the zero locus of $x_0^d + \ldots + x_n^d = 0$ in $\mathbf{P}^n$. On the other hand, the cone defined by $x_1^d + \ldots + x_n^d = 0$ in $\mathbf{P}^n$ is singular. The latter two examples are both projective varieties, defined by homogeneous equations in projective space. An algebraic variety may also be open, obtained for example by removing a finite number of points from a projective variety.

In this section, X will denote a smooth complex projective variety. That is, X is both a smooth complex manifold and a projective algebraic variety. On such manifolds, we have homology and cohomology groups. In homology, our classes are generally defined by cycles, such as geodesics or algebraic cycles. In cohomology, the classes come from differential forms.

Our theme will be to look for differential forms of minimum size. They will be unique and have special properties, allowing one to prove powerful results. For example, we will be able to give restrictions on the topology of algebraic varieties. Below, we list some sample theorems and hint at their proofs.

The dth Betti number $b_d(X)$ of X is defined as the rank of $H_d(X, \mathbf{Z})$.

Theorem 1.1. *The first Betti number $b_1(X)$ is even.*

Theorem 1.1 does not hold for all manifolds. For example, consider the *Hopf surface* $S = (\mathbf{C}^2 - 0)/\mathbf{Z}$ where the action of $n \in \mathbf{Z}$ on $z \in \mathbf{C}^2 - 0$ is given by $n \cdot z = 2^n z$.

Exercise. Show that S is homeomorphic to $S^1 \times S^3$ and use the Künneth formula to conclude that $b_1(S) = 1$.

Corollary 1.2. *The Hopf surface S cannot be defined by algebraic equations.*

We have the following corollary, the proof of which we also leave as an exercise.

Corollary 1.3. *The fundamental group $\pi_1(X)$ is not a free group.*

Exercise. Prove the corollary. *Hint:* Suppose that $\pi_1(X)$ is a free group of rank $n > 1$. Consider a two-fold cover $\tilde{X}$ of X. What is $b_1(\tilde{X})$? Use the fact that is X is a complex projective algebraic manifold then so is $\tilde{X}$.

Theorem 1.4. *The second Betti number $b_2(X)$ is nonzero.*

Theorem 1.5. *Let $M \subset X$ be an algebraic submanifold of complex dimension m. Then $[M] = 0$ in $H_{2m}(X, \mathbf{Z})$.*

Theorem 1.5 does not hold for an arbitrary submanifold. For instance, if X is is a two-holed torus then M may have a geodesic which is trivial in homology.

1.2 Differential forms and the Hodge Decomposition

Let $\mathcal{E}^k = \sum \phi_I dx_I$ be the space of differential k-forms. That is, if $I \subset \{1, \ldots, n\}$ is a subset of k elements then we have a k-form $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$. We have the exterior derivative $d\phi = \sum \frac{\partial \phi_I}{\partial x_k} dx_k \wedge dx_I$. Note that $d^2 = 0$. The kth de Rham cohomology group $H^k_{dR}(X)$ is equal to closed k-forms, i.e. satisfying $d\phi = 0$, modulo exact k-forms, i.e. those in $d\mathcal{E}^{k-1}$. The de Rham isomorphism theorem says that de Rham cohomology is isomorphic to singular cohomology $H^*_{dR}(X) \cong H^*(X, \mathbf{C})$ via the map $\phi \mapsto [c \mapsto \int_C \phi]$.

For complex manifolds, it is more convenient to use complex coordinates. From $z_k = x_k + iy_k$ we obtain 1-forms dz_k and $d\bar{z}_k$.

Exercise. Show that the subspace of 1-forms spanned by the dz_k (and also the subspace spanned by the $d\bar{z}_k$) is invariant under complex change of coordinates.

We then have a basis of k-forms $dz_I \wedge d\bar{z}_J$, and if I has order p and J has order q, the form is said to be of type (p, q). The general form of type (p, q) can then be written $\sum \phi_{IJ} dz_I \wedge d\bar{z}_J$ with I of order p, J of order q, and $\phi_{IJ} \in C^\infty(X)$. Letting $\mathcal{E}^{p,q}$ denote the space of (p, q)-forms, we have a direct sum decomposition

$$\mathcal{E}^k = \oplus_{p+q=k} \mathcal{E}^{p,q}.$$

For example, $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$, which is to say that every $\phi \in \mathcal{E}^1$ has a unique expression of the form $\phi = \sum f_i dz_i + \sum g_i d\bar{z}_i$. Note that $\mathcal{E}^{p,q} = \bar{\mathcal{E}^{q,p}}$.

We can ask the question of whether or not our direct sum decomposition passes to cohomology. That is, if we let H^{pq} denote the image of $\mathcal{E}^{p,q}$ in H^k_{dR} with $k = p+q$, do we have a *Hodge decomposition* $H^k_{dR} = \oplus H^{p,q}$? In fact, do we even have that $H^{p,q} = \bar{H^{q,p}}$? The answer in general is no, because this would force b_1 to be even, whereas $b_1 = 1$ for the Hopf manifold. However, if we think about harmonic theory, we can find conditions under which we do have a Hodge decomposition. In particular, this will immediately imply Theorem 1.1.

1.3 Harmonic Theory

Let M denote a smooth Riemannian manifold with metric $g = \langle \cdot\,, \cdot \rangle$ on TM. Since a basis of T_pM gives rise to a dual basis on T^*_pM, we obtain a metric on $\Lambda^k T^*M$. This gives rise to an inner product on $\mathcal{E}^k$ via

$$(\phi, \psi) = \int_M \langle \phi, \psi \rangle dV,$$

where dV denotes the volume form on M. This allows us to define a norm on $\mathcal{E}^k$ given by $\|\phi\| = (\phi, \phi)$.

We seek closed forms of minimum norm in $[\phi]$. We note that

$$\|\phi + td\psi\|^2 = \|\phi\|^2 + 2t(\phi, d\psi) + O(t^2).$$

Therefore, if $\|\phi\|^2$ is a minimum then $(\phi, d\psi) = 0$ for any $\psi \in \mathcal{E}^{k-1}$. Suppose d has an adjoint d^*, so we have $(d^*\phi, \psi) = 0$ for all ψ. In other words, we have $d^*\phi = 0$. Thus, a closed form of minimum norm satisfies both $d\phi = 0$ and $d^*\phi = 0$. It therefore also satisfies Laplace's equation $\Delta\phi = 0$, where $\Delta = dd^* + d^*d$ is the self-adjoint operator associated to d. That is to say, ϕ is harmonic.

Exercise. Show that if $\Delta\phi = 0$ then $d\phi = 0$ and $d^*\phi = 0$ and that $\|\phi\|^2$ is minimal.

Exercise. Let $M = S^1$. Show that for $(f, g) = \int_0^{2\pi} fg d\theta$, we have $\Delta f = -\frac{d^2 f}{d\theta^2}$. What are the harmonic functions on the circle? Answer this question again using harmonic series. What about for $S^1 \times \cdots \times S^1$?

Theorem 1.6 (Harmonic Theorem). *Let H^k_Δ denote the space of harmonic k-forms. Then the natural map $H^k_\Delta M \to H^k_{dR} M$ is an isomorphism.*

1.4 The Laplacian on a Complex Manifold

We now study the Laplacian Δ on a complex manifold. The operator d can be written as a sum $d = \partial + \bar\partial$, where ∂ is an operator of type $(1,0)$, $\bar\partial$ is an operator of type $(0,1)$, and ∂ and $\bar\partial$ satisfy the relations $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial + \bar\partial\partial = 0$. In coordinates, on a 1-form $\phi = \sum f_i dz_i + \sum g_j d\bar z_j$, we have

$$\begin{aligned}\partial\phi &= \sum \frac{\partial f_i}{\partial z_k} dz_k \wedge dz_i + \sum \frac{\partial g_j}{\partial z_k} dz_k \wedge d\bar z_j,\\ \bar\partial\phi &= \sum \frac{\partial f_i}{\partial \bar z_k} d\bar z_k \wedge dz_i + \sum \frac{\partial g_j}{\partial \bar z_k} d\bar z_k \wedge d\bar z_j.\end{aligned}$$

The operators ∂ and $\bar\partial$ have corresponding adjoints ∂^* and $\bar\partial^*$ of type $(-1,0)$ and $(0,-1)$, respectively, and we have the relation $d^* = \partial^* + \bar\partial^*$. Mimicking the relation, $\Delta = dd^* + d^*d$, we set

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \qquad \text{and} \qquad \Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

We then have

$$\Delta = (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) = \Delta_\partial + \Delta_{\bar\partial} + \text{ cross terms.} \tag{1}$$

Suppose that the cross terms in (1) vanish, so that $\Delta = \Delta_\partial + \Delta_{\bar\partial}$ is an operator of type $(0,0)$. If $\phi = \sum \phi_{p,q}$ is harmonic, then so are the components $\phi_{p,q}$. Moreover, if ϕ is real then we have $\overline{\phi_{p,q}} = \phi_{q,p}$. Therefore, we obtain a Hodge decomposition on the space of harmonic k-forms: $H^k_\Delta = \oplus_{p+q=k} H^{p,q}$ and $\overline{H^{p,q}_\Delta} = H^{q,p}_\Delta$.

Lecture 2

2.1 Kähler Manifolds

A *Kähler manifold* is a complex manifold M with a Riemannian metric compatible with the complex structure on M. We will see that the cross-terms in (1) vanish for Kähler manifolds. Hence, Kahler manifolds have Hodge decompositions. One way of defining a complex structure on X is via an operator J on the tangent bundle TM, which satisfies $J^2 = -1$, $J(dx) = dy$ and $J(dy) = -dx$. Let $g = \langle \cdot\,,\, \cdot \rangle$ denote the Riemannian metric on a Kahler manifold M, and let ∇ denote the Levi-Cevita connection. By compatibility with the complex structure, we mean one of two equivalent statements:

1. $\nabla J = 0$, which is to say that $\nabla(JX) = J\nabla X$ for a vector field X.
2. Set $\omega(X,Y) = g(X, JY)$. Then ω is a positive definite $(1,1)$-form and $d\omega = 0$.

Both $\mathbf{C}^n$ and $\mathbf{P}^n$ are Kahler manifolds, with

$$\omega = i \sum dz_k \wedge d\bar{z}_k$$

and

$$\omega = i\partial\bar{\partial} \log(1 + \|z\|^2),$$

respectively. As a consequence, all projective algebraic manifolds are Kähler.

Exercise. Defining ω as in condition 2 (and assuming condition 1 in part c), show that

a. $\omega(X, Y) = -\omega(Y, X)$

b. ω is a $(1,1)$-form. Hint: Show that $\omega(X, Y) = 0$ if X and Y are both of type $(1,0)$.

c. ω is locally of the form

$$\omega = \sqrt{-1} \sum h_{i,j} dz_i \wedge d\bar{z}_j,$$

where $h_{i,j}$ is locally Hermitian and positive definite. Conclude that $dw = 0$.

2.2 The Kähler Identity and Positivity Considerations

We define an operator L of type $(1,1)$ by $L(x) = \omega \wedge x$ and denote its adjoint by Λ. By definition, we have $(Lx, y) = (x, \Lambda y)$. Later, we shall prove the following *Kähler identity*:

$$[\Lambda, \partial] = i\bar{\partial}^*.$$

It follows immediately from this identity that $[\Lambda, \bar{\partial}] = -i\partial^*$ as well. Now it is not hard to see that the cross-terms in (1) vanish. For instance,

$$\partial\bar{\partial}^* + \bar{\partial}^*\partial = -\sqrt{-1}(\partial\Lambda\partial - \partial^2\Lambda + \Lambda\partial^2 - \partial\Lambda\partial) = 0.$$

Exercise. The following identities hold for a compact Kähler manifold: $\Delta_\partial = \Delta_{\bar{\partial}}$ and $\Delta = 2\Delta_\partial$. (We refer to these identities as *proportionality*).

As a consequence of the Hodge decomposition resulting from the vanishing of the cross-terms in (1), we see that the first Betti number is even. Therefore, Theorem 1.1

holds for all Kähler manifolds. We shall now see that Theorem 1.4 holds as well. On a small local neighborhood U, we may consider $\omega = i \sum dz_k \wedge d\bar{z}_k$. We define

$$\Omega = (idz_1 \wedge d\bar{z}_1) \wedge (idz_2 \wedge d\bar{z}_2) \wedge \ldots \wedge (idz_n \wedge d\bar{z}_n)$$

Note that $\Omega > 0$ in the sense that $\int_U \Omega > 0$. As $\omega^n = n!\Omega$, we conclude that $\int_M \omega^n > 0$. Hence the class $[\omega^i] \neq 0$ for all i with $0 \leq i \leq n$. In particular, the second Betti number $b_2(M)$ is nontrivial.

We now prove the Kähler identity for $(1,1)$-forms.

Idea of proof of the Kähler Identity. We first describe local formulas for Λ, $\bar{\partial}^*$ and ∂. Note that

$$(1, \Lambda(dz_k \wedge d\bar{z}_k)) = (L(1), dz_k \wedge d\bar{z}_k) = (\sqrt{-1} \sum dz_l \wedge d\bar{z}_l, dz_k \wedge d\bar{z}_k) = \sqrt{-1}.$$

So we obtain the formula

$$\Lambda = -\sqrt{-1} \sum \operatorname{Int}(\frac{\partial}{\partial z_k}) \operatorname{Int}(\frac{\partial}{\partial \bar{z}_k}), \tag{2}$$

where Int denotes interior multiplication. Next, we check that

$$(g, \bar{\partial}^*(f d\bar{z}_k)) = (\bar{\partial} g, f d\bar{z}_k) = \int \frac{\partial g}{\partial \bar{z}_k} \bar{f} = -\int g \overline{\frac{\partial f}{\partial z_k}} = -(g, \frac{\partial f}{\partial z_k}).$$

Hence we have seen that

$$\bar{\partial}^*(f dz_1) = \sum \frac{\partial f}{\partial z_l} dz_l \wedge dz_1. \tag{3}$$

Finally, we note that

$$\partial = \sum \operatorname{Ext}(dz_k) \frac{\partial}{\partial z_k}, \tag{4}$$

where Ext denotes exterior multiplication. Putting together equations (2), (3) and (4), we see that

$$\Lambda \partial (f d\bar{z}_1) = \Lambda (\sum \frac{\partial f}{\partial z_k} dz_k \wedge d\bar{z}_k) = -\sqrt{-1} \frac{\partial f}{\partial z_k} = \sqrt{-1} \bar{\partial}^*(f d\bar{z}_1)$$

and $\partial \Lambda (f d\bar{z}_1) = 0$, from which follows the Kähler identity on (1,1)-forms. □

2.3 Consequences of Proportionality

The fact that $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$ on a Kähler manifold has interesting consequences.

Theorem 2.7. *On a compact Kähler manifold, holomorphic p-forms are closed.*

The motivation for the proof of this theorem comes from the case of a Riemann surface. Let $\phi = f\,dz$, where f is holomorphic. Then

$$d\phi = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = 0.$$

Proof. Let ϕ be a holomorphic p-form. Then $\bar\partial\phi = 0$. We also have $\bar\partial^*\phi = 0$ as ϕ has type $(p, 0)$ and $\bar\partial^*$ is an operator of type $(0, -1)$. Hence $\Delta_{\bar\partial}\phi = 0$ and, by proportionality, $\Delta\phi = 0$. Note also that

$$(\Delta\phi, \phi) = (dd^*\phi + d^*d\phi, \phi) = \|d^*\phi\|^2 + \|d\phi\|^2.$$

Therefore we conclude that $d\phi = 0$. □

Let Ω denote the space of holomorphic p-forms on M.

Theorem 2.8 (Dolbeaut Theorem). *We have $H^{p,q}(X) \cong H^q(X, \Omega^p)$.*

Proof. Let $\mathcal{E}^{p,q}$ denote the space of differential (p, q)-forms on M. The qth cohomology group $H^q(\Omega^p)$ of the complex

$$\Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \mathcal{E}^{p,2} \xrightarrow{\bar\partial} \cdots$$

is equal to the group of $\bar\partial$-closed (p, q) forms modulo exact forms. As in the proof of the Harmonic Theorem, this is isomorphic to the group of $\Delta_{\bar\partial}$-harmonic (p, q)-forms. By proportionality, this is the same as the group of harmonic (p, q)-forms. □

Lecture 3

3.1 Algebraic Cycles and the Hodge Conjecture

Let Z denote an algebraic submanifold of dimension k of a complex projective algebraic manifold X of dimension m. We have the following two statements about the integrals of closed $2k$-forms over Z.

Exercise. Prove the following two statements.

1. The class $[Z]$ is nontrivial in $H_{2k}(X, \mathbf{Z})$. In fact, the integral $\int_Z \omega^k$ is positive.

2. Let ϕ be a representative of an algebraic class on X of type (p, q) with $p+q = 2k$. Then $\int_Z \phi = 0$ if $p \neq q$.

The Hodge Conjecture is the converse of the second statement of Exercise 3.1. Note that if c is an algebraic class, then c can be represented as a sum of integer multiples of algebraic cycles, so we can define $\int_c$ by the appropriate sum.

Conjecture 3.9 (Hodge Conjecture). *If c is a nonzero class in $H_{2m}(X, \mathbf{Z})$ and $\int_c \phi = 0$ for every closed form ϕ with $[\phi] \in H_{\mathbf{Z}}^{p,q} = H^{p,q}(X, \mathbf{Z})$ for some $(p, q) \neq (m, m)$ then some multiple of c is represented by an algebraic cycle.*

We may rephrase the Hodge conjecture using Poincaré duality. If Z is an algebraic cycle of complex dimension k and ϕ is a closed form of type $(n-k, n-k)$ then $[\phi]$ is said to be Poincaré dual to $[Z]$ if

$$\int_Z \psi = \int_X \phi \wedge \psi$$

for all $2k$-forms ψ. The Hodge conjecture thus states that every integral $[\phi]$ is Poincaré dual to an algebraic homology class.

The Hodge Conjecture has been proven in certain cases.

Theorem 3.10. *The Hodge conjecture is true in dimension and codimension 1.*

We first prove Theorem 3.10 in dimension 1.

Proof of the Hodge Conjecture in dimension 1. Let $\phi \in H_{\mathbf{Z}}^{1,1}$. We shall construct a line bundle L such that $\phi = c_1(L)$, the first Chern class of L. Then Z will be taken as the divisor of a meromorphic section of L.

The exponential sheaf sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_X \xrightarrow{\exp 2\pi\sqrt{-1}} \mathcal{O}_X^* \to 1$$

yields a long exact sequence in cohomology containing the sequence

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X). \tag{5}$$

The first map in (5) takes a line bundle to its Chern class. We remark that $H^2(\mathcal{O}_X) \cong H^{0,2}(X)$ by Theorem 2.8.

We leave it to the reader to check that the following diagram commutes:

$$\begin{array}{ccc} H^2(X, \mathbf{Z}) & \longrightarrow & H^2(X, \mathcal{O}_X) \\ \downarrow & & \| \\ H^2(X, \mathbf{C}) & \longrightarrow & H^{0,2}, \end{array} \tag{6}$$

where the bottom map is projection. Noting also equation (5), we see by (6) that any $\phi \in H^{1,1}_{\mathbf{Z}}$ maps trivially to $H^2(\mathcal{O}_X)$ and therefore is the image of the Chern class of some line bundle.

Finally, we apply the fact that if D is a divisor of a meromorphic section of a line bundle L, then $\int_D = c_1(L)$. That is, for any $2n-2$ form ψ, we have

$$\int_D \psi = \int_X \alpha \wedge \psi$$

where α is a $(1,1)$-form of class $c_1(L)$. This follows from an explicit description of α and an application of Stokes' theorem. □

To prove Theorem 3.10 in codimension 1, we shall need the Lefschetz Theorem. We define H^k_0, the primitive cohomology of dimension k, to be the kernel of the map $L^{n-k+1} \colon H^k \to H^{2n-k+2}$ given by the exterior product with ω^{n-k+1}.

Theorem 3.11 (Lefschetz Theorem). *Let X be an n-dimensional compact Kähler manifold.*

1. *The map L^{n-k} is an isomorphism for $k \leq n$.*
2. *There is a direct sum decomposition*

$$H^k \cong H^k_0 \oplus LH^{k-2}_0 \oplus \dots .$$

3. *If ϕ is a primitive cohomology class, then so are its Hodge components $\phi_{p,q}$.*

We shall not prove the Lefschetz Theorem, but we will try to illustrate the ideas with key examples. We remark that statement 3 follows from the fact that L has type $(1,1)$.

First, we reduce the problem to the level of forms by proving that $[\Delta, L] = 0$. It suffices to show that $[L, \Delta_\partial] = 0$ by proportionality. We remark first that $[L, \partial] = 0$ and $[L, \bar{\partial}] = 0$. Therfore, we see that

$$\begin{aligned}
[L, \Delta_\partial] &= [L, \partial\partial^*] + [L, \partial^*\partial] \\
&= L\partial\partial^* - \partial\partial^* L + L\partial^*\partial - \partial^*\partial L \\
&= \partial(L\partial^* - \partial^* L) + (L\partial^* - \partial^* L)\partial \\
&= \partial[L, \partial^*] + [L, \partial^*]\partial.
\end{aligned}$$

The Kähler identity yields that $[\Lambda^*, \partial^*] = \sqrt{-1}\bar{\partial}$ and therefore $[L, \partial^*] = \sqrt{-1}\bar{\partial}$. We therefore conclude that

$$[L, \Delta_\partial] = \sqrt{-1}(\partial\bar{\partial} + \bar{\partial}\partial) = \sqrt{-1}(d^2 - \partial^2 - \bar{\partial}^2) = 0.$$

Exercise. Show that the identity $[L, \Lambda] = -1$ holds on $\mathcal{E}^2$ for $n = 3$.

Proof of Theorem 3.11a for $n = 3$ and $k = 2$. Since $\mathcal{E}^2$ and $\mathcal{E}^4$ are vector bundles of the same fiber dimension, it suffices to show that L is injecitve. Assume that $L\alpha = 0$ for some $\alpha \in \mathcal{E}^2$. Then $[L, \Lambda]\alpha = -\alpha$ implies that $L\Lambda\alpha = -\alpha$. That is, there exists a $\beta \in \mathcal{E}^0$ with $L\beta = \alpha$. Finally, note that

$$0 = L^3\beta = \omega^3 \wedge \beta.$$

Since ω^3 is the volume form, we have $\beta = 0$. Hence $\alpha = 0$, and first statement of the Lefschetz Theorem is proven in this case. $\square$

Proof of Theorem 3.11b for $n = 3$ and $k = 3$. Let $\alpha \in \mathcal{E}^3$. Then $L\alpha \in \mathcal{E}^5$, so $L\alpha = L^2\alpha_1$ for some $\alpha_1 \in \mathcal{E}^1$ by Theorem 3.11a for $n = 3$ and $k = 2$. Then

$$L(\alpha - L\alpha_1) = 0,$$

so $\alpha_0 = \alpha - L\alpha_1$ is primitive. Therefore

$$\alpha = \alpha_0 + L\alpha_1$$

with α_0 primitive. But $L^3\alpha_1 = 0$ as it is in $\mathcal{E}^7$, so α_1 is primitive as well. Hence a primitive decomposition exists.

Now suppose that $\alpha_0 + L\alpha_1 = 0$. Applying L, we see that $L^2\alpha_1 = 0$, so $\alpha_1 = 0$ by the first part of the Lefschetz Theorem for $n = 3$ and $k = 2$. Thus $\alpha_0 = 0$ as well, and the decomposition is unique. $\square$

We may now prove the Hodge conjecture in codimension 1.

Proof of the Hodge Conjecture in codimension 1. We shall show that any $\alpha \in H^{n-1,n-1}_{\mathbf{Z}}$ is Poincaré dual to an algebraic homology class. By the Lefschetz Theorem, we have that $\alpha = L^{n-2}\beta$ with $\beta \in H^{1,1}_{\mathbf{Q}}$. By the Hodge Conjecture in dimension 1, we have that β is Poincaré dual to a divisor D. Then α is Poincaré dual to the intersection class of D with $n - 2$ hyperplane sections, as we recall that ω is given by intergrating over a hyperplane. $\square$

Lecture 4

4.1 Hodge-Riemann Bilinear Relations

The Hodge Riemann bilinear relations deal with the behavior of cohomology of a Kähler manifold X with respect to the bilinear form Q defined on H^k by

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta \wedge \omega^{n-k}.$$

The form Q is skew-symmetric if k is odd and symmetric if k is even. We also define a bilinear form h on H^k by

$$h(\alpha,\beta) = \epsilon Q(\alpha,\bar{\beta})$$

where

$$\epsilon = \begin{cases} \sqrt{-1} & k \text{ odd} \\ 1 & k \text{ even.} \end{cases}$$

Theorem 4.12 (HRB). *Let X be a compact Kähler manifold, and let Q and h be as above.*

1. *We have $Q(\alpha,\beta) = 0$ if $\alpha \in H^{p,q}$ and $\beta \in H^{a,b}$ with $(a,b) \neq (q,p)$.*
2. *The bilinear form h is definite on $H_0^{p,q}$ and its sign alternates as q increases (positive on $H_0^{k,0}$).*

The relation HRB1 follows immediately from the type of the forms.

Exercise. Show that the Lefschetz decomposition is h-orthogonal and that, as a consequence of the Hodge-Riemann bilinear relations, Q is nondegenerate on H^k.

We shall prove HRB2 for Riemann surfaces and for algebraic surfaces with $k = 2$.

Proof of HRB2 in two cases. Assume that X is a Riemann surface. On H^1, we have

$$h(\alpha,\beta) = \sqrt{-1}\int_X \alpha \wedge \bar{\beta}.$$

Let $\alpha \in H^{1,0}$, so α has the form $\alpha = f dz$. Then

$$h(\alpha,\alpha) = \sqrt{-1}\int_X |f|^2 dz \wedge d\bar{z} > 0.$$

On $H^{0,1}$, it follows similarly that for $\alpha = f d\bar{z}$ we have $h(\alpha,\alpha) < 0$.

Now assume that X is an algebraic surface. Since the Lefschetz decomposition is orthogonal, we have

$$H_0^2 = H^{2,0} \oplus H_0^{1,1} \oplus H^{0,2}.$$

If $\alpha \in H^2$ then

$$h(\alpha,\alpha) = \int_X \alpha \wedge \bar{\alpha}.$$

An element $\alpha \in H^{2,0}$ has the local form $\alpha = f dz \wedge dw$ for local coordinates z and w. Hence

$$\alpha \wedge \bar{\alpha} = |f|^2(\sqrt{-1}dz \wedge d\bar{z}) \wedge (\sqrt{-1}dw \wedge d\bar{w}),$$

from which it is clear that $h(\alpha, \alpha) > 0$. The positivity on $H^{0,2}$ follows similarly.

Now choose $\alpha \in H_0^{1,1}$. At a point $P \in X$, we have

$$\omega_P = \sqrt{-1}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$$

and

$$\alpha_P = A dz \wedge d\bar{z} + B dw \wedge d\bar{w}$$

Since α is primitive, $[w] \cup [\alpha] = 0$ and hence $w \wedge \alpha = 0$ (for a harmonic representative α). But

$$\omega_P \wedge \alpha_P = (A + B) dz \wedge d\bar{z} \wedge dw \wedge d\bar{w},$$

so $A + B = 0$. Hence α_P has the form

$$\alpha_P = A(dz \wedge d\bar{z} - dw \wedge d\bar{w}).$$

Therefore, we have that

$$\alpha_P \wedge \bar{\alpha}_P = 2|A|^2 dz \wedge d\bar{z} \wedge dw \wedge d\bar{w},$$

and we see that $h(\alpha, \alpha) < 0$. $\square$

Using the Hodge-Riemann bilinear relations, we may define an abstract notion of a Hodge structure (with a group H replacing H_0).

Definition. A *polarized Hodge structure* of weight k is

1. a module H over **Z**, **R** or **C**,
2. a nondegenerate bilinear form Q on H which is symmetric for k even and skew for k odd,
3. a decomposition $H = \oplus_{p+q=k} H^{p,q}$,
4. the Hodge-Riemann bilinear relations.

4.2 Variations of Hodge Structure

We consider a family $X \xrightarrow{\pi} S$ of projective algebraic manifolds, with π holomorphic and $d\pi$ of maximal rank. The fiber X_s of a point $s \in S$ is a smooth projective variety, and these fibers are all mutually isomorphic.

The set $\{H^k(X_s)\}$ of $\mathbf{Z}$, $\mathbf{R}$ or $\mathbf{C}$-modules forms a *local system* V over S. (More precisely, the local system of, say, $\mathbf{C}$-modules is defined by $V = R^k\pi_*\mathbf{C}$, where $R^k\pi_*\mathbf{C}(U) = H^k(\pi^{-1}(U))$ for U open in S, so that $V_s = H^k(X_s)$.) That is, we can represent $\pi_1(S)$ in the outer automorphism group of this vector bundle. That $V \to S$ forms a local system is equivalent to the fact that the transition matrices $f_{\alpha\beta}$ between trivializations on small open neighborhoods U_α can be chosen to be constant.

Another equivalent notion of a local system is the existence of a flat connection. In our case, the connection $\nabla : V \to V \otimes \mathcal{E}^1$, is defined by $(\nabla s)_\alpha = ds_\alpha$ for a section s_α on U_α. The facts that $ds_\beta = f_{\alpha\beta} ds_\alpha$ on $U_\alpha \cap U_\beta$ and $d^2 s_\alpha = 0$ force the flatness $\nabla^2 = 0$.

The local system $V_{\mathbf{C}}$ has a direct sum decomposition $V_{\mathbf{C}} = V^{p,q}$ where

$$V_s^{p,q} = H_0^{p,q}(X_s)$$

and $\overline{V^{p,q}} = V^{q,p}$. We also have a flat bilinear form Q on V_s, which is to say that if α, β are locally constant so that $\nabla\alpha = \nabla\beta = 0$, then $Q(\alpha, \beta)$ is also locally constant. The bilinear form Q satisfies the necessary conditions to make $(V_s, \oplus V_s^{p,q}, Q)$ a polarized Hodge structure of weight k for each s.

Consider the *Hodge filtration* on our vector bundle V given by

$$F^p = \oplus_{a \geq p} V^{a,b}.$$

with $p \geq 0$.

Theorem 4.13. *The following hold for the Hodge filtration (F^p) of V.*

1. *The F^p are holomorphic subbundles of $V_{\mathbf{C}}$.*

2. *(Griffiths transversality) We have the following "horizontality condition":*

$$\nabla \colon F^p \to F^{p-1} \otimes \mathcal{E}^1.$$

The following lemma is the key to the proof of Theorem 4.13.

Lemma 4.14. *Let ϕ be a k-form on X such that $d\phi = 0$ on X_s for all $s \in S$. Let $\tilde{\xi}$ be a vector field on X descending to a vector field ξ on S. Then*

$$\nabla_\xi \phi = L_{\tilde{\xi}}\phi = d\big(\mathrm{Int}(\tilde{\xi}\phi)\big) + \mathrm{Int}(\tilde{\xi})d\phi.$$

Exercise. Prove Lemma 4.14. (Hint: Treat X as a real manifold, choosing coordinates x_i on X. Let $\tilde{\xi} = \partial/\partial x_1$ and $\phi = f dx_I$. Consider two cases: $1 \in I$, $1 \notin I$.)

Proof of Theorem 4.13. Let ϕ be a form of type (p, q), closed on fibers. Note that ∇ may be written as a sum $\nabla = \nabla' + \nabla''$, where

$$\nabla' \colon V \to V \otimes \mathcal{E}^{1,0}$$

and

$$\nabla'' \colon V \to V \otimes \mathcal{E}^{0,1}.$$

Note that $(\nabla'')^2 = 0$ and $\nabla'' s = 0$ if and only if s is a holomorphic section of V. To show that F^p is a holomorphic subbundle of V, it therefore suffices to show that $\nabla''_\xi \colon F^p \to F^p$ for every vector field ξ. Note that $\nabla''_\xi = 0$ for ξ of type $(1,0)$.

So consider $\tilde{\xi}$ of type $(0,1)$. We see that $\mathrm{Int}(\tilde{\xi})\phi$ has type $(p, q-1)$ and thus $d(\mathrm{Int}(\tilde{\xi})\phi)$ is a sum of terms of types $(p+1, q-1)$ and (p, q). Similarly, $\mathrm{Int}(\tilde{\xi}) d\phi$ is a sum of terms of the same form. So by Lemma 4.14, we have

$$\nabla_\xi \colon V^{p,q} \to V^{p+1,q-1} \oplus V^{p,q} \subseteq F^p. \tag{7}$$

Since $\nabla'_\xi = 0$, we have the desired result.

To show Griffiths transversality, we must show that

$$\nabla_\xi \colon F^p \to F^{p-1}$$

for every vector field ξ. By (7), we know this to be true for forms of type $(0,1)$, since $F^p \subset F^{p-1}$. So consider $\tilde{\xi}$ of type $(1,0)$. By considering types, we obtain

$$\nabla_\xi \colon V^{p,q} \to V^{p-1,q+1} \oplus V^{p,q} \subseteq F^{p-1},$$

which gives the result. $\square$

Lecture 5

5.1 The Period Map

We consider $X \xrightarrow{\pi} S$ and $V_{\mathbf{C}} = R^k \pi_* \mathbf{C}$, similarly to Section 4.2. The bundle $V_{\mathbf{C}}$ lifts to a trivial bundle $V_{\mathbf{C}}/\tilde{S} \cong \tilde{S} \times \mathbf{C}^n$ over the universal cover $\tilde{S}$ of S in such a way that $V_{\mathbf{Z}}/\tilde{S} \cong \tilde{S} \times \mathbf{Z}^n$. Over a point $\tilde{s} \in \tilde{S}$, we have

$$F^p_{\tilde{s}} H^k(X_{\tilde{s}}) \subseteq \mathbf{C}^n,$$

and therefore $\tilde{s}$ defines a filtration of $\mathbf{C}^n$. Denoting the space of such filtrations by D, we have a map $\tilde{S} \to D$. Consider the monodromy group Γ, which is the image of the representation of $\pi_1(S)$ in $GL_n(\mathbf{Z})$ (which depends on a choice of integer basis). The monodromy group acts on D, and we can take the quotient to obtain the *period map* $S \to D/\Gamma$. By Theorem 4.13a, the period map is holomorphic. (In the general setting, D is of the form G/V, where G is a Lie group of the form $SO(2p, g)$ or $Sp(g, \mathbf{R})$.)

Let us consider the case where X is a family of Riemann surfaces of genus g. The period map is defined by

$$s \mapsto \{H^{1,0}(X_s) \subset \mathbf{C}^{2g}\}.$$

Let δ_i and γ_i with $1 \leq i \leq g$ be the usual homology classes arising from loops on our g-holed torus X_s (such that $\prod \delta_i \gamma_i \delta_i^{-1} \gamma_i^{-1} \sim 1$), and let ϕ_i with $1 \leq i \leq g$ be a basis of $H^{1,0}$. Define matrices A and B of dimension g by

$$A_{ij} = \int_{\delta_j} \phi_i \qquad \text{and} \qquad B_{ij} = \int_{\gamma_j} \phi_i,$$

respectively. We have

$$\phi_i = \sum A_{ij}\delta_j + \sum B_{ij}\gamma_j,$$

and the period map can now be rewritten as taking a point s to the row space of $P = (A, B)$ modulo choices of bases, or in other words modulo $Sp(g, \mathbf{Z})$.

The following remarks are left as exercises.

Exercise. Let the basis ϕ_i of $H^{1,0}$ be such that $\int_X \phi_i \wedge \phi_j = 0$ for $i \neq j$ and $i \int_X \phi_i \wedge \overline{\phi_i} > 0$ for all i.

1. Show that the matrix A is nonsingular, so the basis $\{\phi_i\}$ may be chosen such that $A = I$.
2. Show that Z is symmetric, given that $A = I$ and $\int_X \phi_i \wedge \phi_j = 0$ for $i \neq j$.
3. Use $\int_X \phi_i \wedge \overline{\phi_i} > 0$ to show that Z has positive imaginary part.

As a consequence of these exercises, we have that

$$S \to \mathcal{H}_g/Sp(g, \mathbf{Z}),$$

where $\mathcal{H}_g$ is the Siegel upper half space (that is, g-dimensional symmetric complex matrices with positive imaginary part).

We now give an example of a nonconstant period map for a family over $\mathbf{P}^1$ with 3 singular fibers.

Example. Consider $y^2 = (x^2 - t)(x - 1)$, and take

$$\phi = \frac{dx}{y} = \frac{dx}{\sqrt{(x^2 - t)(x - 1)}}.$$

Let δ be loop with winding number 1 about $\pm\sqrt{t}$ (and 0 about 1), and let γ be a loop with winding number 1 about 0 and $\sqrt{t}$ (but 0 about $-\sqrt{t}$). Then

$$\int_\delta \phi \sim 2\pi\sqrt{-1}$$

and

$$\int_\gamma \phi \sim 2\int_{\sqrt{t}}^1 \frac{dx}{\sqrt{x^2 - t}} \sim \log t.$$

Therefore the period matrix P is equal to $(1, Z)$ with

$$Z = \frac{\log t}{2\pi\sqrt{-1}} + \text{bounded terms}.$$

In fact, any family of Riemann surfaces over $\mathbf{P}^1$ must have at least 3 singular fibers. To see this, assume this is not the case, that there exists a family with just 2 singular fibers. This follows directly from the complex analytic fact that any holomorphic map $\mathbf{C} \to \mathcal{H}_g$ must be constant.

5.2 Residue Calculus

Consider a holomorphic function $f(z_1, \ldots, z_n)$ in affine coordinates. Consider the n-form

$$\phi = \frac{a dz_1 \wedge \ldots dz_n}{f} \tag{8}$$

with a a holomorhic function and with simple pole along f. Its residue $\text{res}(\phi)$ is class of any holomorphic $n-1$ form, also denoted $\text{res}(\phi)$ such that

$$\phi = \text{res}(\phi)\frac{df}{f}.$$

We have that

$$df = f_1 dz_1 + \ldots + f_n dz_n,$$

and we suppose that f_n is never zero along $f = 0$. Then dz_n is df/f_n up to terms involving dz_i with $i > 1$. Pluggin this into equation (8) and using the definition of $\mathrm{res}(\phi)$, we obtain

$$\mathrm{res}(\phi) = \frac{a dz_1 \wedge \ldots dz_{n-1}}{f_n}.$$

Consider now the following more global situation. Let X be an n-dimensional complex projective algebraic manifold inside $\mathbf{P}^{n+1}$ We have a "tube map":

$$H_n(X) \xrightarrow{T} H_{n+1}(\mathbf{P}^{n+1} - X)$$

given by taking a cycle γ to the boundary of a small tube around it in $\mathbf{P}^{n+1}$. By duality, we obtain an adjoint to T,

$$H^{n+1}(\mathbf{P}^{n+1} - X) \xrightarrow{T^*} H^n(X),$$

which is the residue map.

Exercise. The image of T^* is equal to the primitive cohomology $H^n_0(X)$.

Exercise. Represent T^* on the level of forms. (Hint: Begin in local affine coordinates as above, and projectivize by $z_i = Z_i/Z_0$ and

$$dz_i = \frac{dZ_i}{Z_0} - Z_i \frac{dZ_0}{Z_0^2}.)$$

We have the following theorem on residues. Let $T_\epsilon(\gamma)$ denote (the boundary of) a tube of radius $\epsilon > 0$ around a cycle γ.

Theorem 5.15. *Let Φ be a holomorphic $(n+1)$form and* $\mathrm{res}\,\Phi$ *its residue on X. Then*

$$\lim_{\epsilon \to 0} \int_{T_\epsilon(\gamma)} \Phi = 2\pi\sqrt{-1} \int_\gamma \mathrm{res}\,\Phi.$$

Moreover, $\int_{T_\epsilon(\gamma)} \Phi$ is independent of ϵ for sufficiently small ϵ.

Idea of Proof. In local coordiates, we may consider $T_\epsilon(\gamma) \cong \gamma \times S^1$ and γ to be defined by the equation $z = 0$ for some coordinate function z. The result then follows from the definition of residue, Fubini's theorem on integrals and the fact that $\int dz/z$ around a loop of winding number 1 about $z = 0$ is $2\pi\sqrt{-1}$. □

We apply this discussion to determine the genus of a plane curve of degree d.

Example. Let $F = 0$ define a plane curve of degree d in $\mathbf{P}^2$. Consider

$$\Omega = Z_0 dZ_1 \wedge dZ_2 - Z_1 dZ_0 \wedge dZ_2 + Z_2 dZ_0 \wedge dZ_1,$$

coming from the projectivization of $dz_1 \wedge dz_2$, as in the above exercise. Then

$$\frac{A\Omega}{F}$$

is a rational form with first order pole if the degree of A equals $d - 3$. Then space of homogeneous polynomials of degree d in $n+1$ variables has dimension $\binom{d+n}{n}$. Therefore the dimension of the space of possible choices of A is

$$\binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

But this space is the space of regular differentials on $F = 0$, and so its dimension is the genus of F.