Selmer Schemes IV

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Disclaimer

These lecture slides come with a bibliography at the end. However, there has been no attempt at accurate attribution of mathematical results. Rather, the list mostly contains works the lecturer has consulted during preparation, which he hopes will be helpful for users.

Non-abelian descent?

[Kim1]

From here on, we assume that X is a smooth proper curve of genus ≥ 2. We will focus on the base field $\mathbb{Q}$, even though Netan Dogra has generalised all the arguments to number fields [1].

$$\begin{tikzcd}
X(\mathbb{Q}) \arrow[r] \arrow[d] & X(\mathbb{Q}_p) \arrow[d] \arrow[dr, "j^{DR}"] & \\
H^1_f(G_T, U_n) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U_n) \cong^{D} & U_n^{DR}/F^0
\end{tikzcd}$$

Conjecture (A)

The image of loc_p is non-dense for $n >> 0$.

Theorem

Assuming the conjecture, $X(\mathbb{Q})$ is finite.

Non-abelian descent?

Proof.
By assumption, there is an algebraic function $\alpha \neq 0$ that vanishes on $D(\mathrm{loc}_p(H^1_f(G_T, U_n)))$. Hence,

$$\alpha \circ j^{DR} | X(\mathbb{Q}) = 0.$$

But $\alpha \circ j^{DR}$ is a non-zero convergent power series on each $]y[$, $y \in Y(\mathbb{F}_p)$. So the zero set is finite. □

Define

$$X(\mathbb{Q}_p)_n := \cap_{\alpha \circ D \circ \mathrm{loc}_p = 0} Z(\alpha \circ j^{DR})$$

$$= (j^{DR})^{-1}(\overline{D(\mathrm{loc}_p(H^1_f(G_T, U_n)))}).$$

Non-abelian descent?

[BDKW]

Since the diagrams are compatible over n, we get a decreasing filtration:

$$X(\mathbb{Q}_p) \supset X(\mathbb{Q}_p)_1 \supset X(\mathbb{Q}_p)_2 \supset X(\mathbb{Q}_p)_3 \supset \cdots$$

Note that the conjecture actually implies that $X(\mathbb{Q}_p)_n$ is finite for $n >> 0$.

Conjecture (B)

$$\cap_n X(\mathbb{Q}_p)_n = X(\mathbb{Q})$$

Conjecture (C)

$\cap_n X(\mathbb{Q}_p)_n$ is computable and conjecture (B) is computationally verifiable.

Non-abelian descent?

Key problem:

Find defining equations for

$$\mathrm{loc}_p(H^1_f(G_T, U_n)) \subset H^1_f(G_p, U_n).$$

Are there *canonical* equations related to non-abelian L-functions?

A canonical triviallisation

$$R\Gamma_c(G_T, U) \sim^{\mathcal{L}} 0$$

in a suitable homotopy category?

Should be similar to the annihilation of Selmer groups by p-adic L-functions [CK] or Iwasawa's theorem on the image of global units in local units of cyclotomic fields. [1].

Non-abelian descent?

Let $F/\mathbb{Q}$ be a Galois extension with group $G(F/Q)$. Then

$$H^1_f(G_T, U_n) = H^1_f(G_{F,T}, U_n)^{G(F/\mathbb{Q})}.$$

If $F_m = \mathbb{Q}(\mu_{p^m})$, then

$$H^1_f(G_T, U_n) = [\varinjlim H^1_f(G_{F_m,T}, U_n)]^{\mathbb{Z}_p^\times}.$$

II. Effectivity and the section conjecture

Effectivity and the section conjecture

[Kim2]

Assumptions:

(1) The map

$$H^1_f(G, U_n) \longrightarrow U_n^{DR}/F^0$$

can be effectively computed.

(2) Using (1), we can compute an effective lower bound for the p-adic distances between the points in $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$.

Effectivity and the section conjecture

(3) Thus, we get an effective M such that $X(\mathbb{Q}) \longrightarrow X(\mathbb{Z}/p^M)$ is injective.

(4) Using this, we get effective N, for example, $N = |J_X(\mathbb{Z}/p^M)|$, such that

$$X(\mathbb{Q}) \subset J(\mathbb{Q}) \subset J(\mathbb{Z})/NJ(\mathbb{Q}) \hookrightarrow H^1(G_S, J[N])$$

is injective, where S is the set of all places of bad reduction, and the primes dividing pN.

Effectivity and the section conjecture

(5) Grothendieck's section conjecture [Grothendieck]:

$$X(F) \cong H^1(G_F, \hat{\pi}_1(\bar{X}, b)).$$

Note that for elliptic curves, one conjectures

$$E(F) \otimes \mathbb{Z}_p \cong H^1_f(G_F, \hat{\pi}_1(\bar{E}, b)^{(p)}).$$

Effectivity and the section conjecture

Let n be larger than N and all the primes in S.

– $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $G_n = \pi_1(\mathrm{Spec}(\mathbb{Z}[1/n!]))$.

– $\Delta = \hat{\pi}_1(\bar{X}, b)$ and K_n the intersection of all open subgroups of index $\leq n$. (There are finitely many, and K is normal.)

– $\Delta(n) = \Delta/K_n$. Thus the prime divisors of the order of any element in $\Delta(n)$ is $\leq n$.

– Denote by $\pi(n)$ the quotient of $\hat{\pi}(X, b)$ by K_n, so that we have a pushout diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Delta \arrow[r] \arrow[d] & \hat{\pi}(X, b) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Delta(n) \arrow[r] & \pi(n) \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

Effectivity and the section conjecture

There is a pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & \Delta(n) \arrow[r] \arrow[d] & \pi(n) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta(n) \arrow[r] & \pi_1(\mathfrak{X}_n, b)/K_n \arrow[r] & G_n \arrow[r] & 0
\end{tikzcd}$$

where $\mathfrak{X}_n$ is a smooth projective model for X over $\mathrm{Spec}(\mathbb{Z}[1/n!])$.

Hence, any point $x \in X(\mathbb{Q})$ defines a class in $H^1(G_n, \Delta(n))$.

Effectivity and the section conjecture

We have a commutative diagram

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r, hook] \arrow[d] & H^1(G, \Delta) \arrow[d] \\ H^1(G_n, \Delta(n)) \arrow[r, hook] & H^1(G, \Delta(n)) \end{tikzcd}$$

and hence, a sequence of subsets $H^i(G, \Delta)_n$ consisting of classes whose images in $H^1(G, \Delta(n))$ come from $H^1(G_n, \Delta(n))$.

Effectivity and the section conjecture

Thus we have diagrams

$$
\begin{tikzcd}
 & H^1(G,\Delta)_n \arrow[r, hook] \arrow[d] & H^1(G,\Delta) \arrow[d] \\
 & H^1(G_n,\Delta(n)) \arrow[r, hook] \arrow[d] & H^1(G,\Delta(n)) \\
H^1(G_S,J[N]) \arrow[r, hook] & H^1(G_n,J[N]) &
\end{tikzcd}
$$

Effectivity and the section conjecture

and

$$\begin{tikzcd}
 & & H^1(G_{n+1}, \Delta(n+1)) \arrow[d] \\
 & H^1(G_n, \Delta(n)) \arrow[r, hook] \arrow[d] & H^1(G_{n+1}, \Delta(n)) \arrow[d] \\
H^1(G_S, J[N]) \arrow[r, hook] & H^1(G_n, J[N]) \arrow[r, hook] & H^1(G_{n+1}, J[N])
\end{tikzcd}$$

Effectivity and the section conjecture

Using this, we can define a decreasing sequence of subsets

$$H^1(G_S, J[N])_{n+1} \subset H^1(G_S, J[N])_n$$

consisting of those classes whose images in $H^1(G_i, J[N])$ lift to $H^1(G_i, \Delta(i))$ for all $i \leq n$, i larger than $n_0 = \sup(N, p \in S)$.

Meanwhile, there is an increasing sequence of subsets $X(\mathbb{Q})_n$ of points whose heights are $\leq n$, all of which occur in the 'non-abelian descent sequence':

$$\cdots X(\mathbb{Q})_n \subset X(\mathbb{Q})_{n+1} \subset X(\mathbb{Q})_{n+2} \subset \cdots$$

$$\cdots \subset H^1(G_S, J[N])_{n+2} \subset H^1(G_S, J[N])_{n+1} \subset H^1(G_S, J[N])_n \subset \cdots$$

Effectivity and the section conjecture

Using the section conjecture,

$$X(\mathbb{Q})_n = H^1(G_S, J[N])_n$$

for n sufficiently large, and $X(\mathbb{Q})_n = X(\mathbb{Q})$ at that point.

Effectivity and the section conjecture

To check this, note that the section conjecture implies that the inclusions

$$X(\mathbb{Q}) \subset H^1(G, \Delta)_n \subset H^1(G, \Delta),$$

are all equalities.

From the diagrams

$$\begin{tikzcd}
H^1(G, \Delta)_n \arrow[r, "="] \arrow[d] & H^1(G, \Delta) \arrow[d] \\
H^1(G_n, \Delta(n)) \arrow[r, hook] & H^1(G, \Delta(n))
\end{tikzcd}$$

we have maps

$$H^1(G, \Delta) \longrightarrow H^1(G_n, \Delta(n))$$

and

$$H^1(G, \Delta) = \varprojlim H^1(G, \Delta(n)) = \varprojlim H^1(G_n, \Delta(n)).$$

Effectivity and the section conjecture

Suppose

$$c \in H^1(G_S, J[N])_n$$

for all n. Then the set

$$H^1(\Gamma_n, \Delta(n))_c$$

of classes that lift c is non-empty for all n, and hence,

$$X(\mathbb{Q})_c = H^1(\Gamma, \Delta)_c = \varprojlim H^1(\Gamma_n, \Delta(n))_c$$

is non-empty.

This shows that

$$\cap_n H^1(G_S, J[N])_n = X(\mathbb{Q}).$$

Since all sets are finite, we must have

$$X(\mathbb{Q}) = H^1(G_S, J[N])_n$$

for some n.

III. Remark on non-abelian reciprocity

Diophantine geometry: remark on non-abelian reciprocity

[Kim3]

Given a variety X over a number field F, can one describe the inclusion

$$X(F) \subset X(\mathbb{A}_F)$$

?

For $\mathbb{G}_m$, this is partially achieved by the reciprocity map

$$\mathbb{G}_m(F) \hookrightarrow \mathbb{G}_m(\mathbb{A}_F) \xrightarrow{rec} \mathrm{Gal}^{ab}(\bar{F}/F)$$

For an affine conic

$$C : ax^2 + by^2 = c$$

described by a class $\chi \in H^1(\mathrm{Gal}(\bar{F}/F), \pm 1)$, can replace this by

$$C(F) \hookrightarrow C(\mathbb{A}_F) \longrightarrow \mathrm{Hom}(H^1(\mathrm{Gal}(\bar{F}/F), \mathbb{Q}/\mathbb{Z}(\chi)), \mathbb{Q}/\mathbb{Z}).$$

Diophantine geometry: remark on non-abelian reciprocity

There is a **non-abelian class field theory** with coefficients in a fairly general variety X over a number field F generalising CFT with coefficients in $\mathbb{G}_m$.

This consists (with some simplications) of a filtration

$$X(\mathbb{A}_F) = X(\mathbb{A}_F)_1 \supset X(\mathbb{A}_F)_2 \supset X(\mathbb{A}_F)_3 \supset \cdots$$

and a sequence of maps

$$rec_n : X(\mathbb{A}_F)_n \longrightarrow \mathfrak{G}_n(X)$$

to a sequence of groups such that

$$X(\mathbb{A}_F)_{n+1} = rec_n^{-1}(0).$$

Diophantine geometry: remark on non-abelian reciprocity

$$\begin{tikzcd}
\cdots & rec_3^{-1}(0) \arrow[r, phantom, "\subset"] \arrow[d, equal] & rec_2^{-1}(0) \arrow[r, phantom, "\subset"] \arrow[d, equal] & rec_1^{-1}(0) \arrow[r, phantom, "\subset"] \arrow[d, equal] & X(\mathbb{A}_F) \arrow[d, equal] \\
\cdots & X(\mathbb{A}_F)_4 \arrow[r, phantom, "\subset"] \arrow[d, "rec_4"'] & X(\mathbb{A}_F)_3 \arrow[r, phantom, "\subset"] \arrow[d, "rec_3"'] & X(\mathbb{A}_F)_2 \arrow[r, phantom, "\subset"] \arrow[d, "rec_2"'] & X(\mathbb{A}_F)_1 \arrow[d, "rec_1"'] \\
\cdots & \mathfrak{G}_4(X) & \mathfrak{G}_3(X) & \mathfrak{G}_2(X) & \mathfrak{G}_1(X)
\end{tikzcd}$$

Diophantine geometry: remark on non-abelian reciprocity

Put

$$X(\mathbb{A}_F)_\infty = \cap_{n=1}^{\infty} X(\mathbb{A}_F)_n.$$

Theorem (Non-abelian reciprocity)

$$X(F) \subset X(\mathbb{A}_F)_\infty.$$

For a fixed p, can define $X(\mathbb{Q}_p)_n := pr_p(X(\mathbb{A}_F)_\infty)$

Conjecture

Suppose X is a smooth projective curve of genus ≥ 2. Then

$$X(F) = X(\mathbb{Q}_p)_\infty = \cap_n X(\mathbb{Q}_p)_n.$$

IV. Principal bundles in Diophantine geometry: a little history

Principal bundles in Diophantine geometry: a little history

Weil [Weil1] in 1929 constructed an embedding

$$j : X \hookrightarrow J_X,$$

where J_X is an abelian variety of dimension g.

That is, over $\mathbb{C}$,

$$J_X(\mathbb{C}) = \mathbb{C}^g/\Lambda = H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C}))})^*/H_1(X, \mathbb{Z}).$$

The map j is defined over $\mathbb{C}$ by fixing a basepoint b and

$$j(x)(\alpha) = \int_b^x \alpha \mod H_1(X, \mathbb{Z}),$$

for $\alpha \in H^0(X(\mathbb{C}), \Omega^1_{X(\mathbb{C})})$.

Principal bundles in Diophantine geometry: a little history

But Weil's point was that J_X is also a projective algebraic variety defined over $\mathbb{Q}$, and if $b \in X(\mathbb{Q})$, then the map j is also defined over $\mathbb{Q}$.

The reason is that J_X is a moduli space of line bundles of degree 0 on X and

$$j(x) = \mathcal{O}(x) \otimes \mathcal{O}(-b).$$

The main application is that

$$j : X(\mathbb{Q}) \hookrightarrow J(\mathbb{Q}).$$

Weil also proved that $J(\mathbb{Q})$ is a finitely-generated abelian group, and hoped, without success, that this could be somehow used to study $X(\mathbb{Q})$.

Principal bundles in Diophantine geometry: a little history

In the 1938 paper 'Généralisation des fonctions abéliennes', Weil [Weil2] studied

$$Bun_X(GL_n) = GL_n(K(X))\backslash GL_n(\mathbb{A}_{K(X)})/[\prod_x GL_n(\widehat{\mathcal{O}_x})]$$

as a 'non-abelian Jacobian'.

Proved a number of foundational theorems, including the fact that vector bundles of degree zero admit flat connections, beginning non-abelian Hodge theory.

Principal bundles in Diophantine geometry: a little history

This paper was very influential in geometry, leading to the paper of Narsimhan and Seshadri [NS]:

$$Bun_X(GL_n)_0^{st} \simeq H^1(X, U(n))^{irr}.$$

This was extended by Donaldson [Donaldson], influencing this work on smooth manifolds and gauge theory, and by Simpson [Simpson] to

$$Higgs(GL_n) \simeq H^1(X, GL_n).$$

Serre on Weil's paper:

> *'a text presented as analysis, whose significance is essentially algebraic, but whose motivation is arithmetic'*

Diophantine principal bundles

Go back to Hodge theory of Jacobian:

$$X(\mathbb{C}) \longrightarrow J_X(\mathbb{C}) \simeq \mathrm{Ext}^1_{MHS,\mathbb{Z}}(\mathbb{Z}, H_1(X(\mathbb{C}), \mathbb{Z})).$$

$$X(\mathbb{Q}) \longrightarrow J_X(\mathbb{Q}) \otimes \mathbb{Z}_p \simeq \mathrm{Ext}^1_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),f}(\mathbb{Z}_p, H_1^{et}(\bar{X}, \mathbb{Z}_p)).$$

(Second isomorphism conjectural.)

Also,

$$H_1 = \pi_1^{ab}$$

suggesting the possibility of extending the constructions to non-abelian homotopy and moduli space of non-abelian structures:

– over $\mathbb{C}$, Hain's higher Albanese varieties [Hain];

– over $\mathbb{Q}_p$, p-adic period spaces;

– over global fields, Selmer schemes and variants.

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