p-ADIC L-FUNCTIONS: THE KNOWN AND UNKNOWN

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Contents

1. Iwasawa functions, p-adic measures, and distributions

1.1. **Some preliminaries.** The p-adic L-functions one encounters in life are certain p-adic analytic (or meromorphic) functions that typically arise from spaces of power series, p-adic measures, or p-adic distributions. In this section, we'll explore these spaces.

Problem 1.1. Completed group algebras. Let G be a profinite group. The completed group ring of G (over $\mathbf{Z}_p$) is

$$\mathbf{Z}_p[\![G]\!] := \varprojlim_N \mathbf{Z}_p[G/N]$$

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where N runs over all finite-index subgroups of G and $\mathbf{Z}_p[G/N]$ is the usual group ring of G/N (over $\mathbf{Z}_p$). This is a topological ring. Show that every continuous group homomorphism $\chi : G \to \overline{\mathbf{Q}}_p^\times$ extends 'by linearity' to a continuous ring homomorphism $\mathbf{Z}_p[\![G]\!] \to \overline{\mathbf{Q}}_p$.

Problem 1.2. The ring $\mathbf{Z}_p[\![T]\!]$, a.k.a *the* Iwasawa algebra. Let $\mathbf{Z}_p[\![T]\!]$ be the ring of formal power series in T over $\mathbf{Z}_p$. Let v_p be the usual p-adic valuation on $\mathbf{Z}_p$, so that $v_p(p^n) = n$.

(i) Show that $\mathbf{Z}_p[\![T]\!]$ is a local ring with maximal ideal $\mathfrak{m} = (p, T)$. (Note: in fact, $\mathbf{Z}_p[\![T]\!]$ is a complete Noetherian local ring, i.e. a local ring R with maximal ideal $\mathfrak{m}$ such that R is Noetherian, $\bigcap_{n\geq 1} \mathfrak{m}^n = 0$, and $R \cong \varprojlim_n R/\mathfrak{m}^n$.)

(ii) Let $f(T) = \sum_{n\geq 0} a_n T^n \in \mathbf{Z}_p[\![T]\!]$ be non-zero. Its μ-invariant is

$$\mu(f) := \min_n v_p(a_n),$$

i.e. the least power of p dividing all the coefficients. Its λ-invariant is

$$\lambda(f) := \min\{n : v_p(a_n) = \mu(f)\}$$

i.e. the first coefficient at which the minimum valuation occurs. Prove the following theorem:

Theorem 1.1 (Division algorithm for $\mathbf{Z}_p[\![T]\!]$)**.** *Let $f(T) \in \mathbf{Z}_p[\![T]\!]$ be non-zero with $\mu(f) = 0$. Let $g(T) \in \mathbf{Z}_p[\![T]\!]$. Show that there exist unique $q(T) \in \mathbf{Z}_p[\![T]\!]$ and a polynomial $r(T) \in \mathbf{Z}_p[T]$ of degree $< \lambda(f)$ such that*

$$g = fq + r.$$

(iii) A polynomial $P(T) \in \mathbf{Z}_p[T]$ is called a distinguished polynomial if

$$P(T) = T^d + a_{d-1}T^{d-1} + \cdots + a_0 \quad \text{with } a_i \in p\mathbf{Z}_p.$$

Prove the following theorem:

Theorem 1.2 (p-adic Weierstrass preparation theorem)**.** *Let $f(T) \in \mathbf{Z}_p[\![T]\!]$ be non-zero. Then, there is a unique factorization*

$$f(T) = p^\mu P(T) u(T)$$

where $\mu \in \mathbf{Z}_{\geq 0}$, $P(T)$ is a distinguished polynomial, and $u(T) \in \mathbf{Z}_p[\![T]\!]^\times$. In this factorization, $\mu = \mu(f)$ and $\deg(P) = \lambda(f)$.

(iv) Let $\kappa_n(T) = (1+T)^{p^n} - 1 \in \mathbf{Z}_p[T]$. Show that

$$\mathbf{Z}_p[\![T]\!] \cong \varprojlim_n \mathbf{Z}_p[\![T]\!]/(\kappa_n).$$

Problem 1.3. Let Γ be a topological group isomorphic to the additive group of $\mathbf{Z}_p$. Note that $\mathbf{Z}_p$ is topologically cyclic, i.e. the closure of the cyclic subgroup $\langle 1 \rangle$ is $\mathbf{Z}_p$. A topological generator of Γ is then an element $\gamma_0 \in \Gamma$ such that Γ is the closure of $\langle \gamma_0 \rangle$.

(i) Show that the subgroups of Γ are exactly $\Gamma_n := \langle \gamma_0^{p^n} \rangle$.
(ii) Show that the map

$$\begin{array}{rcl} \mathbf{Z}_p[\![T]\!] & \longrightarrow & \mathbf{Z}_p[\![\Gamma]\!] \\ T & \longmapsto & \gamma_0 - 1 \end{array}$$

is a topological isomorphism. (Hint: set up a collection of commutative diagrams

$$\begin{tikzcd} \mathbf{Z}_p[\![T]\!]/(\kappa_{n+1}) \arrow[r, "\sim"] \arrow[d] & \mathbf{Z}_p[\Gamma/\Gamma_{n+1}] \arrow[d] \\ \mathbf{Z}_p[\![T]\!]/(\kappa_n) \arrow[r, "\sim"] & \mathbf{Z}_p[\Gamma/\Gamma_n] \end{tikzcd}$$

and take inverse limits.)

1.2. **Iwasawa functions and *the* Iwasawa algebra.** Let $\Gamma \cong \mathbf{Z}_p$ as topological groups. From the previous section, every continuous character of $\chi : \Gamma \to \overline{\mathbf{Q}}_p^\times$ transfers to a continuous ring homomorphism $\chi : \mathbf{Z}_p[\![T]\!] \to \overline{\mathbf{Q}}_p$. One choice of Γ is ...oh, wait, now I need to introduce you to the prime 2. I like to say "In Iwasawa theory, $2 = 4$". Basically, every time there should be a 2, you actually see a 4. I attribute this to the following fact (which will be discussed below): the conductor of the teichmüller character is p if p is odd and 4 if p is even. So, really it's that a bunch of times when you think p should appear, what you really mean is the conductor of the teichmüller character. Anyway, to get around this, what is typically done is the following: let

$$q = \begin{cases} p & \text{if } p \text{ is odd} \\ 4 & \text{if } p = 2. \end{cases}$$

Okay, where were we? Right. One choice of Γ is $1+q\mathbf{Z}_p$. And a standard choice of topological generator is $1 + q$. (A standard isomorphism with $\mathbf{Z}_p$ is given by the p-adic logarithm.)

Problem 1.4. p-adic logarithm and exponential. The p-adic logarithm $\log_p$ is initially defined on $1 + p\mathbf{Z}_p$ via the usual power series

$$\log_p(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n}.$$

(i) Show that $\log_p(1 + z)$ converges in $\mathbf{Z}_p$ for $1 + z \in 1 + p\mathbf{Z}_p$ and show that if $1 + z \in 1 + p^n\mathbf{Z}_p$, then $\log_p(1 + z) \in p^n\mathbf{Z}_p$. The logarithm can then be extended in many ways to $\mathbf{Q}_p^\times$. A standard 'branch' to take is called the Iwasawa logarithm and is extended by $\log_p(p) = 0$ and $\log_p(\zeta) = 0$ for $\zeta \in \mu_{p-1}$.

(ii) The p-adic exponential $\exp_p$ is defined, as usual, by

$$\exp_p(z) = \sum_{n \geq 0} \frac{z^n}{n!}.$$

Show that this converges for $z \in p\mathbf{Z}_p$ (in fact, you can use this power series to define a p-adic exponential on the elements of $\mathbf{C}_p$ with valuation $> \frac{1}{p-1}$).

(iii) Show that for $n \in \mathbf{Z}_{\geq 1}$ (or $n \in \mathbf{Z}_{\geq 2}$, for $p = 2$), $\log_p$ and $\exp_p$ give mutually inverse topological isomorphisms $1 + p^n\mathbf{Z}_p \to p^n\mathbf{Z}_p$ (and back).

Problem 1.5. For each $s \in \mathbf{Z}_p$, let $\chi_s : 1 + q\mathbf{Z}_p \to \mathbf{Z}_p^\times$ be given by $\chi_s(u) = u^s := \exp_p(s \log_p(u))$. Show that the induced $\chi_s : \mathbf{Z}_p[\![T]\!] \to \mathbf{Z}_p$ is given by

$$\chi_s(f(T)) = f((1+q)^s - 1).$$

Problem 1.6. Let $F(\mathbf{Z}_p, \mathbf{Z}_p)$ denote the ring of functions from $\mathbf{Z}_p$ to itself. Show that we obtain a ring homomorphism $\mathbf{Z}_p[\![T]\!] \to F(\mathbf{Z}_p, \mathbf{Z}_p)$ given by

$$f(T) \mapsto L_f(s) := \chi_s(f(T)) = f((1+q)^s - 1).$$

A function $L \in F(\mathbf{Z}_p, \mathbf{Z}_p)$ in the image of this map is called an Iwasawa function. It is for this reason that we call $\mathbf{Z}_p[\![T]\!]$ *the* Iwasawa algebra and give it a special notation Λ.

Problem 1.7. A group $\Gamma = \langle \gamma_0 \rangle \cong \mathbf{Z}_p$ is abelian so the map $\gamma \mapsto \gamma^{-1}$ is an automorphism $\iota : \Gamma \to \Gamma$.

(i) Show that ι induces an automorphism $f \mapsto f^\iota$ of Λ given by

$$f^\iota(T) = f((1+T)^{-1} - 1).$$

(ii) Show that the μ- and λ-invariants of f and f^ι are equal.

Problem 1.8 (Optional)**.** Show that every Iwasawa function $L(s)$ is p-adic analytic; specifically, show that there are $a_n \in \mathbf{Z}_p$ such that for all $s \in \mathbf{Z}_p$

$$L(s) = \sum_{n \geq 0} a_n s^n.$$

1.3. **Iwasawa functions and interpolation.** The primary way in which p-adic L-functions are defined is via the interpolation of classical L-values (i.e. values of usual L-functions). We'll explore p-adic interpolation a bit in this section.

Problem 1.9. A continuous function on $\mathbf{Z}_p$ is determined by its values on a dense subset ($\mathbf{Z}_p$ is Hausdorff in the p-adic topology!).

(i) Show that the set of non-positive integers is dense in $\mathbf{Z}_p$.

(ii) What about the set of non-negative integers? The set of non-positive odd integers? The set of non-positive even integers? The set of non-positive integers congruent to 1 (mod $p - 1$)?

Problem 1.10. Getting your hands dirty. From the perspective of $f(T) = \sum_{n \geq 0} a_n T^n \in \Lambda$, the above problem says that you can determine the a_n from the information of the values $f((1+q)^n - 1)$ for $n \in \mathbf{Z}_{\geq 0}$. Let $L_f(s) = f((1+q)^s - 1)$, as above.

(i) Let $p = 3$. Given the following table of values, determine the λ-invariant of $f(T)$.

n	$L_f(n)$
0	0
1	-6
2	24

Congratulations! You've computed the (3-adic) λ-invariant of $\mathbf{Q}(\sqrt{-2})$!

(ii) Let $p = 3$. Given the following table of values, determine the λ-invariant of $f(T)$.

n	$L_f(n)$
0	0
1	-72
2	864

Congratulations! You've computed the (3-adic) λ-invariant of $\mathbf{Q}(\sqrt{-35})$!

(iii) Let $p = 3$. Given the following table of values, determine the λ-invariant of $f(T)$.

n	$L_f(n)$
0	6
1	-36
2	-960

Congratulations! You've computed the (3-adic) λ-invariant of $\mathbf{Q}(\sqrt{-31})$!

(iv) I could keep going, but I'll hold back. Just a remark: if you try to untangle how these exercise are computing Iwasawa invariants of quadratic fields, you'll go through the p-adic L-functions of Dirichlet characters and the Main Conjecture of Iwasawa theory discussed later and you might feel like something looks off; that's correct, because I've used Problem 1.7 to switch to non-negative integers for ease of computation.

Now, interpolating an Iwasawa function at non-negative integers is all well and good when you're interested in p-adic L-functions of Dirichlet characters, but the p-adic L-function $L_p(s, E)$ of an elliptic curve is only related to classical L-values at $s = 1$. And the singleton $\{1\}$ is certainly not dense in $\mathbf{Z}_p$. So, what now? Well, first, we'll have to say a bit about Dirichlet characters.

Problem 1.11. Let F be a field and $N \in \mathbf{Z}_{\geq 1}$. An F-valued Dirichlet character modulo N is a group homomorphism $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to F^\times$. We will also think of it as a function

$\chi : \mathbf{Z}/N\mathbf{Z} \to F^\times$ with $\chi(a) = 0$ for $\gcd(a, N) > 1$, and then also as a function $\chi : \mathbf{Z} \to F^\times$ by precomposing with the reduction mod N map.

Given a Dirichlet character χ modulo N, one obtains a Dirichlet character modulo NM for every $M \in \mathbf{Z}_{\geq 1}$ by precomposing with the natural surjection $(\mathbf{Z}/NM\mathbf{Z})^\times \to (\mathbf{Z}/N\mathbf{Z})^\times$. The conductor of a Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to F^\times$ is the least positive integer $\mathfrak{f}_\chi \mid N$ such that χ can be induced in this way from a Dirichlet character modulo $\mathfrak{f}_\chi$; the integer N is called the level of χ, and χ is called primitive if $N = \mathfrak{f}_\chi$. The character χ is called even if $\chi(-1) = 1$, otherwise it is called odd.

(i) Write down all $\mathbf{C}$-valued Dirichlet characters mod 3. Which of them are primitive? Odd? Even?
(ii) Write down all $\mathbf{C}$-valued Dirichlet characters mod 4. Which of them are primitive? Odd? Even?
(iii) Write down all $\mathbf{Q}_p$-valued Dirichlet characters mod 5, for $p = 3$ and 5. Which of them are primitive? Odd? Even?

Problem 1.12. The teichmüller character. There is a canonical p-adic (i.e. $\overline{\mathbf{Q}}_p$-valued) Dirichlet character of conductor q: the p-adic teichmüller character $\omega : (\mathbf{Z}/q\mathbf{Z})^\times \to \mathbf{Z}_p^\times$. Here's how you can construct it. Let p be odd and let $\mathbf{F}_p = \mathbf{Z}_p/p\mathbf{Z}_p$ be the finite field of order p.

(i) Let $a \in \{0, \ldots, p-1\}$. Show that $\lim_{n\to\infty} a^{p^n}$ converges in $\mathbf{Z}_p$. Denote it by $\omega(a)$.
(ii) Show that $\omega : \mathbf{F}_p \to \mathbf{Z}_p$ is the unique multiplicative section of the quotient map $\pi : \mathbf{Z}_p \to \mathbf{F}_p$ (i.e. $\pi \circ \omega = \mathrm{id}_{\mathbf{F}_p}$ and $\omega(ab) = \omega(a)\omega(b)$, and is the unique such function).
(iii) To deal with $p = 2$, let's think about what we've done. In part (ii), ω gives an isomorphism of $\mathbf{F}_p^\times$ with $\mu_{p-1} \subseteq \mathbf{Z}_p^\times$ (the group of roots of unity in $\mathbf{Z}_p$). Show that the group of roots of unity in $\mathbf{Z}_2$ is $\mu_2 = \{\pm 1\}$. Write down an isomorphism $(\mathbf{Z}/4\mathbf{Z})^\times \to \mu_2 \subseteq (\mathbf{Z}_2)^\times$; this is the 2-adic teichmüller character, and the reason $2 = 4$ in Iwasawa theory.

Problem 1.13. The character $\langle\cdot\rangle$. If χ is any $\overline{\mathbf{Q}}_p$-valued Dirichlet character of conductor p^e, then χ induces a character of $(\mathbf{Z}/p^{e'}\mathbf{Z})^\times$ for every $e' \geq e$, and as such induces a (continuous!) character of $\mathbf{Z}_p^\times = \varprojlim (\mathbf{Z}/p^{e'}\mathbf{Z})^\times$. The teichmüller character then induces a character $\omega : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ surjecting onto $\mu(\mathbf{Z}_p)$. Every element $a \in \mathbf{Z}_p^\times$ can then be written as $a = \omega(a)\langle a\rangle$, where $\langle a\rangle := a/\omega(a)$. Show that $\langle a\rangle = \exp_p(\log_p(a))$ and that ω and $\langle\cdot\rangle$ are the projections in the decomposition $\mathbf{Z}_p^\times = \mu(\mathbf{Z}_p) \times (1 + q\mathbf{Z}_p)$.

Problem 1.14. Characters of the first and second kind. Let $N \in \mathbf{Z}_{\geq 1}$ be relatively prime to p. The Chinese Remainder Theorem gives us that $(\mathbf{Z}/Nqp^e\mathbf{Z})^\times \cong (\mathbf{Z}/N\mathbf{Z})^\times \times (\mathbf{Z}/qp^e\mathbf{Z})^\times$.

(i) Show that the natural surjection $(\mathbf{Z}/qp^e\mathbf{Z})^\times \to (\mathbf{Z}/q\mathbf{Z})^\times$ induces a decomposition
$$(\mathbf{Z}/qp^e)^\times \cong (\mathbf{Z}/q\mathbf{Z})^\times \times U^1_{p^e}$$
where $U^1_{p^e} = \{a \in \mathbf{Z}/qp^e\mathbf{Z} : a \equiv 1 \pmod{q}\}$.

(ii) We thus have a decomposition
$$(\mathbf{Z}/Nqp^e\mathbf{Z})^\times \cong (\mathbf{Z}/N\mathbf{Z})^\times \times (\mathbf{Z}/q\mathbf{Z})^\times \times U^1_{p^e},$$
where the first two factors have order prime to p and the last factor has p-power order. A character χ is called of the first kind if its conductor divides Nq, and of the second kind if its conductor is a power of p and $\chi(a)$ only depends on $\langle a\rangle$. Show that $(1+q\mathbf{Z}_p)/(1+qp^e\mathbf{Z}_p) \cong U^1_{p^e}$ and that every Dirichlet character χ can then be factored as $\chi_1\chi_2$ where χ_1 is a character of the first kind and χ_2 is a character of the second kind.

Problem 1.15. If χ is of the second kind, then it induces a character $\chi : 1+q\mathbf{Z}_p \to \overline{\mathbf{Q}}_p^\times$. As such, χ induces a map
$$\chi : \Lambda \cong \mathbf{Z}_p[\![1+q\mathbf{Z}_p]\!] \to \overline{\mathbf{Q}}_p.$$

(i) Write out explicitly what that map is for $f(T) \in \Lambda$.

(ii) Recall the character χ_s from Definition 1.5. Let $f(T) \in \Lambda$ and fix $k \in \mathbf{Z}$ so that the product $\chi\chi_k$ is another character of $1+q\mathbf{Z}_p$. Show that $f(T)$ is determined by the numbers $(\chi\chi_k)(f)$ as χ varies over all characters of the second kind (or that you can also omit finitely many of these).

(iii) I mentioned above that we can only interpolate the values of the L-function $L(s,E)$ of an elliptic curve at $s=1$. So, what's the fix? Well, you can actually interpolate the *twisted* L-values $L(s,E,\chi)$ for any character χ of the second kind. This then uniquely determines some power series $f(T) \in \Lambda$ and the p-adic L-function $L_p(s,E)$ you may want to have in hand is then the Iwasawa function $L_f(s)$ attached to $f(T)$. Magic!

Really, it's that the domain of a p-adic L-function is the group of continuous characters $\mathfrak{X} = \mathrm{Hom}_{\mathrm{cont}}(1+q\mathbf{Z}_p, \mathbf{C}_p^\times)$ rather than its subgroup (isomorphic to $\mathbf{Z}_p$) of characters χ_s. Show that $\mathfrak{X}$ is topologically isomorphic to the open unit disk in $\mathbf{C}_p$.

(iv) The domain of the characters in $\mathfrak{X}$ can also be enlarged to $\mathbf{Z}_{p,N}^\times := \varprojlim_n (\mathbf{Z}/Np^n\mathbf{Z})^\times$. Show this space of characters is then topologically a finite disjoint union of copies of the open unit disk in $\mathbf{C}_p$.

(v) And, in fact, if you read Tate's thesis, you'll see that his (usual) L-functions do not have domain $\mathbf{C}$, but rather have domain $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{R}^\times, \mathbf{C}^\times)$. Show that this is topologically isomorphic to $\mathbf{C} \times \mathbf{Z}/2\mathbf{Z}$.

1.4. **p-adic distributions and p-adic measures.** There is yet another way to talk about Iwasawa functions: p-adic measures.

Problem 1.16. Distributions. Let X be a profinite topological space given by

$$X = \varprojlim_n X_n,$$

where the X_n are finite sets (with the discrete topology) and the transition maps $\pi_{n+1,n} : X_{n+1} \to X_n$ are surjective. Let B be an abelian group. Let $\mathcal{S}(X)$ denote the Schwartz space of X, i.e. the space of locally constant $\mathbf{Z}$-valued functions on X (with compact support). A B-valued distribution on X is a homomorphism $\mu : \mathcal{S}(X) \to B$. Rather than write $\mu(\varphi)$ for $\varphi \in \mathcal{S}(X)$, we will often write

$$\int_X \varphi(x) d\mu(x).$$

The set of distributions forms an abelian group denoted $\mathcal{D}(X, B)$.

(i) Show that there is a natural bijection between $\mathcal{D}(X, B)$ and the finitely-additive B-valued functions on the collection of compact open subsets of X (hence the integral notation). For a given compact open subset $U \subseteq X$, we will then denote

$$\mu(U) := \int_X \mathbf{1}_U(x) d\mu(x),$$

where $\mathbf{1}_U$ is the characteristic function of U, and we will think of this as the 'measure of U'.

(ii) Suppose that, for each n, you are given an element $\mu_n \in \mathcal{D}(X_n, B)$ and suppose that they satisfy the Distribution Property, i.e. for all n

$$\mu_n(\{x\}) = \sum_{\substack{x' \in X_{n+1} \\ \pi_{n+1,n}(x') = x}} \mu_{n+1}(\{x'\}).$$

Show that the collection of the μ_n determines an element of $\mathcal{D}(X, B)$, and that, in fact,

$$\mathcal{D}(X, B) \cong \varprojlim_n \mathcal{D}(X_n, B).$$

Problem 1.17. p-adic measures. Let B be a p-adic Banach space (i.e. a $\mathbf{Q}_p$-vector space B equipped with a norm $|\cdot|$ satisfying the ultrametric inequality

$$|u + v| \leq \max(|u|, |v|)$$

with respect to which B is complete). An element $\mu \in \mathcal{D}(X, B)$ is then usually called a p-adic distribution. This element is called a p-adic measure if it is bounded, i.e. if

$$\left| \int_X \varphi(x) d\mu(x) \right|$$

is bounded as φ varies over $\mathcal{S}(X)$. Let $\mathcal{D}_0(X, B)$ denote the space of B-valued measures on X. Let $C(X, \mathbf{Q}_p)$ be the space of continuous functions $f : X \to \mathbf{Q}_p$ equipped with the sup norm $|f| = \sup_{x \in X} |f(x)|$. Let $\mathcal{S}(X, \mathbf{Q}_p)$ be the locally constant functions $X \to \mathbf{Q}_p$ equipped with the sup norm.

(i) Show that $\mathcal{S}(X, \mathbf{Q}_p) = \mathcal{S}(X) \otimes_{\mathbf{Z}} \mathbf{Q}_p$.

(ii) Show that $\mathcal{S}(X, \mathbf{Q}_p)$ is dense in $C(X, \mathbf{Q}_p)$.

(iii) Show that every B-valued p-adic measure μ extends uniquely to a function on $C(X, \mathbf{Q}_p)$.

(iv) Conclude that there is an isomorphism of $\mathbf{Q}_p$-Banach spaces

$$\mathcal{D}_0(X, B) \cong \mathrm{Hom}_{\mathrm{cont}}(C(X, \mathbf{Q}_p), B),$$

where these space are equipped with the norm

$$|\varphi| := \sup_{f \neq 0} \frac{|\varphi(f)|}{|f|}.$$

Problem 1.18. Relation to completed group rings/the Amice transform. Let G be a profinite group given by

$$G = \varprojlim_n G/G_n,$$

where $G_{n+1} \leq G_n$ are a collection of open normal subgroups of finite index with trivial intersection.

(i) Show that $\mathcal{D}_0(G, \mathbf{Z}_p) \cong \mathbf{Z}_p[\![G]\!]$. (Hint: consider $\mathcal{D}_0(G/G_n, \mathbf{Z}_p)$ and relate it to $\mathbf{Z}_p[G/G_n]$.)

(ii) Let $\mu \in \mathcal{D}_0(\mathbf{Z}_p, \mathbf{Z}_p)$. Its Amice transform is

$$A_\mu(T) := \int_{\mathbf{Z}_p} (1+T)^x d\mu(x) = \sum_{n \geq 0} T^n \int_{\mathbf{Z}_p} \binom{x}{n} d\mu(x).$$

Show that $\mu \mapsto A_\mu(T)$ is the isomorphism $\mathcal{D}_0(\mathbf{Z}_p, \mathbf{Z}_p) \cong \mathbf{Z}_p[\![T]\!]$.

(iii) Show that if $z \in \mathbf{Z}_p$ with $|z| < 1$, then

$$A_\mu(z) = \int_{\mathbf{Z}_p} (1+z)^x d\mu(x).$$

2. The p-adic L-functions of Dirichlet characters

One of the main goals of this section is to prove the following important result of Kubota–Leopoldt and Iwasawa.

Theorem 2.1. *Let χ be a non-trivial, primitive, even Dirichlet character of the first kind. Then, there is an Iwasawa function $L_p(s,\chi)$ such that for all $n \in \mathbf{Z}_{\leq 0}$,*

$$L_p(n,\chi) = \left(1 - (\chi\omega^{n-1})(p)p^{-n}\right) L(n, \chi\omega^{n-1}).$$

When $\chi = \mathbf{1}$, the trivial character, there is a function $\zeta_p(s)$ such that $(s-1)\zeta_p(s)$ is an Iwasawa function and for all $n \in \mathbf{Z}_{\leq 0}$,

$$\zeta_p(n) = \left(1 - \omega^{n-1}(p)p^{-n}\right) L(n, \omega^{n-1}).$$

2.1. **Preliminaries.**

Problem 2.1. Dirichlet L-functions. Let χ be a ($\mathbf{C}$-valued) Dirichlet character. Its L-function is defined as a function of a complex variable s by the Dirichlet series

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}.$$

(i) Show that this Dirichlet series converges absolutely for $\mathrm{Re}(s) > 1$.
(ii) Show that if χ is non-trivial, then the Dirichlet series converges for $\mathrm{Re}(s) > 0$. (Hint: Ever heard of Dirichlet's test for convergence?)

Problem 2.2. Functional equation for Dirichlet L-functions. Let χ be a Dirichlet character and let $\epsilon \in \{0,1\}$ be given by $\chi(-1) = (-1)^\epsilon$. Then, the completed L-function of χ,

$$\Lambda(s,\chi) := \Gamma_{\mathbf{R}}(s+\epsilon)L(s,\chi),$$

satisfies

$$\Lambda(s,\chi) = \epsilon(s,\chi)\Lambda(1-s,\chi^{-1})$$

where

$$\epsilon(s,\chi) := \frac{\tau(\chi)}{i^\epsilon N^s}$$

is the epsilon factor of χ,

$$\tau(\chi) = \sum_{a=1}^{N} \chi(a)\exp(2\pi i a/N),$$

is the Gauss sum of χ and $\Gamma_{\mathbf{R}}(s)$ is a modification of the Γ-function defined in Problem 4.1 below. Feel free to try to show this.

Problem 2.3. The Bernoulli numbers. The Bernoulli numbers $B_n, n \in \mathbf{Z}_{\geq 0}$ are rational numbers defined by the following identity of formal power series

$$\frac{t\exp(t)}{\exp(t)-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

(i) Use this to work out the first 4 Bernoulli numbers.

(ii) Let $F(t) = \dfrac{t\exp(t)}{\exp(t)-1}$. Relate $F(-t)$ to $F(t)$ and use this to show that $B_n = 0$ for odd $n > 1$.

(iii) For another variable x, let

$$F(t,x) = F(t)\exp(tx)$$

and define the Bernoulli polynomials by

$$F(t,x) = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{2.1}$$

Show that

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$$

and so $B_n(0) = B_n$.

(iv) Use (2.1) to determine the first 3 Bernoulli polynomials.

(v) The Bernoulli numbers are named after Jacob Bernoulli who discovered them in a successful attempt to find a formula for the Faulhaber polynomials. Specifically, for $n, x \in \mathbf{Z}_{\geq 0}$, let

$$S_n(x) = \sum_{a=1}^{x} a^n$$

be the nth Faulhaber polynomial. Show that

$$S_n(x) = \frac{1}{n+1}(B_{n+1}(x) - B_{n+1}(0)).$$

Problem 2.4. The generalized Bernoulli numbers. For χ a Dirichlet character of conductor N, let

$$F_\chi(t) = \sum_{a=1}^{N} \frac{\chi(a)t\exp(at)}{\exp(Nt)-1}$$

and

$$F_\chi(t,x) = F_\chi(t)\exp(tx).$$

The generalized Bernoulli numbers, resp. the generalized Bernoulli polynomials, are defined by

$$F_\chi(t) = \sum_{n\geq 0} B_{n,\chi}\frac{t^n}{n!},$$

resp.

$$F_\chi(t,x) = \sum_{n\geq 0} B_{n,\chi}(x)\frac{t^n}{n!},$$

(i) Show that $F_\chi(-t,-x) = \chi(-1)F_\chi(t,x)$. What does this tell you about the vanishing of $B_{n,\chi}$?

(ii) Show that

$$B_{n,\chi}(x) = \frac{1}{n}\sum_{a=1}^{N}\chi(a)N^nB_n\left(\frac{a-N+x}{N}\right).$$

(iii) For $n, x \in \mathbf{Z}_{\geq 0}$, let $S_{n,\chi}(x) = \sum_{a=1}^{x}\chi(a)a^n$. Show that

$$S_{n,\chi}(xN) = \frac{1}{n+1}(B_{n+1,\chi}(xN) - B_{n+1,\chi}(0)).$$

(iv) Let $\mathbf{Q}(\chi)$ denote the field obtained by adjoining the values of χ to $\mathbf{Q}$. Show that $B_{n,\chi} \in \mathbf{Q}(\chi)$. Similarly, let $\mathbf{Q}_p(\chi)$ be the field obtained by adjoining the values of χ to $\mathbf{Q}_p$ (considering χ as a $\overline{\mathbf{Q}}_p$-valued character). Show that, in $\mathbf{Q}_p(\chi)$,

$$B_{n,\chi} = \lim_{e\to\infty}\frac{S_{n,\chi}(p^eN)}{p^eN}.$$

(v) The reason we are interested in the $B_{n,\chi}$ (and the reason Leopoldt introduced them) is because of the following important result.

Theorem 2.2. *Let χ be a primitive Dirichlet character of conductor N. For $n \in \mathbf{Z}_{\geq 1}$,*

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}$$

and, when n has the same parity as χ,

$$L(n,\chi) = \frac{\tau(\chi)}{N^n}\frac{(-2\pi i)^n}{2(n-1)!}L(1-n,\chi^{-1}).$$

Feel free to try to prove this. There are several ways, though all are somewhat involved. Obtaining the second formula from the first relies only on the functional equation stated above and some properties of the Γ-function; you could try at least deriving the second inequality from the first, thus getting an idea why you can't say anything about the values at positive integers of different parity than χ.

2.2. **Iwasawa's construction.** Iwasawa was the first to show the p-adic Dirichlet L-functions were Iwasawa functions. The basic plan of his method is to construct elements in $\mathbf{Z}_p[U^1_{p^n}]$, using classical Stickelberger elements, whose inverse limit in $\mathbf{Z}_p[\![1+q\mathbf{Z}_p]\!]$ yields the p-adic L-function. Slight modifications and some technicalities are required to actually carry this through.

Problem 2.5. Let χ be an odd ($\mathbf{Q}_p$-valued) Dirichlet character of the first kind of conductor N or Nq, for some N relatively prime to p. For $n \geq 0$, let $q_n = Nqp^n$. For $a \in \mathbf{Z}$, relatively prime to q_0, let $\gamma_n(a)$ denote the image of $a \pmod{q_n}$ under the projection $(\mathbf{Z}/q_n\mathbf{Z})^\times \to U^1_{p^n}$ (see Problem 1.14). Let $K/\mathbf{Q}_p$ be an extension containing the values of χ, let $\mathcal{O}$ be its ring of integers, and let $\Lambda_{\mathcal{O}} := \mathcal{O}[\![T]\!]$. Basically, everything you worked out in §1 for Λ and $\mathbf{Z}_p[\![\Gamma]\!]$ carries over to $\Lambda_{\mathcal{O}}$ and $\mathcal{O}[\![\Gamma]\!]$. Let

$$\xi_{n,\chi} := -\frac{1}{q_n} \sum_{\substack{a=0 \\ (a,q_0)=1}}^{q_n} a\chi(a)\gamma_n(a)^{-1} \in K[U^1_{p^n}]$$

(basically twisted Stickelberger elements).

(i) Show that, under the natural map $K[U^1_{p^{n+1}}] \to K[U^1_{p^n}]$, $\xi_{n+1,\chi}$ gets sent to $\xi_{n,\chi}$.

(ii) Do the same with the elements

$$\eta_{n,\chi} := \left(1 - \frac{1+q_0}{\gamma_n(1+q_0)}\right)\xi_{n,\chi}.$$

(iii) Show that $\eta_{n,\chi} \in \mathcal{O}[U^1_{p^n}]$.

(iv) Show that if $\chi \neq \omega^{-1}$, then $\xi_{n,\chi} \in \mathcal{O}[U^1_{p^n}]$.

Problem 2.6. The previous problem yields compatible elements $\xi_{n,\chi}$ and $\eta_{n,\chi}$. Let

$$\eta_\chi := \varprojlim \eta_{n,\chi} \in \mathcal{O}[\![1+q\mathbf{Z}_p]\!]$$

and, for $\chi \neq \omega^{-1}$, let

$$\xi_\chi := \varprojlim \xi_{n,\chi} \in \mathcal{O}[\![1+q\mathbf{Z}_p]\!]$$

(i) Show that under the decomposition $(\mathbf{Z}_{p,N})^\times \cong \Delta_N \times (1+q\mathbf{Z}_p)$ (where $\Delta_N := (\mathbf{Z}/Nq\mathbf{Z})^\times$ and $(\mathbf{Z}_{p,N})^\times = \varprojlim\, (\mathbf{Z}/Nq_n\mathbf{Z})^\times$), the image of a in the factor $1+q\mathbf{Z}_p$ is $\langle a\rangle$.

(ii) Picking $\langle 1+q_0\rangle$ as a topological generator of $1+q\mathbf{Z}_p$, yields an isomorphism $\Lambda_{\mathcal{O}} \cong \mathcal{O}[\![1+q\mathbf{Z}_p]\!]$ sending $1+T$ to $\langle 1+q_0\rangle$. Determine the element $h(T,\chi)$ of Λ that corresponds to $\varprojlim \left(1 - \frac{1+q_0}{\gamma_n(1+q_0)}\right)$. Write down the Iwasawa function it corresponds to and show that this function has a unique zero at $s=1$, and this zero has order 1.

(iii) For $\chi \neq \omega^{-1}$, let $f(T,\chi) \in \Lambda_{\mathcal{O}}$ be the element corresponding to η_χ. For general χ, let $g(T,\chi) \in \Lambda_{\mathcal{O}}$ be the element corresponding to ξ_χ. When $\chi = \omega^{-1}$, let $f(T,\chi) := g(T,\chi)/h(T,\chi)$ in the fraction field of Λ. Let $\theta = \chi\omega$ so that θ is an even Dirichlet character of the first kind. Show that

$$f((1+q_0)^{1-n}-1,\chi) = (1-(\theta\omega^{-n})(p)p^{n-1})L(1-n,\theta\omega^{-n}),$$

where $(\theta\omega^{-n})$ denotes the primitive character associated to $\theta\omega^{-n}$. (Hint: Part (iv) of Problem 2.4 will show up here.)

(iv) Conclude that for θ a primitive even Dirichlet character of the first kind, the p-adic L-function $L_p(s,\theta)$ whose existence is asserted in Theorem 2.1 is the Iwasawa function associated to $f(T,\theta\omega^{-1})$.

3. The p-adic L-functions of modular forms

We will focus on p-ordinary cuspidal eigenforms of weight 2 with rational coefficients and level prime to p. The analogue of Theorem 2.1 for modular forms is the following result due to Manin, Amice–Vélu, and Višik (with undefined terms defined in the rest of the section).

Theorem 3.1. *Let $f = \sum_{n\geq 1} a_n q^n$ be a cuspidal, holomorphic newform of level $\Gamma_0(N)$ and weight $k = 2$ with rational Hecke eigenvalues and $p \nmid N$. Let $H_p(x) = x^2 - a_p x + p$ be its Hecke polynomial at p. Assume f is p-ordinary, i.e. $p \nmid a_p$. Let α_p and β_p be the roots of the Hecke polynomial (in $\overline{\mathbf{Q}}_p$) with $v_p(\alpha_p) = 0$. Then there is a unique p-adic measure μ_f on $\mathbf{Z}_p^\times$ such that for every finite-order Dirichlet character χ of conductor $p^r > 1$, we have*

$$\int_{\mathbf{Z}_p^\times} \chi d\mu_f = \tau(\chi)\frac{1}{\alpha_p^r}\frac{L(1,f,\overline{\chi})}{\Omega_f^{\chi(-1)}}$$

and

$$\int_{\mathbf{Z}_p^\times} d\mu_f = \left(1 - \frac{1}{\alpha_p}\right)^2 \frac{L(1,f)}{\Omega_f^+} \tag{3.1}$$

$$= \left(1 - \frac{1}{\alpha_p}\right)\left(1 - \frac{\beta_p}{p}\right)\frac{L(1,f)}{\Omega_f^+}. \tag{3.2}$$

3.1. **Preliminaries.** Some (most? all?) problems in this section require some knowledge about modular forms. Suggested sources include Miyake's book or Diamond and Shurman's book.

Problem 3.1. Given a holomorphic cusp form $f = \sum_{n\geq 1} a_n q^n$, its L-function is defined to be

$$L(s,f) := \sum_{n\geq 1}\frac{a_n}{n^s} \quad \text{for } \operatorname{Re}(s) > 1 + \frac{k}{2}.$$

This has an analytic continuation to an analytic function on all of $\mathbf{C}$. Given a Dirichlet character χ, the imprimitive (χ)-twisted L-function of f is

$$L(s,f,\chi) := \sum_{n\geq 1}\frac{\chi(n)a_n}{n^s},$$

which also has analytic continuation to an entire function. The completed imprimitive χ-twisted L-function of f is

$$\Lambda(s,f,\chi) := \Gamma_{\mathbf{C}}(s)L(s,f,\chi).$$

Problem 3.2. Let f be a holomorphic cuspform of weight 2 and level $\Gamma_0(N)$. If you're unfamiliar with modular forms (or just don't want to deal with the specifics right now), you can use the results here as a black box.

(i) Show that $L(1,f) = 2\pi i \int_{i\infty}^{0} f(z)dz$.

(ii) Suppose χ is a primitive Dirichlet character mod p^r. Show that

$$L(1,f,\chi^{-1}) = 2\pi i \frac{\tau(\overline{\chi})}{p^r} \sum_{a\in(\mathbf{Z}/p^r\mathbf{Z})^\times} \chi(a) \int_{i\infty}^{-a/p^r} f(z)dz. \tag{3.3}$$

3.2. **Construction.**

Problem 3.3. The sum on the right-hand side of (3.3) looks like the kind of Riemann sum that shows up in integrating the locally constant function χ against a p-adic measure. Naïvely, we could hope to define

$$\mu_f(a+p^r\mathbf{Z}_p) = 2\pi i \int_{i\infty}^{-a/p^r} f(z)dz.$$

This won't quite work. Let's explore that. For $c \in \mathbf{P}^1(\mathbf{Q})$, let

$$\{c\}^\pm := 2\pi i \left(\int_{i\infty}^{c} f(z)dz \pm \int_{i\infty}^{-c} f(z)dz \right).$$

Then, Shimura proved that there exist $\Omega_f^\pm \in \mathbf{C}^\times$ such that for all $c \in \mathbf{P}^1(\mathbf{Q})$, $\{c\}^\pm \in \mathbf{Q}\Omega_f^\pm$. Define

$$[c]^\pm := \frac{\{c\}^\pm}{\Omega_f^\pm}.$$

Suppose f has level $\Gamma_0(N)$ with $p \nmid$. The T_p Hecke operator acts by

$$T_p f(z) = p\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + pf(pz).$$

Suppose f is an eigenform for the T_p-operator, i.e. $T_pf = a_pf$. Show that

$$a_p\left[\frac{a}{p^n}\right]^\pm = \left(\sum_{j=0}^{p-1}\left[\frac{a+jp^n}{p^{n+1}}\right]^\pm\right) + \left[\frac{a}{p^{n-1}}\right]^\pm. \tag{3.4}$$

Problem 3.4. The equation (3.4) is almost the 'Distribution Property'. One problem is the extra term on the right-hand side. It represents the difference between the T_p operator and the U_p operator. Let's elaborate. If g is modular form of level $\Gamma_0(Np)$, then the U_p operator acts on g by

$$U_p g(z) = p\sum_{j=0}^{p-1} g\left(\frac{z+j}{p}\right).$$

(i) Given $f \in \Gamma_0(N)$ (with $p \nmid N$), show that $f(pz)$ has level $\Gamma_0(Np)$ and $f(z)$ and $f(pz)$ span a U_p-stable subspace of the space of cusp forms of level $\Gamma_0(Np)$ on which the characteristic polynomial of U_p is $H_p(x) = x^2 - apx + p$.

(ii) Let α be a root of $H_p(x)$. Write down the linear combination $f_\alpha(z)$ of $f(z)$ and $f(pz)$ such that $U_p f_\alpha = \alpha f_\alpha$. This is called a p-stabilization of f.

Problem 3.5. Given f as above, let α be the p-adic unit root of $H_p(x)$ and let

$$[c]_\alpha^\pm := \frac{2\pi i}{\Omega_f^\pm}\left(\int_{i\infty}^{c} f_\alpha(z)dz \pm \int_{i\infty}^{-c} f_\alpha(z)dz\right).$$

(i) Define

$$\mu_f^\pm(a+p^r\mathbf{Z}_p) := \frac{1}{\alpha^n}\left[-\frac{a}{p^r}\right]_\alpha^\pm$$

and $\mu_f := \mu_f^+ + \mu_f^-$. Show that μ_f satisfies the Distribution Property and thus gives a p-adic measure.

(ii) Show that μ_f has the properties claimed in Theorem 3.1.

Problem 3.6. Now, let f be primitive of level Np (where $p \nmid N$). In this case, the Hecke polynomial at p is $H_p(x) = x - a_p$, so that we take $\alpha_p = a_p$ and $\beta_p = 0$. Show that μ_f satisfying equation (3.2) still exists.

4. Conjectural p-adic L-functions of motives

4.1. **Deligne's conjecture of special values of L-functions.** Deligne introduces a kind of axiomatic setup for what we mean by a motive; a setup that is sufficient for the purposes of describing their L-functions and the rationality properties of their values at integers. We'll introduce that a bit in this section of problems. This is of great importance in the theory of p-adic L-functions because, for most cases, p-adic L-functions are defined by interpolating values of usual L-functions and these values must thus at least be algebraic so that they can be considered as p-adic numbers.

In Deligne's setup, a motive comes with several *realizations* and we'll introduce these as we go along. A motive is then just thought of as the collection of these realizations (in reality, a motive is a much more complicated thing to define, and a few different definitions exist, but in the end, the point is that the category of motives is supposed to be a universal target for cohomology theories of varieties and the realizations we'll see are then specific cohomologies of, say, smooth projective varieties over $\mathbf{Q}$).

Definition 4.1. The Γ-function is $\Gamma(s) = \displaystyle\int_0^\infty e^{-t}t^s\frac{dt}{t}$ for $\mathrm{Re}(s) > 0$, extended by analytic continuation to a meromorphic function with simple poles at $n \in \mathbf{Z}_{\leq 0}$ with residues $\dfrac{(-1)^n}{(-n)!}$.

Problem 4.1. Given a motive M, its Betti realization is a finite-dimensional $\mathbf{Q}$-vector space $H_B(M)$ equipped with a $\mathbf{Q}$-linear involution $F_\infty : H_B(M) \to H_B(M)$ and a Hodge decomposition

$$H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{m,n\in\mathbf{Z}} H^{m,n}(M)$$

such that $F_\infty : H^{m,n}(M) \xrightarrow{\sim} H^{n,m}(M)$. Let $h^{m,n} := \dim_{\mathbf{C}} H^{m,n}(M)$ and, for $\epsilon \in \{0,1\}$, let $h^{m,m,\epsilon} := \dim_{\mathbf{C}}(H^{m,m}(M))^{F_\infty=(-1)^{m+\epsilon}}$. The dual motive $M^\vee$ has Betti realization the $\mathbf{Q}$-dual of $H_B(M)$ and $H^{m,n}(M^\vee) = H^{-m,-n}(M)$. The L-function of M is defined as a product of factors over the places of $\mathbf{Q}$. The factor at infinity is often called the Γ-factor of the L-function and is defined by

$$\Gamma_M(s) := \left(\prod_{m<n}\Gamma_{\mathbf{C}}(s-m)^{h^{m,n}}\right)\cdot\left(\prod_{m=n}\prod_{\epsilon\in\{0,1\}}\Gamma_{\mathbf{R}}(s-m+\epsilon)^{h^{m,m,\epsilon}}\right)$$

where

$$\Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s) \quad\text{and}\quad \Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma(s/2).$$

Deligne's conjecture states that the value $L(s,M)$ of the L-function of the motive M should be, up to certain specific 'periods', rational at certain integers s, these are called the critical integers. An integer k is critical for M if neither $\Gamma_M(s)$ nor $\Gamma_{M^\vee}(1-s)$ has a pole at $s = k$. A motive is called critical if 0 is a critical integer for M. Finally, let's show something.

(i) Suppose $H^{m,m}(M) = 0$ for all m. Show that k is critical for M if and only if

$$\max_{\substack{m<n \\ h^{m,n}\neq 0}} m < k \leq \min_{\substack{m<n \\ h^{m,n}\neq 0}} n.$$

(ii) Suppose that there is an $m \in \mathbf{Z}$ such that $h^{m,m,0}$ and $h^{m,m,1}$ are both non-zero. Show that there are no critical integers.

(iii) Suppose the only $m \in \mathbf{Z}$ for which $h^{m,m} \neq 0$ is $m = 0$ and only one of $h^{0,0,0}$ and $h^{0,0,1}$ is non-zero. Show that if $h^{0,0,0} \neq 0$, then the critical integers are the negative odd integers and the positive even integers (like the Riemann zeta function!). Show that if $h^{0,0,1} \neq 0$, then the critical integers are the non-positive even integers and the positive odd integers (like the the L-function of an odd Dirichlet character).

Problem 4.2. Given two motives M and M', the Hodge decomposition of their tensor product has

$$H^{m,n}(M \otimes M') = \bigoplus_{\substack{i+i'=m \\ j+j'=n}} H^{i,j}(M) \otimes H^{i',j'}(M).$$

Fix $r \in \mathbf{Z}_{\geq 1}$ and suppose M and M' both have $H^{m,n} \neq 0$ only for $(m,n) = (r,0)$ and $(0,r)$ (such as the motive attached to a holomorphic cuspidal eigenform of weight $r+1$).

(i) Show that $M \otimes M'$ has *no* critical integers (this is like a Rankin product L-function of two modular forms of the same weight).

(ii) Show that $\mathrm{Sym}^2(M)$ has critical integers, and find them.

(iii) What about higher symmetric powers?

(iv) What if M and M' both have Hodge decomposition as stated above, but for two different positive integers r and r', respectively? Can you choose these so that $M \otimes M'$ has critical integers?

Problem 4.3. The Tate motive $\mathbf{Q}(1)$ has Hodge decomposition $H_B(\mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{C} = H^{-1,-1}(\mathbf{Q}(1))$ with $H_B(\mathbf{Q}(1))$ one-dimensional and F_∞ acting by multiplication by -1.

(i) Write down the Γ-factor of $\mathbf{Q}(1)$. At which integers is $\mathbf{Q}(1)$ critical?

(ii) For $n \in \mathbf{Z}_{\geq 1}$, $\mathbf{Q}(n) := \mathbf{Q}(1) \otimes \cdot \otimes \mathbf{Q}(1) = \mathbf{Q}(1)^{\otimes n}$ (the n-fold tensor product), $\mathbf{Q}(-1) := \mathbf{Q}(1)^\vee$, and $\mathbf{Q}(-n) = \mathbf{Q}(-1)^{\otimes n}$. And $\mathbf{Q}(0)$ is usually denoted $\mathbf{Q}$ and you could think of it as $\mathbf{Q}(1) \otimes \mathbf{Q}(-1)$! Given a motive M, the motive $M \otimes \mathbf{Q}(n)$ is called the nth Tate twist of M and denoted $M(n)$. Write $\Gamma_{M(n)}(s)$ it terms of $\Gamma_M(s)$. Show that n is a critical integer for M if and only if $M(n)$ is critical.

Problem 4.4. If χ is a Dirichlet character, then its motive M_χ has Hodge decomposition $H_B(M_\chi) \otimes_{\mathbf{Q}} \mathbf{C} = H^{0,0}(M_\chi)$ with $H_B(M_\chi)$ one-dimensional and F_∞ acting by $\chi(-1)$. Write down the Γ-factor of M_χ. At which integers is M_χ critical?

4.2. **Conjectural p-adic L-function.**

Problem 4.5. To describe the modification of the L-function $L(s, M)$ of a motive M required to p-adically interpolate its L-values requires the p-adic realization of the motive: a p-adic Galois representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Q}_p)$ (here, $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/Q)$ is the absolute Galois group of $\mathbf{Q}$ and the representation is assumed to be continuous). Let V be the vector space on which this representation acts. Greenberg introduced the notion of M being ordinary at p: there is an (exhaustive, separated) descending filtration $\{F^iV\}_{i\in\mathbf{Z}}$ of $G_{\mathbf{Q}_p}$-stable $\mathbf{Q}_p$-subspaces of V such that the inertia subgroup I_p acts on the ith graded piece $\mathrm{gr}^i V := F^iV/F^{i+1}V$ via multiplication by χ_p^i, where χ_p is the p-adic cyclotomic character (χ_p is the p-adic realization of $\mathbf{Q}(1)$) (exhaustive and separated means that $F^iV = V$ for some i and $F^iV = 0$ for some other i). Let

$$H_p(x) = \prod_{i\in\mathbf{Z}} \det(I - \mathrm{Frob}_p\, p^i X : (\mathrm{gr}^i V) \otimes \chi_p^{-i})$$

be the product of the (inverse, twisted) characteristic polynomial of the Frobenius element at p acting on the (unramified) representation of $G_{\mathbf{Q}_p}$ $(\mathrm{gr}^i V) \otimes \chi_p^{-i}$. Let α_j be the inverse roots of $H_p(x)$, i.e.

$$H_p(x) = \prod_j (1 - \alpha_j x).$$

Let

$$\mathcal{E}_p(M) := \prod_{v_p(\alpha_j)\le 1} \left(1 - \frac{1}{\alpha_j}\right) \prod_{v_p(\alpha_j)\ge 1} \left(1 - \frac{\alpha_j}{p}\right)$$

be the interpolation factor. Suppose M is critical at $s = 1$. Deligne's setup produces (conjectural) periods $\Omega_M \in \mathbf{C}$ such that

$$\frac{L(1, M)}{\Omega_M} \in \mathbf{Q}.$$

For simplicity (in some sense) assume p is odd and let Γ^+ denote $\mathbf{Z}_p^\times/\{\pm 1\}$. Also, assume that $V^{G_{\mathbf{Q}_\infty^+}} = 0$ (where $\mathbf{Q}_\infty^+$ is the maximal totally real subfield of $\mathbf{Q}(\mu_{p^\infty})$). The conjecture of Coates and Perrin-Riou then states that there is a p-adic measure μ_M on Γ^+ such that

$$\int_{\Gamma^+} \chi(x)\, d\mu_f(x) = \begin{cases} \mathcal{E}_p(M) \dfrac{L(1, M)}{\Omega_M}, & \chi \text{ trivial} \\ \tau(\chi)^{d_M} \left(\displaystyle\prod_{v_p(\alpha_j) > -1/2} \frac{1}{\alpha_j^r} \right) \dfrac{L(1, M, \overline{\chi})}{\Omega_M}, & \chi \neq 1 \text{ even, of conductor } p^r, \end{cases}$$

where $d_M = \#\{\alpha_j : v_p(\alpha_j) > 0\}$.

(i) Show that the formula I just wrote is actually what Coates' article *Motivic p-adic L-functions* actually says this!

(ii) Show that a motive M is p-ordinary if and only $M(n)$ is p-ordinary.

Problem 4.6. Suppose χ is a primitive Dirichlet character of conductor N with $p \nmid N$. The p-adic realization of M_χ is a character $\rho_{\chi,p}$ of $G_{\mathbf{Q}}$ that is unramified at p and sends Frob_p to $\chi(p)$.

(i) Show that M_χ is ordinary at p.

(ii) For all n such that $M_\chi(n)$ is critical at $s = 1$, write out $\mathcal{E}_p(M_\chi(n))$. (By the way, if M and M' are two motives with p-adic realization ρ_p and ρ'_p, then the p-adic realization is $\rho_p \otimes \rho'_p$.)

(iii) The following is true (and not too hard to show if you just track down the definitions): if $L(s, M)$ is the L-function of M, then $L(s, M(n)) = L(s + n, M)$. Use this to compare the Coates–Perrin-Riou conjecture for $M_\chi(n)$ with Theorem 2.1.

Problem 4.7. If f is a primitive, p-ordinary, weight 2, holomorphic, cuspidal eigenform of level $\Gamma_0(N)$ (with $p \nmid N$) and coefficients in $\mathbf{Q}$ (equivalently, the modular form attached to an elliptic curve over $\mathbf{Q}$ of conductor N with good, ordinary reduction at p), then we said that its motive M_f has Hodge decomposition

$$H_B(M_f) \otimes_{\mathbf{Q}} \mathbf{C} = H^{1,0}(M_f) \oplus H^{0,1}(M_f).$$

Furthermore, $H_B(M_f)$ is two-dimensional. The most important part of its p-adic realization $\rho_{f,p}$ is $\rho_{f,p}|_{G_{\mathbf{Q}_p}}$. Let α be the p-adic unit root of $x^2 - a_p x + p$ and let δ_α be the unramified character of $G_{\mathbf{Q}_p}$ sending Frob_p to α. A theorem of Wiles says that there's a basis in which

$$\rho_{f,p}|_{G_{\mathbf{Q}_p}} = \begin{pmatrix} \chi_p \delta_\alpha^{-1} & * \\ 0 & \delta_\alpha \end{pmatrix}.$$

(i) Show that $\rho_{f,p}$ is p-ordinary in the sense above, and write down the ordinary filtration.

(ii) Write down the interpolation factor $\mathcal{E}_p(M_f)$. Compare to Theorem 3.1.

(iii) Determine which Tate twists of $\mathrm{Sym}^2 M_f$ are critical at $s = 1$. For each of these, write out $\mathcal{E}_p(\mathrm{Sym}^2 M_f)$. Notice anything disturbing? What might you do about it?