HARMONIC MAASS FORMS, MOCK MODULAR FORMS, AND QUANTUM MODULAR FORMS

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In his enigmatic death bed letter to Hardy, written in January 1920, Ramanujan introduced the notion of a *mock theta function.* Despite many works, very little was known about the role that these functions play within the theory of automorphic and modular forms until 2002. In that year Sander Zwegers (in his Ph.D. thesis) established that these functions are "holomorphic parts" of harmonic Maass forms. This realization has resulted in many applications in a wide variety of areas: arithmetic geometry, combinatorics, modular forms, and mathematical physics. Here we outline the general facets of the theory, and we give several applications to number theory: partitions and q-series, modular forms, singular moduli, Borcherds products, extensions of theorems of Kohnen-Zagier and Waldspurger on modular L-functions, and the work of Bruinier and Yang on Gross-Zagier formulae. Following our discussion of these works on harmonic Maass forms, we shall then study the emerging new theory of *quantum modular forms.* Don Zagier introduced the notion of a quantum modular form in his 2010 Clay lecture, and it turns out that a beautiful part of this theory lives at the interface of classical modular forms and harmonic Maass forms.

1. Zwegers's weight 1/2 non-holomorphic Jacobi form

In his thesis, Zwegers constructed weight 1/2 harmonic Maass forms by making use of the transformation properties of Lerch sums. Here we briefly recall some of these important results.

For $\tau \in \mathbb{H}$, $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, Zwegers defined the function

$$(1.1) \qquad \mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v; \tau)} \cdot \sum_{n \in \mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n},$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q := e^{2\pi i \tau}$ and

$$(1.2) \qquad \vartheta(v; \tau) := \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{\pi i \nu} w^\nu q^{\nu^2/2}.$$

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Lemma 1.1. *Assuming the notation above, we have that*

$$\mu(u, v; \tau) = \mu(v, u; \tau),$$
$$\mu(u + 1, v; \tau) = -\mu(u, v; \tau),$$
$$z^{-1}wq^{-\frac{1}{2}}\mu(u + \tau, v; \tau) = -\mu(u, v; \tau) - iz^{-\frac{1}{2}}w^{\frac{1}{2}}q^{-\frac{1}{8}},$$
$$\mu(u, v; \tau + 1) = \zeta_8^{-1}\mu(u, v; \tau) \qquad (\zeta_N := e^{2\pi i/N})$$
$$(\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\mu\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = -\mu(u, v; \tau) + \frac{1}{2i}h(u - v; \tau),$$

where

$$h(z; \tau) := \int_{-\infty}^{\infty} \frac{e^{\pi ix^2\tau - 2\pi xz}dx}{\cosh \pi x}.$$

Remark 1. The integral $h(z; \tau)$ is known as a *Mordell integral*.

Lemma 1.1 shows that $\mu(u, v; \tau)$ is nearly a weight $1/2$ Jacobi form, where τ is the modular variable. Zwegers then uses μ to construct weight $1/2$ harmonic Maass forms. He achieves this by modifying μ to obtain a function $\widehat{\mu}$ which he then uses as building blocks for such Maass forms. To make this precise, for $\tau \in \mathbb{H}$ and $u \in \mathbb{C}$, let $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$, and let

$$(1.3) \qquad R(u; \tau) := \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\nu-\frac{1}{2}}\left\{\mathrm{sgn}(\nu) - E\left((\nu + c)\sqrt{2\mathrm{Im}(\tau)}\right)\right\} e^{-2\pi i\nu u}q^{-\nu^2/2},$$

where $E(x)$ is the odd function

$$(1.4) \qquad E(x) := 2\int_0^x e^{-\pi u^2}du = \mathrm{sgn}(x)(1 - \beta(x^2)),$$

where for positive real x we let $\beta(x) := \int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du$.

Using μ and R, Zwegers defines the real analytic function

$$(1.5) \qquad \widehat{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2}R(u - v; \tau).$$

Theorem 1.2. *Assuming the notation and hypotheses above, we have that*

$$\widehat{\mu}(u, v; \tau) = \widehat{\mu}(v, u; \tau),$$
$$\widehat{\mu}(u + 1, v; \tau) = z^{-1}wq^{-\frac{1}{2}}\widehat{\mu}(u + \tau, v; \tau) = -\widehat{\mu}(u, v; \tau),$$
$$\zeta_8\widehat{\mu}(u, v; \tau + 1) = -(\tau/i)^{-\frac{1}{2}}e^{\pi i(u-v)^2/\tau}\widehat{\mu}\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) = \widehat{\mu}(u, v; \tau).$$

Moreover, if $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then

$$\widehat{\mu}\left(\frac{u}{\gamma\tau + \delta}, \frac{v}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = \chi(A)^{-3}(\gamma\tau + \delta)^{\frac{1}{2}}e^{-\pi i\gamma(u-v)^2/(\gamma\tau+\delta)} \cdot \widehat{\mu}(u, v; \tau),$$

where $\chi(A) := \eta(A\tau)/\left((\gamma\tau + \delta)^{\frac{1}{2}}\eta(\tau)\right).$

Theorem 1.2 shows that $\widehat{\mu}(u, v; \tau)$ is essentially a weight $1/2$ non-holomorphic Jacobi form. In analogy with the classical theory of Jacobi forms, one may then obtain harmonic Maass forms by making suitable specializations for u and v by elements in $\mathbb{Q}\tau + \mathbb{Q}$, and by multiplying by appropriate powers of q. Without this result, it would be very difficult to explicitly construct examples of weight $1/2$ harmonic Maass forms.

Harmonic Maass forms of weight k are mapped to classical modular forms (see Lemma 4.1), their so-called *shadows*, by the differential operator

$$\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \overline{\tau}}}.$$

The following lemma makes it clear that the shadows of the real analytic forms arising from $\widehat{\mu}$ can be described in terms of weight $3/2$ theta functions.

Lemma 1.3. *The function R is real analytic and satisfies*

$$\frac{\partial R}{\partial \overline{u}}(u; \tau) = \sqrt{2}y^{-\frac{1}{2}}e^{-2\pi c^2 y}\vartheta(\overline{u}; -\overline{\tau}),$$

where $c := \mathrm{Im}(u)/\mathrm{Im}(\tau)$. Moreover, we have that

$$\frac{\partial}{\partial \overline{\tau}}R(a\tau - b; \tau) = -\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y}\sum_{\nu \in \mathbb{Z}+\frac{1}{2}}(-1)^{\nu-\frac{1}{2}}(\nu + a)e^{-\pi i \nu^2 \overline{\tau}-2\pi i \nu(a\overline{\tau}-b)}.$$

2. Harmonic Maass forms

In 1949, H. Maass introduced the notion of a *Maass form*[1] He constructed these non-holomorphic automorphic forms using Hecke characters of real quadratic fields, in analogy with Hecke's theory of *modular forms with complex multiplication* (see Ribet's famous paper for a modern treatment).

To define these functions, let $\Delta = \Delta_0$ be the *hyperbolic Laplacian*

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right),$$

where $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. It is a second-order differential operator which acts on functions on $\mathbb{H}$, and it is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$.

A *Maass form* for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have

$$f\left(\frac{az + b}{cz + d}\right) = f(z).$$

(2) We have that f is an eigenfunction of Δ.
(3) There is some $N > 0$ such that

$$f(x + iy) = O(y^N)$$

as $y \to +\infty$.

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[1] In analogy with the eigenvalue problem for the vibrating membrane, Maass referred to these automorphic forms as *Wellenformen*, or *waveforms*.

Furthermore, we call f a *Maass cusp form* if

$$\int_0^1 f(z+x)dx = 0.$$

There is now a vast literature on Maass forms.

This paper concerns a generalization of this notion of Maass form. Following Bruinier and Funke, we define the notion of a harmonic Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ as follows. As before, we let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. We define the weight k hyperbolic Laplacian Δ_k by

$$(2.1) \qquad \Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For odd integers d, define ϵ_d by

$$(2.2) \qquad \epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Definition 2.1. If N is a positive integer (with $4 \mid N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$), then a *weight k harmonic Maass form on* $\Gamma \in \{\Gamma_1(N), \Gamma_0(N)\}$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying the following:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}$, we have

$$M\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k M(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k}(cz+d)^k \ M(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

   Here $\left(\frac{c}{d}\right)$ denotes the extended Legendre symbol, and $\sqrt{z}$ is the principal branch of the holomorphic square root.

(2) We have that $\Delta_k M = 0$.

(3) There is a polynomial $P_M = \sum_{n \le 0} c^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that

$$M(z) - P_M(z) = O(e^{-\epsilon y})$$

   as $y \to +\infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

Remark 2. Maass forms and classical modular forms are required to satisfy moderate growth conditions at cusps, and it is for this reason that harmonic Maass forms are often referred to as "harmonic weak Maass forms". The term "weak" refers to the relaxed condition Definition 2.1 (3) which gives rise to a rich theory. For convenience, we use the terminology "harmonic Maass form" instead of "harmonic weak Maass form".

Remark 3. We refer to the polynomial P_M as the *principal part* of $M(z)$ at ∞. Obviously, if P_M is non-constant, then $M(z)$ has exponential growth at ∞. Similar remarks apply at all cusps.

Remark 4. Bruinier and Funke define two types of harmonic Maass forms based on varying the growth conditions at cusps. For a group Γ, they refer to these spaces as $H_k(\Gamma)$ and $H_k^+(\Gamma)$. Definition 2.1 (3) corresponds to their $H_k^+(\Gamma)$ definition.

Remark 5. Since holomorphic functions on $\mathbb{H}$ are harmonic, it follows that weakly holomorphic modular forms are harmonic Maass forms.

3. Fourier expansions

We shall consider harmonic Maass forms with weight $2 - k \in \frac{1}{2}\mathbb{Z}$ with $k > 1$. Therefore, throughout we assume that $1 < k \in \frac{1}{2}\mathbb{Z}$.

Harmonic Maass forms have distinguished Fourier expansions which are described in terms of the incomplete Gamma-function $\Gamma(\alpha; x)$

$$(3.1) \qquad \Gamma(\alpha; x) := \int_x^\infty e^{-t} t^{\alpha-1} \, dt,$$

and the usual parameter $q := e^{2\pi i z}$. The following characterization is straightforward.

Lemma 3.1. *Assume the notation and hypotheses above, and suppose that N is a positive integer. If $f(z) \in H_{2-k}(\Gamma_1(N))$, then its Fourier expansion is of the form*

$$(3.2) \qquad f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n < 0} c_f^-(n) \Gamma(k - 1, 4\pi|n|y) q^n,$$

where $z = x + iy \in \mathbb{H}$, with $x, y \in \mathbb{R}$.

As Lemma 3.1 reveals, $f(z)$ naturally decomposes into two summands. In view of this fact, we make the following definition.

Definition 3.2. Assuming the notation and hypotheses in Lemma 3.1, we refer to

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n$$

as the *holomorphic part of $f(z)$*, and we refer to

$$f^-(z) := \sum_{n < 0} c_f^-(n) \Gamma(k - 1, 4\pi|n|y) q^n$$

as the *non-holomorphic part of $f(z)$*.

Remark 6. A harmonic Maass form with trivial non-holomorphic part is a weakly holomorphic modular form. We shall make use of this fact as follows. If $f_1, f_2 \in H_{2-k}(\Gamma)$ are two harmonic Maass forms with equal non-holomorphic parts, then $f_1 - f_2 \in M_{2-k}^!(\Gamma)$.

4. The ξ-operator and period integrals of cusp forms

The following lemma plays a central role in the subject to relate spaces of cusp forms to spaces of harmonic Maass forms.

Lemma 4.1. *If $f \in H_{2-k}(N, \chi)$, then*

$$\xi_{2-k} : H_{2-k}(N, \chi) \longrightarrow S_k(N, \overline{\chi})$$

is a surjective map. Moreover, assuming the notation in Definition 3.2, we have that

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^\infty \overline{c_f^-(-n)} n^{k-1} q^n.$$

Thanks to Lemma 4.1, we are in a position to relate the non-holomorphic parts of harmonic Maass forms, the expansions

$$f^-(z) := \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y)q^n,$$

with "period integrals" of modular forms.

To make this connection, we must relate the Fourier expansion of the cusp form $\xi_{2-k}(f)$ with $f^-(z)$. This connection is made by applying the simple integral identity

(4.1) $$\int_{-\bar z}^{i\infty} \frac{e^{2\pi i n\tau}}{(-i(\tau+z))^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi ny)q^{-n}.$$

This identity follows by the direct calculation

$$\int_{-\bar z}^{i\infty} \frac{e^{2\pi i n\tau}}{(-i(\tau+z))^{2-k}} \, d\tau = \int_{2iy}^{i\infty} \frac{e^{2\pi i n(\tau-z)}}{(-i\tau)^{2-k}} \, d\tau = i(2\pi n)^{1-k} \cdot \Gamma(k-1, 4\pi ny) \, q^{-n}.$$

In this way, we may think of the non-holomorphic parts of weight $2-k$ harmonic Maass forms as period integrals of weight k cusp forms, where one applies (4.1) to

$$\int_{-\bar z}^{i\infty} \frac{\sum_{n=1}^\infty a(n)e^{2\pi i n\tau}}{(-i(\tau+z))^{2-k}} \, d\tau,$$

where $\sum_{n=1}^\infty a(n)q^n$ is a weight k cusp form. In short, $f^-(z)$ is the period integral of the cusp form $\xi_{2-k}(f)$.

In view of the surjectivity of spaces of harmonic Maass forms onto spaces of cusp forms guaranteed by Proposition 4.1, combined with the well known importance of cusp forms of various types, we find ourselves asking the following natural question.

Question 4.2. *Suppose that $g(z) \in S_k(N)$ is a cusp form of interest (e.g. in arithmetic geometry, additive number theory, etc), and suppose that $f(z) \in H_{2-k}(N)$ is a "suitable" harmonic Maass form for which*

$$\xi_{2-k}(f) = g.$$

By the discussion above, we have that $f^-(z)$ is morally the Fourier expansion of $g(z)$. The question is: What interesting information about $g(z)$ is unearthed by discovering the q-series $f^+(z)$?

These lectures will aim to answer several forms of this question such as those applications listed in the first paragraph, as well as explain how the structure of these spaces lead naturally to explicit examples of Zagier's theory of *quantum modular forms*, which comes equipped with its own set of applications.