MODULARITY LIFTING THEOREMS - OUTLINE

TOBY GEE

1. Course

The first aim of the course will be to explain modularity/automorphy lifting theorems for two-dimensional p-adic representations, using wherever possible arguments that go over to the n-dimensional case. In particular, I will incorporate Taylor's arguments in [Tay08] that avoid the use of Ihara's lemma. For the most part I will ignore the issues which are local at p, focusing on representations which satisfy the Fontaine–Laffaille condition.

The second aim is to explain the application of these theorems to questions of level raising and lowering for (Hilbert) modular forms, via the method of Khare–Wintenberger.

2. Project

The basic aim of the project will be to use the Khare–Wintenberger method to generalise various results on level raising and lowering from the case of modular forms to Hilbert modular forms (or automorphic forms on a quaternion algebra), under the assumption that $p > 2$. That such generalisations are possible is well-known to the experts, and various versions of them are scattered across the literature, but to the best of my knowledge the definitive statements are not written down anywhere. The statements are frequently useful to researchers outside of the immediate area of modularity lifting theorems, and it would be valuable to have them recorded in one place. (Note that many results on level lowering and level raising for Hilbert modular forms have been proved without the use of modularity lifting theorems, but the results obtained in this way are not definitive.)

The first three items below should all be achievable in the project; the later items are more speculative, and will require some new ideas.

(1) Generalise the main result of [DT94] to Hilbert modular forms, under the usual Taylor–Wiles condition (i.e. that if $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is the representation under consideration, then $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible).
(2) Further generalise this result to allow Hilbert modular forms of weight other than parallel weight two.
(3) Prove the corresponding results for automorphic forms on quaternion algebras.
(4) To what extent can these results be combined with results on the weight part of Serre's conjecture, to allow finer control on the representations at p?
(5) Is it possible to relax the assumption that the Taylor–Wiles condition holds? For example, at least in some cases where the quadratic subextension of $F(\zeta_p)/F$ is CM, it may be possible to use a combination of CM modular

forms, base change and a variant of the Khare–Wintenberger method to get results.

(6) What about the case $p = 2$?

There is also the question of proving multiplicity one results, using the method of [Dia97] (cf. the recent preprint [BD12] for some results for totally real fields).

(7) What are the most general statements about multiplicity one for mod p automorphic forms on quaternion algebras that can be proved via modularity lifting theorems?

3. Reading

There are a lot of excellent introductions to modularity lifting theorems; in particular, the article [DDT97] and the book [CSS97] are still great places to learn the necessary material, and get a sense of the main arguments. The biggest refinements of the basic method (for two-dimensional representations) since the original arguments are

- Diamond and Fujiwara's observation that it is better to patch spaces of modular forms than Galois representations, for which one can see [Dia97].
- Kisin's refinement of this, realising that the spaces of modular forms should be patched together as modules over local deformation rings, and that these deformation rings should be framed. Expositions of this can be found in [Kis07b] and [Kis07a].
- Taylor's argument to avoid the use of Ihara's lemma, which is the main new ingredient in [Tay08]. The argument here is written in the general n-dimensional case, using the framework of [CHT08], but the course lecture notes will explain the simpler case $n = 2$.
- Khare and Wintenberger's method for constructing Galois representations with particular properties. An exposition (and generalisation) of this is in [Kis07b].

All of these topics will be covered in the course notes. For the most part, the notes will use the framework of [CHT08], adapted to work with quaternion algebras rather than rank n unitary groups. The best preparation is probably to look at [DDT97] and [CSS97], followed by [Kis07b] and [Kis07a].

References

[BD12] Christophe Breuil and Fred Diamond, *Formes modulaires de Hilbert modulo p et valeurs d'extensions Galoisiennes*, 2012.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Pub. Math. IHES **108** (2008), 1–181.

[CSS97] Gary Cornell, Joseph H. Silverman, and Glenn Stevens (eds.), *Modular forms and Fermat's last theorem*, Springer-Verlag, New York, 1997, Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995.

[DDT97] Henri Darmon, Fred Diamond, and Richard Taylor, *Fermat's last theorem*, Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), Int. Press, Cambridge, MA, 1997, pp. 2–140.

[Dia97] Fred Diamond, *The Taylor-Wiles construction and multiplicity one*, Invent. Math. **128** (1997), no. 2, 379–391.

[DT94] Fred Diamond and Richard Taylor, *Lifting modular mod l representations*, Duke Math. J. **74** (1994), no. 2, 253–269.

[Kis07a] Mark Kisin, *Modularity for some geometric Galois representations*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, With an appendix by Ofer Gabber, pp. 438–470.

[Kis07b] ______, *Modularity of 2-dimensional Galois representations*, Current developments in mathematics, 2005, Int. Press, Somerville, MA, 2007, pp. 191–230.

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Pub. Math. IHES **108** (2008), 183–239.

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