Overconvergent modular symbols
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Course description: This course will give an introduction to the theory of overconvergent modular symbols. This theory mirrors the theory of overconvergent modular forms in that both spaces encode the same systems of Hecke-eigenvalues. Moreover, the theory of overconvergent modular symbols has the great feature of being easily computable and is intimately connected to the theory of p-adic L-functions.

As the basic reference for the theory of modular symbols and p-adic L-functions, we highly recommend the original paper by Mazur and Swinnerton-Dyer [3] for a beautiful discussion of the theory with constant coefficients. For the generalization of modular symbols to arbitrary coefficient modules, see [1]. For the "classical" treatment of p-adic L-functions associated to cusp forms of "small" non-critical slope, see the paper of Mazur, Tate, and Teitelbaum [4]. Finally, for the definition of overconvergent modular symbols and their application to computing p-adic L-functions, see [5].

The rough outline of the course is as follows:

Lecture 1: Classical modular symbols, Hecke operators, and L-values.

Lecture 2: Distributions and differentials on wide opens leading to overconvergent modular symbols.

Lecture 3: The Control Theorem: Comparing overconvergent modular symbols to classical ones.

Lecture 4: Overconvergent modular symbols and p-adic L-functions.

Student projects: Let $\mathrm{Symb}_\Gamma(X)$ denote the space of X-valued modular symbols where X is some right $\mathrm{SL}_2(\mathbb{Z})$-module and Γ is a congruence subgroup. This space has the following explicit description:

$$\mathrm{Symb}_\Gamma(X) := \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), X),$$

that is, the space of Γ-equivariant linear maps from degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$ to X. Here Γ acts on $\mathbb{P}^1(\mathbb{Q})$ via linear fractional transformations. If we take X to be $V_k(\mathbb{C}) := \mathrm{Sym}^k(\mathbb{C}^2)$ for a non-negative integer k, we obtain the space $\mathrm{Symb}_\Gamma(V_k(\mathbb{C}))$, which is intimately related to the space of modular forms of weight $k+2$ and level Γ (via the Eichler-Shimura correspondence).

1. *Solving the Manin relations*

   For a fixed Γ, one can explicitly determine the structure of $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ as a $\mathbb{Z}[\Gamma]$-module (by solving the Manin relations), and thus give a general description of elements of $\mathrm{Symb}_\Gamma(X)$ for any X.

   (a) "Solve the Manin relations" for $\Gamma_0(11)$ and use this to completely describe $\mathrm{Symb}_{\Gamma_0(11)}(\mathbb{C})$ where $\mathbb{C}$ is viewed as a trivial $\mathrm{SL}_2(\mathbb{Z})$-module. How does Hecke act here?

   (b) For general k, determine the dimension of $\mathrm{Symb}_{\Gamma_0(11)}(V_k(\mathbb{C}))$.

   (c) If you replace, $V_k(\mathbb{C})$ with $V_k(\mathbb{F}_p^2) := \mathrm{Sym}^k(\mathbb{F}_p^2)$, how does this dimension change?

2. *Explicit symbols*

   Looking back to the example of $\mathrm{Symb}_{\Gamma_0(11)}(\mathbb{C})$ (especially the Hecke-eigenvalues), part of this space corresponded to Eisenstein series and the other part to cuspforms.

   (a) Stare at the Eisenstein symbols in $\mathrm{Symb}_{\Gamma_0(11)}(\mathbb{C})$ and determine a simple formula for them.

   (b) Generalize this to $\mathrm{Symb}_{\Gamma_0(11)}(V_k(\mathbb{C}))$.

   (c) Generalize this to $\mathrm{Symb}_\Gamma(V_k(\mathbb{C}))$ for arbitrary Γ.

(d) Generalize this to $\mathrm{Symb}_\Gamma(\mathcal{D}_k(\mathbb{Z}_p))$ – i.e. to the overconvergent case!

(e) Can you find other explicit modular symbols?
Ideas: use the θ-operator, look at the case of CM modular forms, ...

3. *Slopes of overconvergent modular symbols*

In [2], Buzzard and Calegari prove an explicit formula for the slopes of the eigenvalues of U_2 acting on the space of 2-adic overconvergent modular forms of tame level 1 and weight 0 which starts off as: 0,1,3,7,13,... . Moreover, they formulate a general conjecture about these slopes for arbitrary weights. Since the systems of Hecke-eigenvalues in space of overconvergent modular forms are the same as the systems of Hecke-eigenvalues in space of overconvergent modular symbols, there should be a direct proof of this fact in the overconvergent modular symbols case.

Project: Prove directly that the Buzzard-Calegari slope formula holds in the case of overconvergent modular symbols. Moreover, prove their conjecture for arbitrary weights!

Ideas: pick a good basis of $\mathcal{D}_k(\mathbb{Z}_p)$ and explicitly compute the infinite matrix attached to U_2. Alternatively, try to "see" the first few of these slopes occurring in space of modular symbols with values in the first few finite-approximation modules to $\mathcal{D}_k(\mathbb{Z}_p)$.

4. *Families of modular symbols*

In [5], Pollack and Stevens (that's us!) give an explicit algorithm for computing overconvergent modular symbols. Generalize this picture to handle *families* of overconvergent modular symbols. Compute these symbols and thus compute two-variable p-adic L-functions. Ideas: Determine the appropriate analogue of finite approximation modules *in families*.

5. *Pairings on overconvergent modular symbols*

If Γ is torsion-free, then there is a canonical isomorphism $\mathrm{Symb}_\Gamma(X) \cong H^1_c(\Gamma, X)$ where H^*_c denotes compactly supported cohomology. Thus if $X \times Y \longrightarrow K$ is a Γ-invariant pairing of $K[\Gamma]$-modules, then Lefschetz duality gives a pairing $\mathrm{Symb}_\Gamma(X) \times H^1(\Gamma, Y) \longrightarrow H^2_c(\Gamma, K) = K$. Taking $X = Y = V_k := \mathrm{Sym}^k(\mathbb{Q}_p^2)$ for $k \in \mathbb{Z}^{\geq 0}$ there is a natural Γ-invariant pairing $V_k \times V_k \longrightarrow K$ and this induces a pairing

$$\mathrm{Symb}_\Gamma(V_k) \times \mathrm{Symb}_\Gamma(V_k) \longrightarrow \mathbb{Q}_p.$$

which gives an analog for modular symbols of the well-known Petersson inner product for modular forms.

Project: Show that the above pairing lifts to a pairing on overconvergent modular symbols

$$\mathrm{Symb}_\Gamma(\mathcal{D}_k) \times \mathrm{Symb}_\Gamma(\mathcal{D}_k) \longrightarrow \mathbb{Q}_p$$

which varies p-adically continuously as a function of k. Compute this pairing explicitly for some concrete examples of overconvergent modular symbols.

References

[1] A. Ash and G. Stevens, *Modular Forms in characteristic ℓ and special values of their L-functions*, Duke Math J., **53** (1986), no 3, 849-868.

[2] Kevin Buzzard and Frank Calegari, *Slopes of Overconvergent 2-adic Modular Forms*, Compositio Mathematica, **141** (2005), no 3, 591-604.

[3] B. Mazur and P. Swinnerton-Dyer, *Arithmetic of Weil curves*, Invent. Math. **25** (1974), 161.

[4] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), no. 1, 148.

[5] R. Pollack and G. Stevens, *Overconvergent modular symbols and p-adic L-functions*, to appear in Annales Scientifiques de l'Ecole Normale Superieure.