OVERCONVERGENT MODULAR SYMBOLS

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1. Introduction

The theory of overconvergent modular symbols was created by Glenn Stevens over 20 years ago, and since then the subject has had many generalizations and applications central to modern number theory (*e.g.* overconvergent cohomology, eigenvarieties of reductive groups, families of p-adic L-functions, just to name a few). In these notes[1], rather than give a systematic development of the general theory, we aim to convey the idea that spaces of overconvergent modular symbols (despite being infinite dimensional!) are fairly concrete and can be computed quite explicitly. We do not provide many formal proofs in this article, but instead only sketch arguments, perhaps only in specific cases, and challenge the reader to work out the details.[2] We attempt to keep the tone of these notes quite informal in part to mimic the tone of the corresponding lectures series and in part to keep the barrier to entry to the theory as low as possible.[3]

The structure of these notes is as follows: in the next section, we give a detailed discussion of classical modular symbols and L-values, and work through a computation with modular symbols of level 11. In the third section, we discuss overconvergent modular symbols, their connection to classical modular symbols (*i.e.* Stevens' control theorem), and their relation to p-adic L-functions. In the fourth section, we discuss how to approximate p-adic distributions which leads to a method of computing overconvergent modular symbols as well as a proof of the control theorem. In the fifth section, we return to the connection between p-adic L-functions and overconvergent modular symbols, and in the final section, we close with some numerical examples of overconvergent eigensymbols of level 11.

2. Modular symbols and L-values

2.1. **Introductory example.** Let f be the function on the upper half-plane defined by the q-expansion

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$$

[1]These notes arose from a series of lectures given by the author and Glenn Stevens at the *Arizona Winter School* in March 2011, and then were further refined after a second lecture series by the author at the summer school attached to the conference *Computations with Modular Forms* in August 2011 at Heidelberg University.

[2]Formal treatments of overconvergent modular symbols are already available in the literature (see [11, 12, 3]).

[3]We note that the background needed to study overconvergent modular symbols is significantly less substantial than the background needed to study the corresponding theory of overconvergent modular forms. See section 3.1 for a brief comparison between the two theories.

with $q = e^{2\pi i z}$. Then $f \in S_2(\Gamma_0(11))$, *i.e.* f is a weight two cusp form for $\Gamma_0(11)$. We consider the period integrals

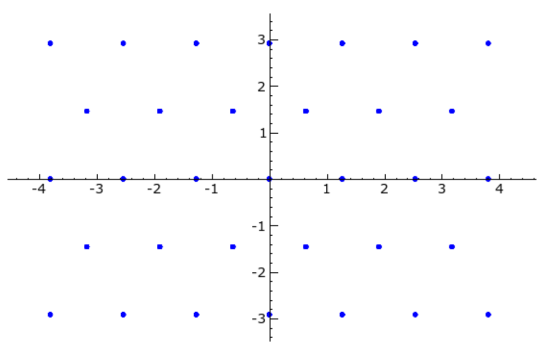
$$2\pi i \int_r^s f(z)dz$$

where r and s vary over $\mathbb{P}^1(\mathbb{Q})$. These are path integrals in the upper-half plane along the semicircle connecting r to s, or along a vertical line if either r or s is infinite.

Let's do a little numerical experiment. Randomly choose a 100 pairs of r's and s in $\mathbb{Q}$, and compute the corresponding period integrals.[4]. Here are the first 10 period integrals:

$$\begin{aligned}
1.2692093042795534216887946170 0... &+ 0.000000000000000000000000000000...i\\
0.000000000000000000000000000000... &+ 2.9176332338769904586617792260 0...i\\
0.6346046521397767108443973085 00... &+ 1.4588166169384952293308896130 0...i\\
3.1730232606988835542219865425 0... &- 1.4588166169384952293308896130 0...i\\
1.9038139564193301325331919255 0... &+ 1.4588166169384952293308896130 0...i\\
0.000000000000000000000000000000... &+ 0.000000000000000000000000000000...i\\
1.2692093042795534216887946170 0... &- 2.9176332338769904586617792260 0...i\\
-1.9038139564193301325331919255 0... &+ 1.4588166169384952293308896130 0...i\\
-3.1730232606988835542219865425 0... &+ 1.4588166169384952293308896130 0...i\\
3.1730232606988835542219865425 0... &- 1.4588166169384952293308896130 0...i
\end{aligned}$$

Plotting these points in the plane gives the following picture:



Not so random, eh? As you might guess, this collection of period integrals spans a lattice in $\mathbb{C}$. Moreover, one can explicitly write down generators of this lattice – namely, Ω_E^+ and $\frac{1}{2}\Omega_E^+ + \Omega_E^-$ where $\Omega_E^\pm$ are the Néron periods of the elliptic curve $X_0(11)$.

This period lattice is intimately related to the L-series of f. For instance, we have

$$2\pi i \int_{i\infty}^0 f(z)dz = L(f,1) = \frac{1}{5} \cdot \Omega_E^+. \tag{1}$$

[4]More honestly, only choose pairs of rational numbers which are $\Gamma_0(11)$-equivalent under linear fractional transformations. See footnote 6 for more details.

The first equality above is true much more generally.[5] The spirit of the second equality is true generally, but the presence of the factor of $\frac{1}{5}$ is very specific to our particular modular form f, and in general, the exact value which appears is related to the Birch and Swinnerton-Dyer conjecture.[6]

The period lattice also contains the information of all twisted L-values. Namely, if χ is a Dirichlet character of conductor N, we have

$$L(f,\chi,1) = \frac{\tau(\chi)}{N} \sum_{a \bmod N} \overline{\chi}(a) \cdot 2\pi i \int_{i\infty}^{-a/N} f(z)dz \tag{2}$$

where $\tau(\chi)$ is the Gauss sum attached to χ. Thus, twisted L-values can be recovered from a simple linear combination of period integrals.

This initial discussion is meant to convince you that these period integrals are quite interesting values as they (a) encode L-values and (b) posses some hidden algebraic structure. We now seek for an axiomatic (and algebraic!) way to describe them.

2.2. **Modular symbols.** Let Δ_0 denote the collection of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$. (To connect to the previous discussion, think of the divisor $\{s\} - \{r\}$ as the path in the upper-half plane connecting r to s.) We then have a map ψ_f from Δ_0 to $\mathbb{C}$ defined by

$$\{s\} - \{r\} \mapsto 2\pi i \int_r^s f(z)dz.$$

Here, f is any cusp form of weight 2 on Γ a congruence subgroup. Of course, we have only defined ψ_f on elements of Δ_0 of the form $\{s\} - \{r\}$. But every element of Δ_0 is a sum of such elements, and so we extend ψ_f accordingly. We thus have constructed

$$\psi_f \in \mathrm{Hom}(\Delta_0, \mathbb{C})$$

where Hom here denotes additive maps.

The modularity of the function $f(z)$ tells us that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any matrix in Γ. In particular, the change of variables $u = \gamma z = \frac{az+b}{cz+d}$ yields following simple equality:

$$\int_{\gamma r}^{\gamma s} f(z)dz = \int_{\gamma r}^{\gamma s} f(\gamma z)(cz+d)^{-2}dz = \int_r^s f(z)dz. \tag{3}$$

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[5]Here is a heuristic argument for this equality:

$$2\pi i \int_{i\infty}^{0} f(z)dz = 2\pi i \int_{i\infty}^{0} \sum_n a_n e^{2\pi i n z} dz = 2\pi i \sum_n a_n \int_{i\infty}^{0} e^{2\pi i n z} dz$$
$$= \sum_n \frac{a_n}{n} e^{2\pi i n z}\Big|_{i\infty}^{0} dz = \sum_n \frac{a_n}{n} = L(f,1)$$

However, this argument suffers from serious convergence issues.

[6] The reason this period integral is not a $\mathbb{Z}$-multiple of Ω_E^+ is related to the fact that ∞ and 0 are not $\Gamma_0(11)$-equivalent. Computing the period integrals between *all* cusps would yield a lattice which contains our lattice with finite index.

To algebraically encode this symmetry of period integrals, we endow Δ_0 with the structure of a left $\mathrm{SL}_2(\mathbb{Z})$-module via linear fractional transformations. Then equation (3) converts into the fact that

$$\psi_f \in \mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$$

where the subscript of Γ indicates maps which satisfy $\varphi(\gamma D) = \varphi(D)$ for all $\gamma \in \Gamma$ and all $D \in \Delta_0$. We call $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ the space of $\mathbb{C}$-*valued modular symbols of level* Γ.[7]

2.3. **Back to $\Gamma_0(11)$.** Returning to the example of $f \in S_2(\Gamma_0(11))$, we have a $\mathbb{C}$-valued modular symbol ψ_f of level $\Gamma := \Gamma_0(11)$ built out of the period integrals of f. Does the abstractly defined modular symbol space $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ contain any thing else other than multiples of ψ_f? Well, first let's point out that $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is equipped with an involution ι given by

$$\iota(\varphi)(D) = \varphi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) D)$$

since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ. So ι breaks $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ into plus and minus subspaces, and we get modular symbols ψ_f^+ and ψ_f^- such that $\psi_f = \psi_f^+ + \psi_f^-$. Moreover, the symbols $\psi_f^\pm$ take values which are (non-trivial) rational multiples of $\Omega_E^\pm$.

So is $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ bigger than 2-dimensional? Well, we could hope to produce more such modular symbols by just taking other cusp forms of level $\Gamma_0(11)$. However, $S_2(\Gamma_0(11))$ is 1-dimensional so this won't yield anything new.

Here's a completely different approach where one can explicitly write down more modular symbols in a very simple fashion. Namely, let $\Delta = \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$ – *i.e.* we dropped the degree zero requirement. If we can define a Γ-invariant function on Δ, then by restriction we get an element of $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$.

What is a Γ-invariant function on Δ? Well, it's a function constant on Γ-orbits of Δ. But Γ-orbits of Δ are simply the cusps of $X_0(11)$! Since 11 is prime, there are 2 cusps, the orbits of 0 and ∞, and thus there are two linear independent functions on the cusps. However, the constant function is killed after restriction to Δ_0, and so this approach will only yield one more dimension of modular symbols. Namely, define

$$\varphi : \Delta \to \mathbb{C}$$

by

$$\varphi(r) = \begin{cases} 1 & \text{if } r \text{ is } \Gamma\text{-equivalent to } \infty \\ 0 & \text{if } r \text{ is } \Gamma\text{-equivalent to } 0, \end{cases}$$

and then restricting φ to Δ_0 gives a (non-zero) element of $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$. (Note that flipping 0 and ∞ in the above definition just negates φ after restricting to Δ_0.)

So now we have that $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is at least 3-dimensional. Any more? Well, we can again try to write $\varphi = \varphi^+ + \varphi^-$ as before. However, you should check that $\varphi^- = 0$, and this doesn't yield a new modular symbol.

Any more? Well...I'm all out of ideas for making new modular symbols. This seems like a good moment to try to prove that there are no more such symbols, and or at least to try to find some upper bound for the dimension of this space. For instance, if we knew generators of Δ_0 as a $\mathbb{Z}[\Gamma]$-module, we would be in great shape. (Note that we just easily determined generators of Δ as a $\mathbb{Z}[\Gamma]$-module.)

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[7]We note that this definition of modular symbols is *dual* to the definition of modular symbols in sources such as [9, 10].

2.4. **Generators of Δ_0 as a $\mathbb{Z}[\Gamma]$-module.** Let's begin by at least writing down a set of generators of Δ_0 over $\mathbb{Z}$ (ignoring the Γ-action). Most naively, note that the set of $\{s\} - \{r\}$ as r and s vary over $\mathbb{P}^1(\mathbb{Q})$ generate over $\mathbb{Z}$. Even better, I claim that the set of $\{\frac{b}{d}\} - \{\frac{a}{c}\}$ for a, b, c, d satisfying $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ generate over $\mathbb{Z}$. Don't believe me? Well note that

$$\left\{\frac{29}{11}\right\}-\left\{\frac{0}{1}\right\}=\left\{\frac{29}{11}\right\}-\left\{\frac{8}{3}\right\}+\left\{\frac{8}{3}\right\}-\left\{\frac{-5}{-2}\right\}+\left\{\frac{5}{2}\right\}-\left\{\frac{3}{1}\right\}+\left\{\frac{3}{1}\right\}-\left\{\frac{-2}{-1}\right\}+\left\{\frac{2}{1}\right\}-\left\{\frac{1}{0}\right\}+\left\{\frac{1}{0}\right\}-\left\{\frac{0}{-1}\right\}$$

(Here $\frac{1}{0}$ just means ∞.) Indeed, this is Manin's continued fraction trick; the rational numbers appearing between $\frac{29}{11}$ and $\frac{0}{1}$ are just the convergents in the continued fraction expansion of $\frac{29}{11}$. This trick works generally yielding a $\mathbb{Z}$-generating set of Δ_0 indexed by $\mathrm{SL}_2(\mathbb{Z})$. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, write $[\alpha]$ for the divisor $\{\frac{b}{d}\} - \{\frac{a}{c}\}$ – we will call such divisors *unimodular*.

Now we incorporate the action of Γ. A quick computation shows that if $\beta \in \mathrm{SL}_2(\mathbb{Z})$, then $[\beta\alpha] = \beta \cdot [\alpha]$ (do it!). That is, multiplying on the left by β and then taking the associated divisor is the same as first taking the associated divisor and then acting with β by linear fractional transformations.

This simple formula tells us the following: if $\alpha_1, \cdots, \alpha_d$ are a system of right coset representations for $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$, then $[\alpha_1], \cdots, [\alpha_d]$ are $\mathbb{Z}[\Gamma]$-generators of Δ_0. Indeed, if $[\beta]$ is one of our $\mathbb{Z}$-generators of Δ_0, write $\beta = \gamma\alpha_i$ for some i and some $\gamma \in \Gamma$. Then $[\beta] = \gamma \cdot [\alpha_i]$. In particular, a modular symbol is uniquely determined by its values on the finite list of divisors $[\alpha_1], \cdots, [\alpha_d]$.

Great! So we now have a finite list of $\mathbb{Z}[\Gamma]$-generators of Δ_0. In our particular case of $\Gamma = \Gamma_0(11)$, we would then need 12 coset representations as $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = 12$. Hmph. Twelve is indeed finite, but is a large number to start with. Let's take a simpler example:

2.5. **On to $\Gamma_0(2)$.** The index of $\Gamma_0(2)$ in $\mathrm{SL}_2(\mathbb{Z})$ is 3, and the following are right coset representatives:

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

(Note that two matrices represent the same right coset in $\Gamma_0(2)$ if their bottom rows, thought of as elements as $\mathbb{P}^1(\mathbb{F}_2)$, agree.) Their associated divisors are:

$$\{0\} - \{\infty\}, \{-1\} - \{0\}, \{\infty\} - \{-1\}. \tag{4}$$

Are these linearly independent over $\mathbb{Z}[\Gamma_0(2)]$? Nope. Not even over $\mathbb{Z}$ as their sum is 0. Any other relations? Well, note that

$$\{-1\} - \{0\} = -(\{0\} - \{-1\}) = -(\{\gamma(-1)\} - \{\gamma(0)\}) = -\gamma(\{-1\} - \{0\}).$$

where $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$.

Now, let's say we have φ some $\mathbb{C}$-valued modular symbol of level $\Gamma_0(2)$. Then φ is determined by its values on the three divisors in (4) by the arguments of the section 2.4. Moreover, by $\Gamma_0(2)$-invariance, we have

$$\varphi(\{-1\} - \{0\}) = -\varphi(\gamma(\{-1\} - \{0\})) = -\varphi(\{-1\} - \{0\}),$$

and thus φ vanishes on $\{-1\} - \{0\}$. Then, since our three divisors sum to 0, we get

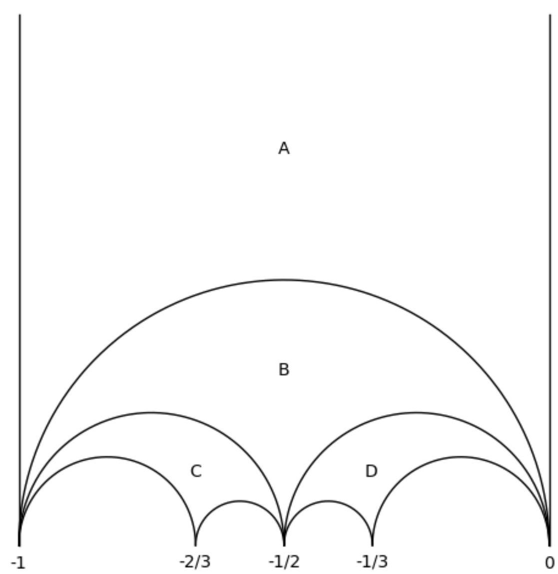
$$0 = \varphi(\{0\} - \{\infty\}) + \varphi(\{\infty\} - \{-1\}).$$

Therefore, the value of φ on $\{\infty\} - \{0\}$ determines the value of φ on $\{-1\} - \{\infty\}$, and thus on all divisors. In particular, this space of modular symbols is at most

1-dimensional. The same trick of writing down a symbol on Δ and restricting to Δ_0 again works, and we see that this space is exactly 1-dimensional.[8]

2.6. **Back to $\Gamma_0(11)$.** How to generalize this to the $\Gamma_0(11)$ case? Well, let me tell you where the three generating divisors for $\Gamma_0(2)$ in (4) came from. First note that the ideal triangle connecting ∞ to 0 to -1 is a fundamental domain for $\Gamma_0(2)$. Taking differences between the vertices of this triangle then gives rise to our 3 divisors. The fact that the three divisors sum to 0 comes from going around the triangle. The relation involving $\{0\} - \{-1\}$ arises from the fact that the bottom edge of the triangle is identified with itself modulo $\Gamma_0(2)$ – *i.e.* from the gluing data describing how to wrap the fundamental domain to get $X_0(2)$. So for $X_0(11)$, let's make a nice fundamental domain and see what happens!

Again, consider the ideal triangle with vertices ∞, 0 and -1 (triangle A in the figure below). We note that this triangle contains three copies of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ and this region sits inside of a fundamental domain for $\Gamma_0(11)$.[9] To extend this region to a fundamental domain, we consider the ideal triangle sitting directly below triangle A – *i.e.* the ideal triangle with vertices -1, 0 and $-\frac{1}{2}$ (triangle B in the figure below). A simple computation shows that the union of these two triangles is still contained within a fundamental domain for $\Gamma_0(11)$. To further extend this region, we consider the two ideal triangles below this second triangle – *i.e.* the triangle with vertices $-\frac{1}{2}$, $-\frac{2}{3}$ and -1 (triangle C), and the triangle with vertices 0, $-\frac{1}{3}$ and $-\frac{1}{2}$ (triangle D). Another computation shows that this region is still contained within a fundamental domain for $\Gamma_0(11)$. In fact, since this region now contains 12 copies of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ – 4 ideal triangles each containing 3 such fundamental domains – and since $\Gamma_0(11)$ has index 12 in $\mathrm{SL}_2(\mathbb{Z})$, the union of these 4 triangles is in fact a full fundamental domain for $\Gamma_0(11)$.



[8]Note that there are no cusp forms of weight 2 and level $\Gamma_0(2)$, and so it should not be a surprise that there are no symbols of the form ψ_f.

[9]See [11, Section 2.3] for the details of the following construction.

How does this fundamental domain help us in computing modular symbols? Well, we will use this picture to find $\mathbb{Z}[\Gamma_0(11)]$-generators of Δ_0. Starting with the oriented path $\infty \to 0$, consider all $\mathrm{SL}_2(\mathbb{Z})$ translates of this path. Exactly 12 such translates will be contained within our fundamental domain (as long as we give the boundary of the domain a fixed orientation, say clockwise). These 12 paths are:

$$\infty \to 0,\ 0 \to -\frac{1}{3},\ -\frac{1}{3} \to -\frac{1}{2},\ -\frac{1}{2} \to -\frac{2}{3},\ -\frac{2}{3} \to -1,\ -1 \to \infty,$$
$$-1 \to 0,\ 0 \to -1,\ -1 \to -\frac{1}{2},\ -\frac{1}{2} \to -1,\ -\frac{1}{2} \to 0,\ 0 \to -\frac{1}{2}.$$

We generated this list by simply going around the boundary of our fundamental domain (in the clockwise direction), and then taking each internal path in our domain with both orientations.

Associated to an oriented path $r \to s$, we consider the degree 0 divisor $\{s\} - \{r\}$. Taking the corresponding degree 0 divisors of the 12 paths above gives:

$$\{0\} - \{\infty\},\ \left\{-\frac{1}{3}\right\} - \{0\},\ \left\{-\frac{1}{2}\right\} - \left\{-\frac{1}{3}\right\},\ \left\{-\frac{2}{3}\right\} - \left\{-\frac{1}{2}\right\},$$
$$\{-1\} - \left\{-\frac{2}{3}\right\},\ \{\infty\} - \{-1\}, \{-1\} - \{0\},\ \{0\} - \{-1\},$$
$$\{-1\} - \left\{-\frac{1}{2}\right\},\ \left\{-\frac{1}{2}\right\} - \{-1\},\ \left\{-\frac{1}{2}\right\} - \{0\},\ \{0\} - \left\{-\frac{1}{2}\right\}.$$

We claim these 12 unimodular divisors generate Δ_0 as a $\mathbb{Z}[\Gamma_0(11)]$-modules. Indeed, as argued in section 2.4, it suffices to see that the span of these divisors contains every divisor arising from an $\mathrm{SL}_2(\mathbb{Z})$ matrix. To this end, consider $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$ and its associated path $\frac{a}{c} \to \frac{b}{d}$. This path must be $\Gamma_0(11)$-equivalent to one of the 12 paths inside of our fundamental domain listed above. Thus the divisor $\{\frac{b}{d}\} - \{\frac{a}{c}\}$ is $\Gamma_0(11)$-equivalent to one of the above 12 unimodular divisors listed above, and we are done.

We now have a finite generating set of divisors. But you may ask why didn't we just randomly choose a set of coset representatives which would have also given a generating set of unimodular divisors as in section 2.4? The reason is that we will be able to read off all of the $\mathbb{Z}[\Gamma_0(11)]$-relations of the above generating set of divisors via our fundamental domain. For starters, every divisor which arises from a path not on the boundary of our fundamental domain can be expressed in terms of divisors arising from the boundary. For instance, the path $0 \to -1$, which is in the interior of our fundamental domain, is homologous to the the union of the paths:

$$0 \to -\frac{1}{3},\ -\frac{1}{3} \to -\frac{1}{2},\ -\frac{1}{2} \to -\frac{2}{3},\ -\frac{2}{3} \to -1$$

and thus

$$\{-1\} - \{0\} = \{-1\} - \left\{-\frac{2}{3}\right\} + \left\{-\frac{2}{3}\right\} - \left\{-\frac{1}{2}\right\} + \left\{-\frac{1}{2}\right\} - \left\{-\frac{1}{3}\right\} + \left\{-\frac{1}{3}\right\} - \{0\}$$

Great! So in fact it suffices to just take the 6 divisors arising from the boundary of the domain as a set of $\mathbb{Z}[\Gamma_0(11)]$-generators.

Further relations come from how the boundary is glued together. For instance, if we reverse the orientation of the edge $0 \to -\frac{1}{3}$, this reversed edge will be $\Gamma_0(11)$-equivalent to another boundary edge (with the clockwise orientation). Namely,

$$\begin{pmatrix} 7 & 2 \\ -11 & -3 \end{pmatrix} \cdot \left(-\frac{1}{3} \to 0 \right) = -\frac{1}{2} \to -\frac{2}{3},$$

and thus,

$$-\begin{pmatrix} 7 & 2 \\ -11 & -3 \end{pmatrix} \left(\left\{ -\frac{1}{3} \right\} - \{0\} \right) = \left\{ -\frac{2}{3} \right\} - \left\{ -\frac{1}{2} \right\}.$$

In particular, the generator $\left\{-\frac{2}{3}\right\} - \left\{-\frac{1}{2}\right\}$ is redundant as it is in the $\mathbb{Z}[\Gamma_0(11)]$-span of $\left\{-\frac{1}{3}\right\} - \{0\}$. So now we are down to 5 generators!

Similarly, we have

$$-\begin{pmatrix} 8 & 3 \\ -11 & -4 \end{pmatrix} \cdot \left(\left\{ -\frac{1}{2} \right\} - \left\{ -\frac{1}{3} \right\} \right) = \{-1\} - \left\{ -\frac{2}{3} \right\}$$

and

$$-\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot (\{0\} - \{\infty\}) = \{\infty\} - \{-1\}.$$

In particular, we see that the 3 divisors

$$D_1 = \{0\} - \{\infty\}, \ D_2 = \left\{ -\frac{1}{3} \right\} - \{0\}, \ \text{ and } D_3 = \left\{ -\frac{1}{2} \right\} - \left\{ -\frac{1}{3} \right\}$$

generate Δ_0 over $\mathbb{Z}[\Gamma_0(11)]$.

Super! This means that any modular symbol on $\Gamma_0(11)$ is uniquely determined by its values on D_1, D_2, and D_3. In particular, $\mathbb{C}$-valued modular symbols on $\Gamma_0(11)$ are at most 3-dimensional. Since we have already written down 3 independent symbols, we must have found them all![10]

2.7. **Higher weight case.** Now let's take an arbitrary weight cusp form f in $S_k(\Gamma)$ with Γ a congruence subgroup. The relevant period integrals attached to f are of the form

$$\int_r^s f(z) z^j \, dz$$

[10]For completeness, let's mention that D_1, D_2 and D_3 do not generate *freely* over $\mathbb{Z}[\Gamma_0(11)]$. Indeed, there is one relation between these divisors that is easy to see via the fundamental domain. Namely, if we sum up over the boundary of the fundamental domain, we get 0. Explicitly, we have

$$\begin{aligned} &(\{0\} - \{\infty\}) + (\{\infty\} - \{-1\}) + \left(\{-1\} - \left\{ -\frac{2}{3} \right\} \right) \\ &\quad + \left(\left\{ -\frac{2}{3} \right\} - \left\{ -\frac{1}{2} \right\} \right) + \left(\left\{ -\frac{1}{2} \right\} - \left\{ -\frac{1}{3} \right\} \right) + \left(\left\{ -\frac{1}{3} \right\} - \{0\} \right) = 0 \end{aligned}$$

and re-writing this in terms of our generators gives

$$\begin{aligned} &\left(1 - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right) (\{0\} - \{\infty\}) + \left(1 - \begin{pmatrix} 7 & 2 \\ -11 & -3 \end{pmatrix} \right) \left(\left\{ -\frac{1}{3} \right\} - \{0\} \right) \\ &\qquad + \left(1 - \begin{pmatrix} 8 & 3 \\ -11 & -4 \end{pmatrix} \right) \left(\left\{ -\frac{1}{2} \right\} - \left\{ -\frac{1}{3} \right\} \right) = 0. \end{aligned}$$

Further, one can check that this is the only relation these divisors satisfy. This relation plays no role when considering modular symbols with trivial coefficients, but once the coefficients have a non-trivial matrix action, this more complicated relation must be considered.

where j ranges between 0 and $k-2$. We should thus beef up the associated modular symbol to encode all of these periods, and we do so by changing the space where the symbols take values.

Namely, let $V_g(\mathbb{C}) = \mathrm{Sym}^g(\mathbb{C}^2)$ realized as the space of homogeneous polynomials of degree g in $\mathbb{C}[X,Y]$. Moreover, we endow this space with a right action (seriously, a right action) of $\mathrm{SL}_2(\mathbb{Z})$ by setting

$$(P\big|\gamma)(X,Y) = P\left((X,Y)\cdot\gamma^*\right) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $\gamma^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and $P \in V_g(\mathbb{C})$. In fact, this action makes sense for any matrix with non-zero determinant.

We then define

$$\psi_f(\{s\} - \{r\}) = 2\pi i \int_r^s f(z)(zX+Y)^{k-2}dz \in V_{k-2}(\mathbb{C})$$

which gives an element of

$$\mathrm{Hom}(\Delta_0, V_{k-2}(\mathbb{C})).$$

As before we exhibit some Γ-invariant property. Namely, for any $\gamma \in M_2(\mathbb{Z})$ with non-zero determinant, define a right action on $\mathrm{Hom}(\Delta_0, V_{k-2}(\mathbb{C}))$ via

$$(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma.$$

A simple computation (do it!), which again uses the modularity of f, yields that

$$\psi_f \in \mathrm{Hom}_\Gamma(\Delta_0, V_{k-2}(\mathbb{C}))$$

where the subscript Γ denotes the subspace of maps which are invariant under the above action of Γ. Explicitly, these are the maps such that

$$\varphi(\gamma D) = \varphi(D)|\gamma^{-1}$$

for $\gamma \in \Gamma$ and $D \in \Delta_0$.

Before going further, let's take a moment to discuss our choice of normalizations in the above actions. First note that the standard $\mathrm{GL}_2^+(\mathbb{Q})$-action on modular forms given by:

$$(f|\gamma)(z) = \det(\gamma)^{k-1}(cz+d)^{-k}f(\gamma z) = \det(\gamma)^{k-1}(cz+d)^{-k}f\left(\frac{az+b}{cz+d}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is definitely a *right* action as one acts on the variable of the function on the left. So it's not so strange to use a right action on the space of modular symbols as well. Moreover, the association of f goes to ψ_f should be equivariant under our chosen matrix actions – and indeed the action given above on $\mathrm{Sym}^g(\mathbb{C}^2)$ is exactly the one that makes this association equivariant.

2.8. ***L*-values.** As mentioned before, the modular symbol ψ_f should know special values of the L-series of f. Generalizing equation (1), we have the following relation between L-values and period integrals:

$$2\pi i \int_{i\infty}^0 f(z)z^j dz = \frac{j!}{(-2\pi i)^j} L(f, j+1)$$

for $0 \le j \le k-2$.

If we set $\sum_{j=0}^{k-2} c_j X^j Y^{k-2-j} \in \mathrm{Sym}^{k-2}(\mathbb{C}^2)$ equal to the value of ψ_f at $\{0\}-\{\infty\}$, we can then relate the coefficient c_j to the $L(f, j+1)$. Namely, we have

$$\begin{aligned}\psi_f\left(\{0\}-\{\infty\}\right) &= 2\pi i \int_{i\infty}^{0} f(z)(zX+Y)^{k-2}dz \\ &= 2\pi i \int_{i\infty}^{0} f(z) \sum_{j=0}^{k-2} \binom{k-2}{j} z^j X^j Y^{k-2-j} dz \\ &= \sum_{j=0}^{k-2} \binom{k-2}{j} \left(2\pi i \int_{i\infty}^{0} f(z) z^j dz \right) X^j Y^{k-2-j} \\ &= \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{j!}{(-2\pi i)^j} L(f, j+1) X^j Y^{k-2-j},\end{aligned}$$

which implies that

$$c_j = \binom{k-2}{j} \frac{j!}{(-2\pi i)^j} L(f, j+1).$$

Note that this matches our previous formula when $j = 0$.

2.9. **Modular symbols in general.** So far we've considered modular symbols with values in $\mathbb{C}$ (with trivial action) and with values in $\mathrm{Sym}^g(\mathbb{C}^2)$. Let's write down the general theory here as laid out in [1]. To this end, let $\Delta_0 := \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the set of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. Then Δ_0 has the structure of a left $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Q})]$-module where $\mathrm{GL}_2(\mathbb{Q})$ acts via standard linear fractional transformations.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let V be a right $\mathbb{Z}[\Gamma]$-module. We define a right action of Γ on $\mathrm{Hom}(\Delta_0, V)$ by

$$(\varphi\big|\gamma)(D) := \varphi(\gamma D)\big|\gamma$$

for $\varphi : \Delta_0 \to V$, $D \in \Delta_0$ and $\gamma \in \Gamma$.

For φ in $\mathrm{Hom}(\Delta_0, V)$, we say that φ is a *V-valued modular symbol* on Γ if $\varphi\big|\gamma = \varphi$ for all $\gamma \in \Gamma$; we denote the space of all V-valued modular symbols on Γ by $\mathrm{Symb}_\Gamma(V)$.[11]

The modules V considered in these notes will have the additional structure of a right action by the semi-group $S_0(p)$ where

$$S_0(p) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \text{ such that } (a,p) = 1,\ p|c \text{ and } ad - bc \neq 0 \right\}.$$

Given this additional structure, one can define a Hecke-action on $\mathrm{Symb}_\Gamma(V)$; if ℓ is a prime, then the Hecke operator T_ℓ is given by the double coset $\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell \end{smallmatrix}\right) \Gamma$. For example, if $\Gamma = \Gamma_0(N)$ and $\ell \nmid N$, then

$$\varphi|T_\ell = \varphi|\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a=0}^{l-1} \varphi|\left(\begin{smallmatrix} 1 & a \\ 0 & \ell \end{smallmatrix}\right).$$

[11]We note that if $\mathcal{H}$ denotes the upper-half plane and $\widetilde{V}$ is the associated locally constant sheaf of V on $\mathcal{H}/\Gamma$, then there is a canonical isomorphism

$$\mathrm{Symb}_\Gamma(V) \cong H^1_c(\mathcal{H}/\Gamma, \widetilde{V}),$$

provided that the order of any torsion element of Γ acts invertibly on V (see [1, Prop 4.2]). In these notes, we however focus on the explicit description of modular symbols given by maps rather than by cohomology classes.

If $q|N$, we write U_q for T_q, and we have

$$\varphi|U_q = \sum_{a=0}^{q-1} \varphi| \left(\begin{smallmatrix} 1 & a \\ 0 & q \end{smallmatrix}\right).$$

We further remark that when $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ, this matrix acts as an involution on $\mathrm{Symb}_\Gamma(V)$. When 2 acts invertibly on V, we then have a natural decomposition

$$\mathrm{Symb}_\Gamma(V) \cong \mathrm{Symb}_\Gamma(V)^+ \oplus \mathrm{Symb}_\Gamma(V)^-$$

into ± 1-eigenspaces for this action.

2.10. **Exercises with Hecke operators.** Now that we have Hecke operators defined in general, it's a great exercise to go back and see how Hecke acts on modular symbols of level $\Gamma_0(11)$. We know that this space is 3-dimensional with 2 dimensions coming from cusp forms and 1 dimension coming from a symbol which extends to Δ. For the moment though, forget that we know this. Instead, let's just use the fact that we know that a modular symbol of level $\Gamma_0(11)$ is determined by it's values on the three divisors D_1, D_2 and D_3 defined in section 2.6.

Define three modular symbols φ_1, φ_2 and φ_3 with the property that $\varphi_i(D_j) = \delta_{ij}$. These three modular symbols will then be a basis of the space of all $\mathbb{C}$-valued modular symbols of level $\Gamma_0(11)$. However, there is no reason to expect that these symbols will be eigensymbols under the Hecke action. Indeed, let's start to compute the action of T_2 on these symbols. Namely,

$$\begin{aligned}(\varphi_1|T_2)(D_1) &= \varphi_1((\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})\, D_1)|\,(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}) + \varphi_1((\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix})\, D_1)|\,(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}) + \varphi_1((\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})\, D_1)\,(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}) \\ &= \varphi_1((\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix})\, D_1) + \varphi_1((\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix})\, D_1) + \varphi_1((\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix})\, D_1) \\ &= \varphi_1(\{0\} - \{\infty\}) + \varphi_1(\{1/2\} - \{\infty\}) + \varphi_1(\{0\} - \{\infty\}) \\ &= 2 + \varphi_1(\{1/2\} - \{\infty\})\end{aligned}$$

To determine $\varphi_1(\{1/2\} - \{\infty\})$, we must write $\{1/2\} - \{\infty\}$ as a sum of unimodular divisors, and then express these divisors in terms of our generating divisors. We have

$$\begin{aligned}\{1/2\} - \{\infty\} &= (\{1/2\} - \{1\}) + (\{1\} - \{\infty\}) \\ &= (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (\{-1/2\} - \{0\}) + (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (\{0\} - \{\infty\}) \\ &= (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (\{-1/2\} - \{-1/3\} + \{-1/3\} - \{0\}) + (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (\{0\} - \{\infty\}) \\ &= (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (D_1 + D_2 + D_3)\,.\end{aligned}$$

Thus,

$$\varphi_1(\{1/2\} - \{\infty\}) = \varphi_1((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot (D_1 + D_2 + D_3)) = \varphi_1\,(D_1 + D_2 + D_3) = 1$$

Completing the above computation, we then have

$$(\varphi_1|T_2)(D_1) = 2 + \varphi_1(\{1/2\} - \{\infty\}) = 3.$$

This computation tells us that when we write $\varphi_1|T_2$ in the basis φ_1, φ_2 and φ_3, the coefficient of φ_1 is 3. Equivalently, the matrix for T_2 in this basis has a 3 in its upper left corner.

Repeating this computation for $(\varphi_i|T_2)(D_j)$ for the remaining pairs of i and j (try it!), we get that the matrix for T_2 (viewed as multiplying on the right on row vectors) is

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

and thus the eigenvalues of T_2 are 3, -2 and -2.

The repeated root of -2 is easy to explain. This is simply the eigenvalue of T_2 acting on the cuspidal newform f on $\Gamma_0(11)$. To explain the eigenvalue of 3, observe from the above matrix for T_2 that φ_1 is in fact an eigensymbol. Try to compute the action of a general Hecke operator T_ℓ on φ_1 to determine a general formula for these Hecke-eigenvalues.[12]

To complete this exercise, we should find an eigenbasis for T_2, and further find a simultaneous eigenbasis for T_2 and the involution ι (do it!). Doing so would recover the modular symbols ψ_f^+/Ω_E^+ and ψ_f^-/Ω_E^- up to some rational multiple. This is remarkable as the above construction is completely *algebraic* (*i.e.* without any reference to period integrals!).

2.11. **Eichler-Shimura.** The following theorem (essentially of Eichler and Shimura) gives a complete relation between classical modular forms and classical modular symbols. Namely:

Theorem 2.1. *Let $\Gamma = \Gamma_1(N)$. There is an isomorphism*

$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) \cong M_k(\Gamma, \mathbb{C}) \oplus S_k(\Gamma, \mathbb{C})$$

which respects the action of Hecke on both sides.[13]

Note that it is not at all surprising that there are two copies of the cusp forms appearing. Indeed, attached to each cusp form f, there are two modular symbols ψ_f^+ and ψ_f^-. You may ask, why are Eisenstein series appearing? Well, look back to the maps we originally wrote down on Δ and then restricted to Δ_0. These will account for Eisenstein series (as you may have already noted if you did the exercises of section 2.10!).

2.12. **p-adic L-functions.** As it will be relevant in the final section, we mention now a connection to p-adic L-functions. Namely, one can construct the p-adic L-function of f out of the modular symbol ψ_f.

Before doing this, a few words on what p-adic L-functions are, at least in the case of weight 2 forms. The job of the p-adic L-function of f is to interpolate the special values of the L-series of f. Namely, the p-adic L-function should "know" the values $L(f, \chi, 1)$ where χ runs over all Dirichlet characters of conductor p^n. On the surface, this makes no sense because these values are complex numbers and p-adic L-functions should live in the p-adic world. To remedy this, consider the modular symbol $\psi_f^\pm$ all of whose values are algebraic multiples of some fixed complex number

[12]Hint: φ_1 is nothing other than the modular symbol defined earlier which extends to all of Δ!

[13]For a reference, see [12, Footnote 5].

$\Omega_f^{\pm}$ as we noted for the $\Gamma_0(11)$ example.[14] Set $\varphi_f^{\pm} = \frac{\psi_f^{\pm}}{\Omega_f^{\pm}}$ which is a modular symbol taking values in $\overline{\mathbb{Q}}$. Also, set $\varphi_f = \varphi_f^+ + \varphi_f^-$.

From equation (2), it follows that the L-values $\dfrac{L(f,\chi,1)}{\Omega_f^{\pm}}$ are all algebraic. (Here the sign of the period depends on whether χ is even or odd.) Since these L-values are algebraic, we can fix embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and into $\overline{\mathbb{Q}}_p$ and consider these L-values as either complex or p-adic numbers.

So what kind of p-adic gadget can encode all of these twisted L-values? Well, in the spirit of Tate's thesis, one thinks of the association:

$$\chi \mapsto \frac{L(f,\chi,1)}{\Omega_f^{\pm}},$$

and then thinks of L-functions as functions on spaces of characters. In the p-adic world, the relevant character space is $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{C}_p)$. Note that every Dirichlet character of p-power conductor is in this space. So the p-adic L-function should be able to take as an input any $\mathbb{C}_p$-valued character on $\mathbb{Z}_p^\times$ and return a p-adic number. Even better, it will be able to take as an input any "nice enough" function on $\mathbb{Z}_p^\times$, and return a p-adic number. That is, the p-adic L-function will be a distribution – *i.e.* something in the dual of a space of nice p-adic functions.

All of this will be made more precise in the next section, but for now, we want to build a gadget which takes in nice functions on $\mathbb{Z}_p^\times$ and spits out numbers. Moreover, when you input a finite-order character on $\mathbb{Z}_p^\times$, it spits out the relevant twisted L-value. To accomplish this, it suffices to write down a measure on $\mathbb{Z}_p^\times$ which we now do (at least in the case of p-ordinary[15] weight two modular forms). That is, let α be the unique unit root of $x^2 - a_p(f)x + p$ and define

$$\mu_f(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha^n}\varphi_f(\{a/p^n\} - \{\infty\}) - \frac{1}{\alpha^{n+1}}\varphi_f(\{a/p^{n-1}\} - \{\infty\}).$$

With this definition, the p-adic L-function μ_f satisfies the following interpolation property: for χ a non-trivial finite order character of conductor p^n, we have

$$\int_{\mathbb{Z}_p^\times} \chi \; d\mu_f = \frac{1}{\alpha^n}\tau(\chi)\frac{L(f,\overline{\chi},1)}{\Omega_f^{\pm}}.$$

Many questions should arise: where does the formula defining μ_f come from? Is μ_f even additive: *i.e.* if one takes an open of the form $a + p^n\mathbb{Z}_p$ and writes it as a disjoint union of opens each of the form $b+p^{n+1}\mathbb{Z}_p$, is the above formula compatible with such a union? Lastly, how does it connect to L-values? We will attempt to answer all of these questions in the final section of the paper. For now, we'll just say, "you'll see", "yes" and "equation (2)", and point out that the p-adic L-function of f is being built out of the data of the modular symbol attached to f evaluated at infinitely many different divisors.

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[14]We are side stepping the appropriate normalization of this period which would be needed to get μ-invariants correct.

[15]A p-ordinary modular form is a normalized eigenform f such that $a_p(f)$ is a p-adic unit. Note that $a_p(f)$ is *a priori* only in $\overline{\mathbb{Q}}$, but again we are implicitly viewing it as a p-adic number via our fixed embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ from above.

3. Distributions leading to overconvergent modular symbols

3.1. **A brief story about p-adic families.** Although we won't discuss p-adic families in detail in this article, the motivation behind overconvergent modular symbols (and overconvergent modular forms) is too closely intertwined with families that we digress briefly to explain this connection.

In the mid 80s, Hida constructed p-adic families of ordinary eigenforms (see [6, 7]). The constructions and methods of this theory relied crucially on the following observation: the dimension of the subspace of ordinary modular forms in $S_k(\Gamma)$ is independent of the weight k. (Here Γ is some congruence subgroup contained in $\Gamma_1(p)$.) This fact allowed Hida to p-adically interpolate the finite-dimensional spaces of ordinary forms as the weight varied.

The phenomenon of the dimension of ordinary forms being independent of the weight of course cannot occur for arbitrary cusp forms as the dimension of the full space $S_k(\Gamma)$ depends heavily upon k (growing unboundedly as k increases). Thus, one can't hope for a nice (finite, flat) family over weight space interpolating $S_k(\Gamma)$.

To circumvent this problem, Coleman had the ingenious idea to pass to a much larger space, namely $M_k^\dagger(\Gamma)$, the space of overconvergent modular forms. This is an infinite-dimensional space (on which U_p acts completely continuously). Moreover, it contains $M_k(\Gamma)$, the space of classical forms. And, doubly moreover, Coleman proves that an overconvergent eigenform is classical if the p-adic valuation of its U_p-eigenvalue is strictly less than $k-1$ (see [4]).[16]

Now at least there is a hope that the spaces $M_k^\dagger(\Gamma)$ can be put into a nice family as they all at least have the same dimension, namely infinity! And indeed this is exactly what Coleman does (see [5]), and he succeeds in making p-adic families of eigenforms (although, unlike Hida theory, there are forms in these families of classical weight which are not themselves classical).

In this section, we will describe Steven's analogue of overconvergent modular forms (called overconvergent modular symbols) along with his analogue of Coleman's control theorem. To this end, we are going to replace the spaces $V_k(\mathbb{C})$ with p-adic spaces whose dimensions don't move around with k. In particular, we will write down spaces $\mathcal{D}_k$ of p-adic distributions (which are infinite-dimensional spaces). Moreover, these spaces will admit surjective maps from $\mathcal{D}_k$ to $V_k(\mathbb{Q}_p)$. We then replace the space $\mathrm{Symb}_\Gamma(V_k(\mathbb{Q}_p))$ with the space $\mathrm{Symb}_\Gamma(\mathcal{D}_k)$ which will be the space of overconvergent modular symbols. And indeed Stevens succeeded in varying these spaces p-adically.[17]

3.2. **Distributions.** We start by defining the simplest of the distribution spaces which we will need for this article. Namely, let $\mathbf{A}$ denote the collection of power series with coefficients in $\mathbb{Q}_p$ which converge on the unit disc of $\mathbb{C}_p$. That is:

$$\mathbf{A} = \{f(z) \in \mathbb{Q}_p[[z]] \ : \ f(z) = \sum_n a_n z^n \text{ and } |a_n| \to 0 \text{ as } n \to \infty\}.$$

---

[16] A classical form of level Γ cannot have a U_p-eigenvalue with p-adic valuation larger than $k-1$ (see section 3.7 for details). So it's only forms of slope $k-1$ that are not completely explained by this theorem. These are called the *critical slope* forms.

[17] We also note that spaces of overconvergent modular symbols contain the same system of Hecke-eigenvalues as the corresponding space of overconvergent modular forms (see [12, Theorem 7.1]).

Note that $\mathbf{A}$ is a Banach space under the norm:

$$||f|| = \max_n |a_n|$$

where $f(z) = \sum_n a_n z^n$. We then define our space of distributions $\mathbf{D}$ by

$$\mathbf{D} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{A}, \mathbb{Q}_p).$$

Note that $\mathbf{D}$ is a Banach space under the norm

$$||\mu|| = \sup_{\substack{f \in \mathbf{A} \\ f \neq 0}} \frac{|\mu(f)|}{||f||}.$$

3.3. **Moments of distributions.** The distribution space $\mathbf{D}$ is quite concrete. Indeed, $\mathbf{D}$ can be identified with the space of bounded sequences in $\mathbb{Q}_p$ endowed with the sup norm. To get this identification, we just note that the $\mathbb{Q}_p$-span of all monomials $\{z^j\}_{j=0}^{\infty}$ is dense in $\mathbf{A}$ (check this!). Thus, a distribution $\mu \in \mathbf{D}$ is uniquely determined by its values on these monomials. In particular, we get an injective map

$$\mathbf{D} \xrightarrow{M} \prod_{j=0}^{\infty} \mathbb{Q}_p.$$

$$\mu \mapsto \{\mu(z^j)\}_j$$

We call $\{\mu(z^j)\}_j$ the sequence of *moments* attached to μ.

We claim that the image of M is exactly the collection of bounded sequences. Indeed, if $\{\alpha_n\}$ is a bounded sequence of elements in $\mathbb{Q}_p$, we can just define a distribution $\mu \in \mathbf{D}$ by simply setting

$$\mu(z^n) = \alpha_n.$$

Then μ extends linearly to a functional on $\mathbf{A}$ by setting

$$\mu\left(\sum a_n z^n\right) = \sum a_n \alpha_n,$$

which converges since $|a_n| \to 0$ and $\{\alpha_n\}$ is bounded. We leave it for you to check that for any $\mu \in \mathbf{D}$, its associated sequence of moments is a bounded sequence.

The upshot of this subsection is thus following: although $\mathbf{D}$ has the complicated definition as the dual of convergent power series on the closed unit disc of $\mathbb{C}_p$, it has the very concrete realization as the collection of bounded sequences in $\mathbb{Q}_p$.

3.4. **The action of $\Sigma_0(p)$.** Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \text{ such that } p \nmid a, p \mid c \text{ and } ad - bc \neq 0 \right\}$$

For each non-negative integer k, we define a weight k action of $\Sigma_0(p)$ on $\mathbf{A}$ by

$$(\gamma \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in \mathbf{A}$. Then $\Sigma_0(p)$ acts on $\mathbf{D}$ on the right by

$$(\mu\big|_k \gamma)(f) = \mu(\gamma \cdot_k f)$$

where $\mu \in \mathbf{D}$. When we view $\mathbf{A}$ or $\mathbf{D}$ as a $\Sigma_0(p)$-module endowed with a weight k action, we write $\mathbf{A}_k$ or $\mathbf{D}_k$.

Note that by "transport of structure" we have also defined a $\Sigma_0(p)$-action on the space of bounded sequences in $\mathbb{Q}_p$ via our identification from the last section. However, don't expect anything special; this action is just a mess in the language of sequences.

3.5. **Finite-dimensional quotients.** For an integer $k \geq 0$, consider

$$V_k := V_k(\mathbb{Q}_p) := \mathrm{Sym}^k(\mathbb{Q}_p^2),$$

the space of homogeneous polynomials of degree k in X and Y with coefficients in $\mathbb{Q}_p$. We recall that we endow the space $V_k(\mathbb{Q}_p)$ with the structure of a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module by

$$(P\big|\gamma)(X,Y) = P\left((X,Y)\cdot\gamma^*\right) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $P \in V_k(\mathbb{Q}_p)$.

There is a $\Sigma_0(p)$-equivariant map

$$\begin{aligned} \rho_k : \mathbf{D}_k &\to V_k(\mathbb{Q}_p) \\ \mu &\mapsto \int (Y - zX)^k \; d\mu(z) \end{aligned}$$

where the integration takes place coefficient by coefficient. That is

$$\rho_k(\mu) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu(z^j) X^j Y^{k-j} \in V_k(\mathbb{Q}_p).$$

(Check that this map is really $\Sigma_0(p)$-invariant![18])

3.6. **Specialization.** Let $\Gamma_0 = \Gamma_0(Np)$ and $\Gamma = \Gamma_0(N)$. We refer to $\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)$ as a space of weight k overconvergent modular symbols.[19] These spaces of overconvergent modular symbols naturally map to the space of classical modular symbols. Indeed, the map $\rho_k : \mathbf{D}_k \to V_k$ induces a map

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k) \to \mathrm{Symb}_{\Gamma_0}(V_k)$$

which we refer to as the *specialization map*. Note that ρ_k^* is Hecke-equivariant as ρ_k is $\Sigma_0(p)$-equivariant.[20]

The source of the specialization map is infinite-dimensional while the target is finite-dimensional, and so the kernel is certainly huge. We will soon see that this kernel is almost completely describable in terms of the action of U_p, and a classical eigensymbol whose U_p-eigenvalue is not too divisible by p uniquely lifts to an overconvergent eigensymbol. We first give a brief digression on slopes of classical modular forms (*i.e.* on the valuation of U_p-eigenvalues).

[18]Here's a more natural way to see the map ρ_k. Consider the span of the functions $1, z, \ldots, z^k$ inside of $\mathbf{A}_k$. This span is stable under the weight k action, and its dual is simply V_k. Thus dualizing this inclusion gives a map from $\mathbf{D}_k$ to V_k which can be seen to be precisely the map ρ_k.

[19]Note that $\mathbf{D}_k$ is equipped with a natural Γ_0-action, but this action does not extend to Γ.

[20]Recall that $V_k = \mathrm{Sym}^k(\mathbb{Q}_p^2)$ and so modular symbols in $\mathrm{Symb}_{\Gamma_0}(V_k)$ correspond to modular forms of weight $k+2$, and not weight k.

3.7. **Slopes of modular forms.** If g is an eigenform of weight $k+2$ on Γ_0, we say that the *slope* of g is the valuation of its U_p-eigenvalue. The following is the key fact which will be crucial in what follows.

Fact: The slope of an eigenform on Γ_0 of weight $k+2$ is at most $k+1$.

The reason for this is actually not too hard. An eigenform on Γ_0 is either new at p or old at p. Forms that are new at p, automatically have that $a_p(g) = \pm p^{\frac{k}{2}}$ (see [2]), and thus have slope $\frac{k}{2}$. So we only need to worry about p-old forms.

If g is our p-old form, then g is in the span of $f(z)$ and $f(pz)$ where f is some eigenform on Γ. In fact, a simple and pleasant computation with q-expansions (do it!) shows that the span of $f(z)$ and $f(pz)$ is stable under the Hecke operator U_p. Moreover, the characteristic polynomial of U_p on this space is $x^2 - a_p(f)x + p^{k+1}$ where $a_p(f)$ is the eigenvalue of f under T_p, the p-th Hecke operator on Γ. Since $a_p(g)$ is a root of this polynomial, we immediately see that its valuation is bounded by $k+1$ as desired.

3.8. **Comparison theorem.** Returning to the specialization map

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k) \to \mathrm{Symb}_{\Gamma_0}(V_k),$$

we just saw that U_p acts on the target with slope at most $k+1$. Thus, the entire subspace of the source on which U_p acts with slope strictly larger than $k+1$ is necessarily in the kernel of this map[21].

The following control theorem of Stevens says that apart from the critical slope cases (*i.e.* slope $k+1$)[22], this is precisely what happens. Namely:

Theorem 3.1 (Stevens)**.** *We have*

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)^{(<k+1)} \longrightarrow \mathrm{Symb}_{\Gamma_0}(V_k)^{(<k+1)}$$

is an isomorphism. That is, the specialization map restricted to the subspace where U_p acts with slope strictly less than $k+1$ is an isomorphism.

This theorem should be viewed as the overconvergent modular symbol analogue of Coleman's theorem on overconvergent forms of small slope being classical. Indeed, Stevens' theorem says that a classical eigensymbol of small enough slope lifts uniquely to an overconvergent eigensymbol.[23] A proof of Theorem 3.1 will be sketched in the next section. For a complete argument see [14, Theorem 0.9], [11, Theorem 5.12] or [12, Theorem 5.4].

3.9. **The truth about our distribution spaces.** To be more honest, the space $\mathbf{D}$ is not really the space we are ultimately interested in (although it will be the space we primarily work with). Indeed, we ultimately want to see p-adic L-functions as special values of these overconvergent modular symbols. However, p-adic L-functions actually live in a smaller space of distributions than $\mathbf{D}$. Indeed,

[21]So far we've only defined slopes of modular forms. But if M is any $\mathbb{Z}_p$-module with an action of U_p, one defines the slope of an eigenvector of U_p as the p-adic valuation of the associated eigenvalue. Moreover, for any real number h, we write $M^{(<h)}$ to denote the subspace on which U_p acts with slope less than h.

[22]What happens in the critical slope case is more subtle. See [12] and [3].

[23]Note that in Coleman's world, classical forms are a subspace of overconvergent forms. However, in our setting, classical modular symbols are a quotient of overconvergent modular symbols.

distributions in $\mathbf{D}$ can only be evaluated on functions which are expressible as a single convergent power series on the entire closed unit disc, while p-adic L-functions need to be evaluated against, for instance, finite-order characters on $\mathbb{Z}_p^\times$. These locally constant functions (if non-trivial) certainly cannot be represented by a single convergent power series.

The class functions on $\mathbb{Z}_p$ we will ultimately need to be integrating against are the "locally analytic" ones; that is, $f : \mathbb{Z}_p \to \mathbb{C}_p$ is locally analytic if for every point y of $\mathbb{Z}_p$, the function f is expressible as a power series on some ball around y in $\mathbb{Z}_p$. (Since $\mathbb{Z}_p$ is compact, one only needs finitely many power series to represent f on $\mathbb{Z}_p$.) Let $\mathcal{A}$ denote the collection of all locally analytic functions on $\mathbb{Z}_p$. Note that finite-order characters on $\mathbb{Z}_p^\times$ (considered as functions on $\mathbb{Z}_p$ by extending by 0) are locally analytic since they are locally constant. One can also check that any character on $\mathbb{Z}_p^\times$ (not necessarily finite-order) is also locally analytic.

We want to define the space of "locally analytic" distributions to be the (continuous) dual of $\mathcal{A}$. However, we haven't given a topology yet to $\mathcal{A}$. This is a little bit tricky because $\mathcal{A}$ is not a Banach space.

To proceed, for each $r \in |\mathbb{C}_p^\times|$, we set

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p \mid \text{there exists some } a \in \mathbb{Z}_p \text{ with } |z - a| \leq r\}.$$

For example, if $r \geq 1$ then $B[\mathbb{Z}_p, r]$ is the closed disc in $\mathbb{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbb{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points 0, 1, ... , $p-1$. As r increases, $B[\mathbb{Z}_p, r]$ becomes a union of more and more discs with smaller and smaller radii but still covering $\mathbb{Z}_p$.

Let $\mathbf{A}[r]$ denote the collection of $\mathbb{Q}_p$-rigid analytic functions on $B[\mathbb{Z}_p, r]$. For example, if $r \geq 1$

$$\mathbf{A}[r] = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] \text{ such that } \{|a_n| \cdot r^n\} \to 0 \right\}.$$

In particular, $\mathbf{A}[1]$ is nothing other than the space $\mathbf{A}$ we've been studying this section. If $r = \frac{1}{p}$, then an element of $\mathbf{A}[r]$ is a function on $B[\mathbb{Z}_p, r]$ which when restricted to each of the p discs of radius $\frac{1}{p}$ is representable by a convergent power series with coefficients in $\mathbb{Q}_p$. The norm on $\mathbf{A}[r]$ is given by the supremum norm. That is, if $f \in \mathbf{A}[r]$ then

$$||f||_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|_p.$$

For $r_1 > r_2$, there is a natural restriction map $\mathbf{A}[r_1] \to \mathbf{A}[r_2]$ which is injective. Since $\mathbb{Z}_p$ is contained in $B[\mathbb{Z}_p, r]$ for any $r > 0$, there is a natural restriction map $\mathbf{A}[r] \to \mathcal{A}$ which is also injective. Moreover, since each element in $\mathcal{A}$ is representable by finitely many power series, any such element is in the image of $\mathbf{A}[r]$ for some r. Thus,

$$\mathcal{A} = \varinjlim_{r} \mathbf{A}[r]$$

as r tends to 0 in the limit. We thus endow $\mathcal{A}$ with the inductive limit topology which is the strongest topology making all of the inclusions $\mathbf{A}[r] \to \mathcal{A}$ continuous. Lastly, we set $\mathcal{D}$ equal to the continuous $\mathbb{Q}_p$-dual of $\mathcal{A}$, which we will call the space of locally analytic distributions on $\mathbb{Z}_p$.

Since we have a map $\mathbf{A} \to \mathcal{A}$ by restriction to $\mathbb{Z}_p$, dualizing gives an injective map $\mathcal{D} \to \mathbf{D}$. The injectivity of this map follows from:

Fact: The span of the monomials $\{z^j\}$ is dense in $\mathcal{A}$.

Thus, we can and do identify $\mathcal{D}$ as a subspace of $\mathbf{D}$; that is, every locally analytic distribution is in fact a rigid analytic distribution.

We note that in a completely analogous way, we can endow $\mathcal{A}$ and $\mathcal{D}$ with a weight k action by $\Sigma_0(p)$, and again we write $\mathcal{A}_k$ and $\mathcal{D}_k$. It is also true that the same formula defining ρ_k also gives a $\Sigma_0(p)$-map from $\mathcal{D}_k$ to $V_k(\mathbb{Q}_p)$. Lastly, Steven's control theorem will hold true if we replace $\mathbf{D}_k$ with $\mathcal{D}_k$. Indeed, for any $h \in \mathbb{R}$, the natural map

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k)^{(<h)} \to \mathrm{Symb}_{\Gamma_0}(\mathbf{D}_k)^{(<h)}$$

is an isomorphism (see [12, Lemma 5.3]). Thus any $\mathbf{D}_k$-valued U_p-eigensymbol of finite slope will automatically take values in $\mathcal{D}_k$.

3.10. **Connecting to p-adic L-functions.** We close this section by mentioning a connection to p-adic L-functions.

Theorem 3.2. *Let g be a cuspidal eigenform on Γ_0 of non-critical slope,*[24] *and let φ_g be the corresponding modular symbol. If Φ_g is the unique overconvergent eigensymbol lifting φ_g (by Theorem 3.1), then the restriction of $\Phi_g(\{0\} - \{\infty\})$ to $\mathbb{Z}_p^\times$ is the p-adic L-function of g.*[25]

This theorem gives a construction of the p-adic L-function in one fell swoop as opposed to what was done in section 2.12 when the p-adic L-function was defined by gathering together the data of the classical modular symbol evaluated on infinitely many different divisors.

4. The Control Theorem: Comparing overconvergent modular symbols to classical ones

We aim to give an explicit proof of the control theorem, at least for the slope 0 subspace.

4.1. **Finite approximation modules.** We would like to be able to approximate distributions in a systematic way (with a finite amount of data). Doing so will (a) lead us to an explicit proof of Steven's control theorem, and (b) allow us to represent distributions on a computer, and thus allow for numerical computations with overconvergent modular symbols.

A first guess on how to form an approximation of a distribution μ with integral moments is to fix two integers M and N, and consider the first M moments of μ modulo p^N. Unfortunately, these approximations are not stable under the action of $\Sigma_0(p)$; that is, given such an approximation of μ, one cannot compute the corresponding approximation of $\mu|\gamma$ to the same accuracy. Indeed, the collection of

[24] Note that we are assuming that g is an eigenform on Γ_0 and not on Γ. If one is starting off with a form on Γ, to form its p-adic L-function, one must choose a p-stabilization of this form to Γ_0. See section 5.1 for details.

[25] We note that this theorem is off by a sign from the corresponding theorems in [11] and [12]. The underlying reason for this discrepancy is that in this paper we defined $\psi_f(\{s\} - \{r\})$ in terms of the path integral from r to s rather than from s to r. But, more importantly, we should mention that "the" p-adic L-function of g depends upon the choice of Ω_g^+ and Ω_g^-, each of which is only well-defined up to a p-adic unit. Thus, any ambiguity of sign can be absorbed into the choice of these periods.

distributions whose first M moments vanish is not stable under the action of $\Sigma_0(p)$. To see this, let's work out a little example:

Let $k=0$ and let μ_4 denote the distribution which takes the value 1 on z^4 and 0 on all other monomials. Let $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ -p & 1 \end{smallmatrix}\right)$, and we compute

$$(\mu_4|\gamma)(z) = \mu_4\left(\frac{z}{1-pz}\right) = \mu_4\left(z \cdot \sum_{j=0}^{\infty}(pz)^j\right) = p^3$$

$$(\mu_4|\gamma)(z^2) = \mu_4\left(\frac{z^2}{(1-pz)^2}\right) = \mu_4\left(z^2 \cdot \left(\sum_{j=0}^{\infty}(pz)^j\right)^2\right) = 3p^2$$

$$(\mu_4|\gamma)(z^3) = \mu_4\left(\frac{z^3}{1-pz}\right) = \mu_4\left(z^3 \cdot \left(\sum_{j=0}^{\infty}(pz)^j\right)^3\right) = 3p$$

So even though the first 4 moments of μ_4 vanish, the same is not true of $\mu_4|\gamma$. However, do note that the early moments of $\mu_4|\gamma$ are highly divisible by p, and this divisibility trails off as we consider later moments.

This phenomenon holds quite generally. Let

$$\mathbf{D}_k^0 = \left\{\mu \in \mathbf{D}_k \mid \mu(x^j) \in \mathbb{Z}_p \text{ for all } j \geq 0\right\}$$

be the collection of distributions with all integral moments, and consider the subspace

$$\mathrm{Fil}^M \mathbf{D}_k^0 = \left\{\mu \in \mathbf{D}_k^0 \text{ such that } \mu(z^j) \in p^{M+1-j}\mathbb{Z}_p\right\}.$$

whose moments satisfy this trail-off of divisibility. That is, for $\mu \in \mathrm{Fil}^M \mathbf{D}_k^0$, the 0-th moment of μ is divisible by p^{M+1}, the first is divisible by p^M, the second is divisible by p^{M-1}, and so on. A direct computation (try it!) shows that $\mathrm{Fil}^M \mathbf{D}_k^0$ is stable under the weight k action of $\Sigma_0(p)$ for $k \geq 0$.

4.2. **Approximating distributions.** We now use the filtration $\left\{\mathrm{Fil}^M \mathbf{D}_k^0\right\}$ to systematically approximate distributions.

Definition 4.1. We define the M-th *finite approximation module* of $\mathbf{D}_k^0$ to be

$$\mathcal{F}_k(M) := \mathbf{D}_k^0/\,\mathrm{Fil}^M(\mathbf{D}_k^0).$$

Proposition 4.2. *We have that $\mathcal{F}_k(M)$ is a $\Sigma_0(p)$-module and*

$$\begin{aligned}\mathcal{F}_k(M) &\xrightarrow{\sim} (\mathbb{Z}/p^{M+1}\mathbb{Z}) \times (\mathbb{Z}/p^M\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p\mathbb{Z}) \\ \overline{\mu} &\mapsto \left(\mu(z^j) + p^{M+1-j}\mathbb{Z}_p\right)_j\end{aligned}$$

is an isomorphism. In particular, $\mathcal{F}_k(M)$ is a finite set.

Proof. Since $\mathrm{Fil}^M(\mathbf{D}_k^0)$ is a $\Sigma_0(p)$-module, $\mathcal{F}_k(M)$ is also a $\Sigma_0(p)$-module. The fact that this map is an isomorphism follows directly from the definition of the filtration and the fact that the moment map identifies $\mathbf{D}_k^0$ with the set of sequences in $\mathbb{Z}_p$. $\square$

By the above proposition, we can approximate $\mu \in \mathbf{D}_k^0$ with a finite amount of data by taking its image in $\mathcal{F}_k(M)$. Moreover, if one knows the image of μ in every $\mathcal{F}_k(M)$, then one can recover μ as one can recover all of its moments. Great!

4.3. **Lifting modular symbols.** Let's assume for simplicity that we are working in weight 2 (*i.e.* $k = 0$), and drop k from the notation. We seek to show that the specialization map

$$\mathrm{Symb}_{\Gamma_0}(\mathbf{D})^{\mathrm{ord}} \longrightarrow \mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$$

is an isomorphism. Here the superscript ord denotes the subspace where U_p acts invertibly (*i.e.* with slope 0), and so this statement is a special case of the control theorem.

To gain some intuition, let's assume this theorem is true, and from that, try to construct the unique lift of a given eigensymbol. Let φ denote some Hecke-eigensymbol in the target, $\mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$. Then take some overconvergent lift Ψ of φ in $\mathrm{Symb}_{\Gamma_0}(\mathbf{D})$. Note that we are **not** assuming that Ψ is a U_p-eigensymbol (and thus not necessarily in the slope zero subspace). Since the source of specialization is infinite-dimensional and the target is finite-dimensional, there will be lots and lots of choice for such a Ψ.

The operator U_p is compact (see, for instance, [3, Prop 3.6]), and so has an infinite collection of eigenvalues which tend to 0 in $\mathbb{Z}_p$. Say the eigenvalues are

$$\lambda_1, \lambda_2, \ldots, \lambda_n \ldots,$$

ordered by valuation, with corresponding eigensymbols,

$$\Phi_1, \Phi_2, \ldots \Phi_n \ldots,$$

with Φ_1 our sought after eigensymbol lifting φ. In particular, λ_1 is then the U_p-eigenvalue of φ.

Now write Ψ as an infinite linear combination of these symbols:

$$\Psi = \Phi_1 + \Phi_2 + \cdots + \Phi_n + \ldots.^{26}$$

Applying the operator U_p/λ_1 repeatedly to Ψ, and looking at its eigen-expansion we get

$$\Psi\big|(U_p/\lambda_1)^M = \Phi_1 + (\lambda_2/\lambda_1)^M \Phi_2 + \cdots + (\lambda_n/\lambda_1)^M \Phi_n + \ldots.$$

Note that $\Psi\big|(U_p/\lambda_1)^M$ is still a lift of φ as specialization is Hecke-equivariant – that is,

$$\rho^*(\Psi\big|(U_p/\lambda_1)^M) = \rho^*(\Psi)\big|(U_p/\lambda_1)^M = \varphi\big|(U_p/\lambda_1)^M = \varphi.$$

Since φ is an ordinary eigensymbol, λ_1 is a p-adic unit. To simplify matters, let's assume that λ_1 is the only unit eigenvalue of U_p.[27] In particular, as we continually apply U_p/λ_1, the higher terms in the eigen-expansion of Ψ get p-adically small, and thus we get a convergence:

$$\left\{\Psi\big|(U_p/\lambda_1)^M\right\} \to \Phi_1,$$

and we have "constructed" the desired symbol.

To turn the above argument into a real proof, we need to (a) not assume the theorem we want to prove, and (b) deal with all of the convergence issues that arise in these eigen-expansions. We do this in the following steps.

[26] Because of convergence issues, there is no reason that such an expansion should even exist, but let's just imagine so anyway.

[27] If this were not true, we could use the other Hecke operators to kill off the other overconvergent eigensymbols of slope 0 in the expansion of Ψ without changing the fact that Ψ lifts φ.

A) **Prove that the specialization map is surjective.** We need this statement to know the existence of the lift Ψ of φ. This fact is not too hard to establish – the source is infinite-dimensional and the target is finite-dimensional, so the surjectivity statement is pretty reasonable.

B) **Prove directly that the sequence $\left\{\Psi|(U_p/\lambda)^M\right\}$ converges**. Here λ is the U_p-eigenvalue of φ. This also is not too hard. The underlying reason is the following lemma.

Lemma 4.3. *If Φ is in the kernel of specialization, then*

$$||\Phi|U_p|| \leq \frac{1}{p}||\Phi||.$$

In particular, any U_p-eigensymbol in the kernel of specialization has slope at least 1.

We'll assume this lemma for now (in fact it's just an easy computation). Back to the main argument, we'll check that $\left\{\Psi|(U_p/\lambda)^M\right\}$ is Cauchy. To this end, we note that $\Psi - \Psi|(U_p/\lambda)^j$ is in the kernel of specialization for any j (as both symbols lift φ). Thus,

$$\Psi|(U_p/\lambda_1)^{M_1} - \Psi|(U_p/\lambda_1)^{M_2} = \left(\Psi - \Psi|(U_p/\lambda_1)^{M_2-M_1}\right)\Big|(U_p/\lambda_1)^{M_1}$$

tends to 0 for M_1, M_2 large as λ_1 is a unit and the right hand side is U_p applied many times to an element in the kernel of specialization. This proves the desired Cauchy statement. Let Φ denote the limit of $\left\{\Psi|(U_p/\lambda)^M\right\}$.

C) **Prove that Φ is a Hecke-eigensymbol lifting φ.** That Φ lifts φ is clear as $(U_p/\lambda)^M\Psi$ lifts φ for every M. That Φ is an U_p-eigensymbol is clear as

$$\Phi|U_p = \left(\lim_{M\to\infty}\Psi|(U_p/\lambda)^M\right)|U_p = \lambda\lim_{M\to\infty}\Psi|(U_p/\lambda)^{M+1} = \lambda\Phi.$$

The other eigenvalues are also easily checked. (Do it!)

4.4. **Lifting symbols – take II (a la M. Greenberg).** In the previous section, we punted on the issue of simply lifting φ to *some* overconvergent symbol – not even an eigensymbol. In fact, a lot was swept under the rug here (*i.e.* in step A). It's not too hard to check this lift exists (using a little cohomology – see [12, Lemma 5.1]), but to directly write down a lift is involved (though completely worked out in [11]).

However, Matthew Greenberg found a method which sidesteps these difficulties and does steps A through C from the past section in one fell swoop (see [8]). Let's explain. Let φ be a U_p-eigensymbol in $\mathrm{Symb}_{\Gamma_0}(\mathbb{Q}_p)^{\mathrm{ord}}$. The idea is to successively lift φ to a U_p-eigensymbol in $\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(M))^{\mathrm{ord}}$ for $M = 1, 2, \ldots$. Since $\varprojlim_M \mathcal{F}(M) = \mathbf{D}^0$ this would suffice to produce an eigenlifting of φ. Note also that this is exactly the kind of thing one would want to do if you were programming a computer.

Let's start with $M = 1$. To write down an element Ψ_1 of $\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(1))^{\mathrm{ord}}$, we need to give the 0-th moment of $\Psi_1(D)$ modulo p^2 and the 1st moment of $\Psi_1(D)$ modulo p for each divisor $D \in \Delta_0$. Since we are trying to write down a lift of φ, our hands are forced on the 0-th moments. Indeed, $\Psi_1(D)(\mathbf{1})$ should just be the reduction of $\varphi(D)$ modulo p^2. As for the 1st moments, there is no clear choice. So just randomly assign values to $\Psi_1(D)(z) \in \mathbb{Z}/p\mathbb{Z}$.

The result is an element

$$\Psi_1 \in \mathrm{Maps}(\Delta_0, \mathcal{F}(1)).$$

Here Maps means simply that, set maps. We've lost the homomorphism property when we randomly assigned the first moments, and we've certainly lost the Γ_0-invariance.

To somehow fix this random choice, we apply U_p. (This shouldn't be so unreasonable considering the arguments from the last section.) Though, a word of warning here: U_p is a well-defined operator on modular symbols, *i.e.* it is independent of double coset representatives of $\Gamma_0 \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_0$. On $\mathrm{Maps}(\Delta_0, \cdot)$, this is no longer true. So we just pick coset representations. That is U_p is *defined* as the operator

$$U_p := \sum_{a=0}^{p-1} \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right).$$

Now the magic: the element $\Phi_1 := \Psi_1|(U_p/\lambda)$ lies in

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(1)) \subseteq \mathrm{Maps}(\Delta_0, \mathcal{F}(1));$$

that is, Φ_1 is both additive and Γ_0-invariant. Moveover, Φ_1 is independent of any choices made!

Let's see why this is true. First, we'll check that Φ_1 is in fact a homomorphism. To see this, consider

$$\Phi_1(D) + \Phi_1(D') - \Phi_1(D + D') \tag{5}$$

$$= \left(\sum_{a=1}^{p-1} \Psi_1(\gamma_a D) + \Psi_1(\gamma_a D') - \Psi_1(\gamma_a D + \gamma_a D') \right) \Big| \gamma_a \tag{6}$$

where $\gamma_a = \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$. By construction, $\Psi_1(\gamma_a D) + \Psi_1(\gamma_a D') - \Psi_1(\gamma_a D + \gamma_a D')$ has vanishing 0-th moments as these moments are built out of the values of φ which is a bona-fide modular symbol. However, we have no control over the 1st moments.

To understand what is going on, let's take an arbitrary element μ of $\mathcal{F}(1)$ with vanishing 0-th moment and act γ_a on it. We have

$$(\mu|\gamma_a)(\mathbf{1}) = \mu(\gamma_a \cdot \mathbf{1}) = \mu(\mathbf{1}) = 0,$$

and thus $\mu|\gamma_a$ still has vanishing first moment. Further,

$$(\mu|\gamma_a)(x) = \mu(\gamma_a \cdot x) = \mu(a + px) = \mu(a) + p\mu(x) = 0$$

as $\mu(a) = a\mu(\mathbf{1}) = 0$ and p kills whatever value $\mu(x)$ is taking. (Remember the 1st moment lies in $\mathbb{Z}/p\mathbb{Z}$!) This means that any element in $\mathcal{F}(1)$ with vanishing first moment is killed by γ_a and hence the expression in (5) vanishes. In particular, Φ_1 is a homomorphism!

A similar argument proves that Φ_1 is Γ-invariant. Just consider $\Phi|\gamma - \Phi$ and argue just as before (remembering that $\varphi|\gamma = \varphi$).

And so we've done it! We've formed a lift of φ with values in $\mathcal{F}(1)$. To see how general this is, let's try to form a $\mathcal{F}(2)$-valued lift Ψ_2 of Φ_1. For any divisor $D \in \Delta_0$, we seek to define the 0-th, 1st, and 2nd moments of $\Psi_2(D)$ modulo p^3, p^2 and p respectively. The 0-th moment is easy – just reduce $\varphi(D)$ modulo p^3. For the 1st moment, we want to be lifting Φ_1. This means the value we choose for $\Psi_2(D)(x)$

should be congruent to $\Phi_1(D)(x)$ modulo p – pick any value modulo p^2 that works. Lastly, we have no information on 2nd moment, so again, pick randomly.

As before, we set $\Phi_2 := \Psi_2|U_p \in \mathrm{Maps}(\Delta_0, \mathcal{F}(2))$. To check that Φ_2 is a homomorphism, the identical argument as above reduces us to checking that any element of $\mathcal{F}(2)$ with 0-th moment equal to 0 and first moment divisible by p is killed by γ_a. Let's compute!

Fix μ any such element of $\mathcal{F}(2)$. We have

$$(\mu|\gamma_a)(\mathbf{1}) = \mu(\gamma_a \cdot \mathbf{1}) = \mu(\mathbf{1}) = 0,$$

Further,

$$(\mu|\gamma_a)(x) = \mu(\gamma_a \cdot x) = \mu(a + px) = \mu(a) + p\mu(x) = p\mu(x) = 0$$

as $\mu(x)$ is divisible by p and thus $p\mu(x)$ is 0 modulo p^2. Lastly,

$$(\mu|\gamma_a)(x^2) = \mu(\gamma_a \cdot x) = \mu((a + px)^2) = \mu(a^2) + 2ap\mu(x) + p^2\mu(x^2) = 0$$

as desired.

As you might imagine, this just keeps working. The underlying fact that is needed is that if μ is in $\mathcal{F}(M)$ with vanishing 0-th moment and with vanishing projection to $\mathcal{F}(M-1)$ then $\mu|\gamma_a = 0$. We leave the details to you (plus the generalizations to higher weight and higher (non-critical) slope).

5. Overconvergent modular symbols and p-adic L-functions

5.1. Motivating the construction of p-adic L-functions. The p-adic L-function of a eigenform $f \in S_2(\Gamma_0(N))$ is a distribution $\mu_f \in \mathcal{D}$ such that when μ_f is evaluated at some Dirichlet character χ the result should be $\dfrac{L(f,\chi,1)}{\Omega_f^{\pm}}$ up to some explicit controllable constant.

When $p \nmid N$ and f is a p-ordinary form (that is, when $a_p(f)$ is a p-adic unit), a formula was given in section 2.12 for $\mu_f(a + p^n\mathbb{Z}_p)$.[28] We now take a little bit of time to motivate this formula, and in the process, derive the basic properties of μ_f.

The starting point is equation (2) which relates L-values to period integrals. Rewriting this formula is terms of modular symbols gives

$$\tau(\overline{\chi})\frac{L(f,\chi,1)}{\Omega_f^{\pm}} = \sum_{a \bmod p^n} \overline{\chi}(a) \cdot \varphi_f^{\pm}\left(\{a/p^n\} - \{\infty\}\right). \tag{7}$$

Staring at this above expression just right, one sees the right hand side as a Riemann sum. Indeed, think of $\overline{\chi}$ as a function on $\mathbb{Z}_p$, and cover $\mathbb{Z}_p$ by the opens $a + p^n\mathbb{Z}_p$. Then think of $\varphi_f^{\pm}(\{a/p^n\} - \{\infty\})$ as the measure of $a + p^n\mathbb{Z}_p$. The right hand side then reads as the sum of measures of discs covering $\mathbb{Z}_p$ times the value of $\overline{\chi}$ on centers of those discs. Moreover, this Riemann sum equals an L-value, which is exactly what we are after.

However, if μ is a measure on $\mathbb{Z}_p$, then it must be true that

$$\mu(a + p^n\mathbb{Z}_p) = \sum_{j=0}^{p-1} \mu(a + jp^n + p^{n+1}\mathbb{Z}_p) \tag{8}$$

[28] Writing $\mu_f(a + p^n\mathbb{Z}_p)$ is a bit of an abusive of notation. What is meant here is the value obtained when evaluating μ_f on the characteristic function of $a + p^n\mathbb{Z}_p$.

as $a+p^n\mathbb{Z}_p$ is the disjoint union of opens of the form $a+jp^n+p^{n+1}\mathbb{Z}_p$. Could it be that the values of the modular symbol $\varphi_f^{\pm}(\{a/p^n\}-\{\infty\})$ satisfy such an additivity property? Amazingly, the answer is nearly yes, and results from the fact that φ_f is a T_p-eigensymbol.

Indeed, we know that $\varphi_f|T_p = a_p\varphi_f$. Thus,

$$\begin{aligned}
a_p\cdot&\varphi_f(\{a/p^n\}-\{\infty\})\\
&=(\varphi_f|T_p)(\{a/p^n\}-\{\infty\})\\
&=(\varphi_f|\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right))(\{a/p^n\}-\{\infty\})+\sum_{j=0}^{p-1}(\varphi_f|\left(\begin{smallmatrix} 1 & j\\ 0 & p\end{smallmatrix}\right))(\{a/p^n\}-\{\infty\})\\
&=\varphi_f(\{a/p^{n-1}\}-\{\infty\})+\sum_{a=0}^{p-1}\varphi_f(\{(a+jp^n)/p^{n+1}\}-\{\infty\}).
\end{aligned}$$

Note that this formula is nearly what we are looking for except for two points: (a) the factor of a_p in the front of the left hand side, and (b) the first term of the right hand side which arose from acting by $\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)$.

Let's start with the second issue. We first note that if we had been using the Hecke operator U_p instead of T_p, then this problem wouldn't be present. Indeed, U_p is defined by only p terms and is exactly missing that extra troublesome matrix $\left(\begin{smallmatrix} p & 0\\ 0 & 1\end{smallmatrix}\right)$. The reason we are using T_p instead of U_p is that we are working at a level prime to p. To get around this, we can force p into the level through the process of p-stabilization.

Namely, our form f is an eigenform for the full Hecke algebra for level $\Gamma_0(N)$. However, if we instead think of f as an eigenform with level $\Gamma_0(Np)$, it would no longer be an eigenform at p (although it is still an eigenform away from p). As usual, we also consider the form $f(pz)$ which has level $\Gamma_0(Np)$ and is an eigenform away from p. As discussed in section 3.7, the span of $f(z)$ and $f(pz)$ is stable under U_p and the characteristic polynomial of U_p on this two-dimensional space is nothing other than the Hecke polynomial $x^2-a_p(f)x+p$. (Have you done this exercise yet??) Performing a little linear algebra (*i.e.* diagonalizing), one sees that if α and β are the roots of $x^2-a_p(f)x+p$, and if we set

$$f_\alpha = f(z)-\beta f(pz) \text{ and } f_\beta = f(z)-\alpha f(pz),$$

then $f_\alpha|U_p = \alpha f_\alpha$ and $f_\beta|U_p = \beta f_\beta$ – *i.e.* f_α and f_β are U_p-eigenforms with eigenvalues α and β respectively.

Since we are assuming that f is p-ordinary, a_p is a p-adic unit. Thus exactly one of the two roots α and β is also a p-adic unit as their sum is a_p and their product is p. Let α be this unit root so that f_α is a form of slope 0 with the same Hecke-eigenvalues as f away from p.[29]

Returning to modular symbols, we can consider the modular symbol φ_{f_α} attached to f_α. Since f_α is a U_p-eigensymbol, we get the following formula (analogous to what we derived above but without the extra bothersome term):

$$\alpha\cdot\varphi_{f_\alpha}(\{a/p^n\}-\{\infty\})=\sum_{a=0}^{p-1}\varphi_{f_\alpha}(\{(a+jp^n)/p^{n+1}\}-\{\infty\}).$$

[29]Note then that f_β is a critical slope form, and in fact, all critical slope forms arise in this manner.

This relation is nearly what we had hoped for except for the the presence of the α on the left hand side. But, this is easily dealt with. Indeed, we set

$$\mu_f(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha^n}\varphi_{f_\alpha}(\{a/p^n\} - \{\infty\}),$$

and an easy computation shows that μ_f satisfies the additivity relation in (8).[30] Further, since $f_\alpha = f(z) - \beta f(pz)$, a simple computation shows that $\varphi_{f_\alpha} = \varphi_f - \frac{1}{\alpha}\varphi_f | \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Plugging this into the definition of μ_f yields the formula stated in section 2.12 (and hopefully explains where that crazy formula comes from!). Lastly, the interpolation property follows directly from (7) which in turn followed from (2).

5.2. **Proving that $\Phi_{f_\alpha}(\{0\} - \{\infty\})$ is the p-adic L-function.** In this section, we prove Theorem 3.2 for weight 2 ordinary forms. That is, let Φ_{f_α} be the unique overconvergent modular symbol in $\mathrm{Symb}_{\Gamma_0}(\mathcal{D})$ lifting φ_{f_α}. We will show that $\Phi_{f_\alpha}(\{0\} - \{\infty\}) = \mu_f$, the p-adic L-function of f.

Note that

$$\begin{aligned} \Phi_{f_\alpha}(\{0\} - \{\infty\}) &= \alpha^{-n}(\Phi_{f_\alpha}|U_p^n)(\{0\} - \{\infty\}) \\ &= \alpha^{-n} \sum_{b=0}^{p^n-1} \Phi_{f_\alpha}\left(\{b/p^n\} - \{\infty\}\right) \Big| \left(\begin{smallmatrix} 1 & b \\ 0 & p^n \end{smallmatrix}\right). \end{aligned}$$

Evaluating at the characteristic function of $a + p^n\mathbb{Z}_p$ gives

$$\begin{aligned} &\Phi_{f_\alpha}(\{0\} - \{\infty\})(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\ &\qquad = \alpha^{-n} \sum_{b=0}^{p^n-1} \left(\Phi_{f_\alpha}\left(\{b/p^n\} - \{\infty\}\right) \Big| \left(\begin{smallmatrix} 1 & b \\ 0 & p^n \end{smallmatrix}\right)\right)(\mathbf{1}_{a+p^n\mathbb{Z}_p}) \\ &\qquad = \alpha^{-n} \sum_{b=0}^{p^n-1} \Phi_{f_\alpha}\left(\{b/p^n\} - \{\infty\}\right)\left(\left(\begin{smallmatrix} 1 & b \\ 0 & p^n \end{smallmatrix}\right) \cdot \mathbf{1}_{a+p^n\mathbb{Z}_p}\right) \\ &\qquad = \alpha^{-n}\Phi_{f_\alpha}\left(\{a/p^n\} - \{\infty\}\right)\left(\mathbf{1}_{\mathbb{Z}_p}\right), \end{aligned}$$

as

$$\left(\begin{smallmatrix} 1 & b \\ 0 & p^n \end{smallmatrix}\right) \cdot \mathbf{1}_{a+p^n\mathbb{Z}_p} = \begin{cases} \mathbf{1}_{a+p^n\mathbb{Z}_p} & \text{if } b \equiv a \pmod{p^n} \\ 0 & \text{otherwise} \end{cases}.$$

But the specialization map for $k = 0$ is simply taking total measure. Thus, we get

$$\begin{aligned} \Phi_{f_\alpha}(\{0\} - \{\infty\})(\mathbf{1}_{a+p^n\mathbb{Z}_p}) &= \alpha^{-n}\rho^*(\Phi_{f_\alpha})\left(\{a/p^n\} - \{\infty\}\right) \\ &= \alpha^{-n}\varphi_{f_\alpha}\left(\{a/p^n\} - \{\infty\}\right) \end{aligned}$$

which agrees exactly with the definition of the p-adic L-function from the previous section.

[30] In fact, this definition only tells us the value of μ_f on locally constant functions. However, since the values given are bounded (α is a unit!), standard arguments via Riemann sums allow us to integrate μ_f against any continuous function (and thus against any locally analytic function).

6. Some examples

We close this paper with a few explicit computations of overconvergent eigensymbols. Let's begin by once again letting f denote the unique normalized newform in $S_2(\Gamma_0(11))$, let ψ_f denote the corresponding $\mathbb{C}$-valued modular symbol built out of period integrals, and let $\varphi_f = \psi_f^+/\Omega_E^+ + \psi_f^-/\Omega_E^-$. Following the algorithms (and notation) of section 2.6, recall that every modular symbol of level $\Gamma_0(11)$ is determined by its values on the divisors D_1, D_2, and D_3. Explicitly, we have

$$\varphi_f(D_1) = \frac{1}{5},\ \varphi_f(D_2) = \frac{-3}{2},\ \varphi_f(D_3) = \frac{1}{2}.$$

For notational convenience, we will express the values of φ_f on D_1, D_2 and D_3 as:

$$\varphi_f \longleftrightarrow \left[\frac{1}{5}, -\frac{3}{2}, \frac{1}{2}\right].$$

We now set $p = 11$ (so we won't have to worry about p-stabilizations), and we seek to construct an overconvergent lift of φ_f. Following Greenberg's trick, we will lift φ_f to a $\mathcal{F}(1)$-valued modular symbol. Recall that $\mathcal{F}(1)$ is isomorphic to $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ where the first factor should be thought of as the 0-th moment mod p^2 and the second factor is the 1st moment mod p. To compute this lift, we first lift randomly, and then apply U_p. The result is the $\mathcal{F}(1)$-valued modular symbol

$$\Phi_1 \longleftrightarrow [(97,3),(59,4),(61,4)].$$

Here we are denoting an element of $\mathcal{F}(1)$ as an ordered pair (a,b) where a is an element of $\mathbb{Z}/p^2\mathbb{Z}$ and b is in $\mathbb{Z}/p\mathbb{Z}$.

We observe that Φ_1 does indeed "lift" φ_f in the sense the 0-th moments of the values of Φ_1 match the values of φ_f modulo p^2. For instance, the 0-th moment of $\Phi_1(D_1)$ is 97 and the difference

$$97 - \varphi_f(D_1) = 97 - \frac{1}{5} = \frac{484}{5} = 11^2 \cdot \frac{4}{5}$$

is indeed divisible by 11^2.

Repeating Greenberg's lifting trick yields the following data:

$$\Phi_2 \longleftrightarrow [(1065,47,6),(664,37,5),(666,70,1)]$$
$$\Phi_3 \longleftrightarrow [(11713,894,72,6),(7319,1126,27,5),(7321,1159,100,9)]$$
$$\Phi_4 \longleftrightarrow [(128841,7549,798,50,4),(80524,1126,1237,60,2),(80526,13138,463,86,6)]$$
$$\vdots$$
$$\begin{aligned}\Phi_{10} \longleftrightarrow\ & [(228249336489,4107437612,751372925,96227115,15695904,1666537,14032,10489,929,102,0),\\ & (142655835304,17332646489,213170204,172173501,12969871,150949,143485,7580,1257,2,2),\\ & (142655835306,5630575713,733658311,186492667,19306282,1166702,17021,8513,1017,57,6)]\end{aligned}$$

So that's a lot of numbers. We note that using, for instance, SAGE [13] to do these computations takes less than a second to compute Φ_{10}. Note also that the value $\Phi_{10}(D_1)$ gives an approximation of the moments of the 11-adic L-function of f. We further note that if one tried to compute the 11-adic L-function of f directly from the modular symbol φ_f via Riemann sums, it would take a very very long time. Moreover, to get 1 additional digit of accuracy would require 11 times as much computation. By contrast, using this overconvergent modular symbols algorithm, to get one additional digit only requires lifting and applying U_{11} one more time.

Writing this L-function in the T-variable (as in [11, section 9]) gives

$$L_p(f,T) \approx 1490719231T + 433614230T^2 + 133205004T^3 + \dots.$$

Note that $L_p(f,T)$ vanishes at $T = 0$ – this is a trivial zero coming from the fact that $X_0(11)$ has split multiplicative reduction at 11. Further note that the leading term 1490719231 is an 11-adic unit. Thus, the μ-invariant of f is 0 and the λ-invariant is 1.

We close with one final example. Recall the symbol φ_1 defined by

$$\varphi_1 \longleftrightarrow [1,0,0]\,.$$

This symbol is an Eisenstein eigensymbol. Lifting as before yields:

$$\begin{aligned}
&\Phi_1 \longleftrightarrow [(1,0),(0,4),(0,2)]\\
&\Phi_2 \longleftrightarrow [(1,0,0),(0,81,5),(0,101,9)]\\
&\Phi_3 \longleftrightarrow [(1,0,0,0),(0,444,27,9),(0,222,64,9)]\\
&\Phi_4 \longleftrightarrow [(1,0,0,0,0),(0,9761,148,9,3),(0,12201,185,97,6)]\\
&\Phi_5 \longleftrightarrow [(1,0,0,0,0,0),(0,53684,8134,493,3,1),(0,26842,2847,339,50,9)]\\
&\quad\vdots\\
&\Phi_{10} \longleftrightarrow\\
&\quad [(1,0,0,0,0,0,0,0,0,0,0),\\
&\quad (0,17291616401,261994188,87331396,16119018,58323,19441,1600,977,3,1),\\
&\quad (0,21614520501,327492735,100232625,9457894,329206,93798,3776,380,10,0)]
\end{aligned}$$

Note that in this case the value of Φ_{10} on $\{0\}-\{\infty\}$ is simply the Dirac distribution at 0. The associated 11-adic L-function (after restriction to $\mathbb{Z}_p^\times$) thus vanishes!

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