Project ideas for "The parametrization of rings of small rank"

November 25, 2008

1. **Cubic Rings**

   (a) Warm up problem: What binary cubic form f corresponds to the cubic ring $\mathbb{Z}^3$? To $\mathbb{Z}[\sqrt[3]{n}]$?

   (b) Consider the form $\mathrm{Tr}(x^2)$ on the cubic ring $R = R(f)$. Now restrict this form to the sublattice of R consisting of elements of trace 0. What is the interpretation of this quadratic form in terms of the corresponding binary cubic f?

   (c) Write down some examples of cubic rings inside Galois cubic fields. Do they all have three automorphisms? What are the associated binary cubics? What can you say about the $\mathrm{Tr}(x^2)$ form for a cubic ring having three automorphisms? Can you use this to give an explicit parametrization of such "C_3-cubic rings"?

   (d) Show that the cubic ring given by a binary cubic form lies in the field generated by the coordinates of the points cut out in $\mathbb{P}^1$ by the form. What if the field is quadratic?

   (e) What can you say about the integers represented by a binary cubic form f, making use of the relationship with the corresponding cubic ring $R(f)$?

2. **Quartic Rings**

   (a) Warm up: What pair of ternary quadratic forms corresponds to $\mathbb{Z}^4$? To $\mathbb{Z}[\sqrt[4]{n}]$? To $\mathbb{Z}[\sqrt{a}, \sqrt{b}]$? Or your favorite quartic ring?

   (b) Find pairs of ternary quadratic forms corresponding to quartic rings that have some special kind of structure – for example, those lying inside $K \oplus \mathbb{Q}$ where K is a cubic field. How does this relate to cubic rings and binary cubic forms? What about other types of special structure, such as the various possible Galois groups? Can you find nice representatives for the pairs of ternary quadratic forms corresponding to these? Can you find a parametrization space for quartic rings with one of these structures?

(c) Show that the quartic ring given by a pair of ternary quadratic forms lies in the field generated by the coordinates of the points cut out in $\mathbb{P}^2$ by these forms. In particular, what happens in some of the special cases considered in part (b)?

(d) What can you say about the pairs of integers represented by a pair of ternary quadratic forms in terms of the corresponding quartic ring?

3. **Quintic Rings**

(a) Give examples of forms corresponding to some examples of quintic rings.

(b) Can you find a parametrization space for quintic rings with some special structure (as in 2(b))?

4. **Noncommutative Rings**

(a) Consider some of the analogous questions for quaternion and octonion rings!

(b) By taking the $\mathrm{Tr}(x^2)$ form on a quartic ring (restricted again to the trace 0 part of the ring), one obtains a ternary quadratic form, which corresponds to a quaternion ring! What is the relation between this quaternion ring and the original quartic ring?