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The Southwest Center for Arithmetic Geometry
AWS 2008: *Special Functions and Transcendence*

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1. Historical introduction to irrationality

1.1 *Early history.*
Irrationality of $\sqrt{2}$.
Irrationality of π by Lambert.
Continued fraction expansion of e by Euler.

1.2 *Fourier.*
Fourier's proof of the irrationality of e.
Extensions by Liouville.

1.3 *Introduction to Hermite's work.*
Proof of the irrationality of e^r for r a non-zero rational number.
Proof of the irrationality of π.
Proof of the irrationality of $e^r \notin \mathbb{Q}(i)$ for $r \in \mathbb{Q}(i)^\times$.

2. Historical introduction to transcendence methods.

2.1 *Proof of the transcendence of e, following Hermite.*

2.2 *Transcendence theory up to 1900.*
Hermite, Lindemann, Weierstraß.
Cantor. Metric Theory.

3. The role of auxiliary functions in transcendence proofs.

3.1 *Padé approximants.*
Padé approximants for the exponential function: type II (Hermite) and I (Hermite–Mahler).

3.2 *Interpolation methods.*
Weierstraß question on transcendental entire functions with many algebraic values. Interpolation series.
Polya (1914): integer valued entire functions. Gel'fond, Fukasawa, Masser, Gramain.
Interpolation by Hermite and R. Lagrange.
Gel'fond (1929): transcendence of e^π.

3.3 *Auxiliary functions arising from Dirichlet's box principle.*
Thue–Siegel's Lemma (Dirichlet's box principle), Gel'fond–Schneider's solution of Hibert's seventh problem (1934).

3.4 *Laurent's interpolation determinants.*

3.5 *Bost slope inequalities, Arakelov's theory.*

4. The main open problem in transcendental number theory: Schanuel's Conjecture.

4.1 *Statement of the Conjecture.*
List of known special cases.

4.2 *Conjecture on algebraic independence of logarithms of algebraic numbers.*
Special cases.
Six exponentials Theorem, four exponentials Conjecture, *strong* versions (D. Roy).
Rank of matrices whose entries are logarithms of algebraic numbers.
Density statements (Mazur's Conjecture).

4.3 *Some other consequences of Schanuel's Conjecture.*

4.4 *Quantitative version: measures of algebraic independence.*

4.4 *Roy's Conjecture.*
Gel'fond Criterion for algebraic independence. Variants with multiplicities.
Statement of Roy's Conjecture and equivalence with Schanuel.

`Rough outlines of the project`

a) Study the known constructions of transcendental functions with algebraic values.

`References`: K. Mahler [4], F. Gramain, Stäckel, Surroca...

b) Check the new proof of the irrationality of $\zeta(3)$ recently obtained by Tanguy Rivoal [6] involving interpolation by Hermite and R. Lagrange. If possible deduce new results.

c) (Expanding a remark by S. Lang – [3]). Define $K_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define K_n as the algebraic closure of the field generated over K_{n-1} by the numbers e^x, where x ranges over K_{n-1}. Let Ω_+ be the union of K_n, $n \geq 0$. Show that the numbers

$$\pi,\ \log \pi,\ \log\log \pi,\ \log\log\log \pi, \ldots$$

are algebraically independent over Ω_+.

d) Try to get a (conjectural) generalisation involving the field Ω_- defined as follows. Define $E_0 = \overline{\mathbb{Q}}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let Ω_- be the union of L_n, $n \geq 0$.'

References

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[2] H. Lambert – *Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques*, Mémoires de l'Académie des Sciences de Berlin, 17 (1761), 1768, p. 265–322; lu en 1767; Math. Werke, t. II.

[3] S. Lang – *Introduction to transcendental numbers*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966. *Collected papers. Vol. I*, Springer-Verlag, New York, 2000, 1952–1970.

[4] K. Mahler, – *Lectures on transcendental numbers*, Lecture Notes in Mathematics, Vol. **546** Springer-Verlag, Berlin-New York (1976).

[5] I. Niven – *Irrational numbers*, Carus Math. Monographs **11** (1956).

[6] T. Rivoal – *Applications arithmétiques de l'interpolation lagrangienne*, IJNT, to appear.

[7] S.A. Shirali – *Continued fraction for e*, Resonance, vol. **5** N°1, Jan. 2000, 14–28.
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[8] B. Sury – *Bessels contain continued fractions of progressions*, Resonance, vol. **10** N°3, March 2005, 80–87.
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