OUTLINE AND REFERENCES FOR PROJECT: AUTOMORPHISMS OF RIEMANN SURFACES

PAULA TRETKOFF

Project assisted by Ahmad El-Guindy, Texas A&M,
http://www.math.tamu.edu/~elguindy/

Project outline: Throughout my course, arithmetic groups play a special role. For example, $\mathrm{PSL}(2,\mathbb{Z})$ is an arithmetic group. We define *arithmetic Riemann surfaces* to be those uniformized by *arithmetic discrete subgroups* of $\mathrm{PSL}(2,\mathbb{R})$. In other words, they are quotients of the upper half plane by such groups. Schwarz [13] showed that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite, and Hurwitz showed that its order is at most the sharp bound $84(g-1)$. It is also well known that there are infinitely many genera for which the bound $84(g-1)$ is not attained. It therefore makes sense to consider the maximal order $N(g)$ of the group of automorphisms of any Riemann surface of genus g. By work of Accola [1] and Maclachlan [12], combined with the result of Hurwitz, we have sharp bounds: $N(g)$ is bounded above by $84(g-1)$ and below by $8(g+1)$. All extremal surfaces for Hurwitz's upper bound are arithmetic, whereas all those for the Accola–Maclachlan lower bound are *non-arithmetic*, that is, not arithmetic.

The aim of the project is to study a paper of Belolipetsky and Jones [4], in which the arithmetic case is studied separately. Let $N(g,\mathrm{ar})$ and $N(g,\mathrm{nar})$ be the maximal orders of the automorphism groups of the arithmetic and non-arithmetic surfaces of genus g, respectively. The non-arithmetic analog of Hurwitz's upper bound, obtained by Belolipetsky in [3], is $156(g-1)/7$; this bound is sharp, and the least genus attaining it is 50. Therefore $N(g,\mathrm{nar})$ is between $8(g+1)$ and $156(g-1)/7$. In their paper, Jones and Belilopetsky obtain an arithmetic analog of the Accola-Maclachlan lower bound. Namely, they show that, for each $g \geq 2$, there is an arithmetic surface of genus g with $4(g-1)$ automorphisms. Moreover, this bound is attained for infinitely many g, starting with 24. Therefore $N(g,\mathrm{ar})$ is between $4(g-1)$ and $84(g-1)$.

Typeset March 2, 2008 [14:03].
1991 *Mathematics Subject Classification.* 11J91, 33C65.
Key words and phrases. Arithmetic groups, Riemann surfaces.

Among the most interesting compact Riemann surfaces are those with a group G of automorphisms which is relatively large compared with g. The general problem of determining all such surfaces S and groups G is very difficult, but it tends to be easier when the Euler characteristic $\chi = 2(1-g)$ of S has a simple numerical form. A second paper of Belolipetsky and Jones [5] considers the simplest case, where $g = p+1$ for some prime p. In this situation, Accola [2] has determined the possibilities for G where $|G| = 8(g+1)$. These results are extended in [5] to the case $|G| = \lambda(g-1)$ for each $\lambda > 6$. In this paper, the authors show that, if p is sufficiently large (with respect to λ), then S and G lie in one of six easily described infinite families. You are encouraged to study this paper as well and, if possible, to expand the list of examples of Riemann surfaces with large groups of automorphisms compared to their genera g.

The references for these two papers are as follows:

MR2132170 (2005k:30077) Belolipetsky, Mikhail; Jones, Gareth A. A bound for the number of automorphisms of an arithmetic Riemann surface. Math. Proc. Cambridge Philos. Soc. 138 (2005), no. 2, 289–299. (Reviewer: Jane Gilman) 30F35 (14H37 14H55) (on the arxiv at math.GR/0306105)

MR2203507 (2007a:14032) Belolipetsky, Mikhail(RS-AOSSI); Jones, Gareth A.(4-SHMP-SM) Automorphism groups of Riemann surfaces of genus $p+1$, where p is prime. (English summary) Glasg. Math. J. 47 (2005), no. 2, 379–393. 14H37 (30F35) (on the arxiv at math.GR/0306106)

Also obtainable, along with other papers of Belolipetsky, at
http://front.math.ucdavis.edu/author/M.Belolipetsky

Suggested Reading on Prerequisite Material

Suggested reference on Riemann surfaces:

Reference [9]: Jones, G.A., Singerman, D.: Complex Functions, An algebraic and geometric viewpoint, Camb. U. Press, Cambridge, 1987, reprinted 1997. ISBN 0-521-31366-X

Suggested reference on arithmetic groups:

Reference [10]: Katok, S.: Fuchsian Groups, Chicago Lectures in Math., Chicago, 1992. ISBN 0-226-42583-5

Additional references on Riemann surfaces:

Reference [7]: Farkas, H.M., Kra, I.: Riemann surfaces, Grad. Texts in Math. 71, Springer-Verlag, 1980.
Robert Gunning's book in progress *Function Theory of Compact Riemann Surfaces*, available on his website:
http://www.math.princeton.edu/~gunning/book.pdf
Chapter 3 on "Holomorphic differentials" is relevant to both my project and my course. Appendix E on "Complex Tori" is relevant to my course.
Related results we will not study can be found in: [1], [2], [3], [11] [12].
[1], [11] are available online from Trans. Amer. Math. Soc.

Getting started!

The paper of Belolipetsky and Jones [4] is very nicely written, which helps considerably for our concentrated project at the AWS 2008. Everything is explained, and the background results are stated clearly, often with explicit references. At the AWS, we will get right into working through the paper itself, hopefully with time to move on to [5]. We will go rapidly over the prerequisite theorems, but accept them as facts without further ado. The extent of your investment in the project is up to you. It can be a motivation to brush up on, or to learn, some Riemann surface and group theory. It can provide a springboard for possible further research on these or related topics.

As minimal preparation before coming to the AWS, you need to read up on some basic results. You are encouraged to study [4] in advance, but I won't assume at the AWS that you have read it, and, if you are short on time, you should concentrate on some background reading rather than the paper itself. Of course, the AWS participants have very diverse mathematical backgrounds, so this will mean something different to each of you. I provide some suggested references above, but there are many, many other possibilities. I will bring with me to the AWS the references [9] and [10], which contain most of the prerequisite material.

Before reading on, print out [4] from the arXiv: math.GR/0306105

You may also like to print out my Lecture Notes for the AWS 2008 (available on the AWS website: http://math.arizona.edu/~swc/aws/08/). There is some intersection with the material for this project.

To get started, you need to know:

– the definition of a Riemann surface and its genus (see p2 of [4], and Chapter 4 of [9])): we only deal with *compact* Riemann surfaces of *genus* $g \geq 2$.

– the definition of a Fuchsian group (of the first kind) (see p2 of [4], Chapter 5 of [4], and Chapter 2 of [10]).

– that a Fuchsian group acts on the upper half plane by fractional linear, or Möbius, transformations (Chapter 5 of [9]).

– the uniformization theorem in the case of compact Riemann surfaces of genus $g \geq 2$ (see p2 of [4], Chapter 4 of [9]).

– the fact that every cocompact Fuchsian group ($\Gamma\backslash\mathcal{H}$ is compact) has a presentation in terms of generators and relations given by its signature (see p2 of [4], Chapter 5 of [9], Chapter 4 of [10]).

– the definition of the automorphism group of a Riemann surface (Chapter 4 of [9]).

– Hurwitz's upper bound on the number of automorphisms of a compact Riemann surface (Chapter 5 of [9]).

– the definition of an arithmetic group (see p3 of [4], Chapter 5 of [10]).

– the definition of a triangle group. You should also look at Takeuchi's list of arithmetic triangle groups (just the results, not the proofs unless you are interested; see p238, Chapter 5 of [4]. It is worth a trip to the library to photocopy [14]).

– that the Hurwitz upper bound is attained only for arithmetic Riemann surfaces. This essentially follows from Theorem 5.11.2, Chapter 5 of [9], together with the fact that $(2, 3, 7)$ is an arithmetic signature (see p3 of [4], Example 2.1), that is, corresponds to an arithmetic triangle group.

– hyperbolic area, or measure (see §5.5, Chapter 5 of [9]): you really just need to know the formula for $\mu(\Gamma)$ on p2 of [4], but it is nice to understand where it comes from.

– that the automorphism group of a compact Riemann surface is the quotient of the normalizer of its surface group by the surface group (see p3 of [4], and Theorem 5.9.4, Chapter 5 of [9]).

– the discreteness of arithmetic volumes: see Section 8 of [6], whose full text is available on www.numdam.org

– Sylow's theorems for finite groups: for a short account see:

http://www.math.dartmouth.edu/~ ppollack/sylow.pdf

– that the *exponent* of a group is the smallest positive integer e such that, in the given group, the identity $g^e = 1$ holds.

– that [4] uses the notation $\mathbb{Z}_p$ for $Z/p\mathbb{Z}$, p prime.

– that the first integer homology group of a Riemann surface is the abelianization of its fundamental group.

– Maschke's Theorem: see books on basic representation theory. It states that if G is a finite group, and k a field of characteristic not dividing $|G|$, then any representation of G over k is completely reducible.

–Dirichlet's Theorem on arithmetic progressions: for any two positive coprime integers a and d, there are infinitely many primes of the form $a + nd$, where $n \geq 0$.

References

[1] Accola, R.D.: On the number of automorphisms of a closed Riemann surface. Trans. Amer. Math. Soc. 131 (1968), 398408.

[2] Accola, R.D.: Topics in the Theory of Riemann Surfaces, Lecture Notes in Math. 1595 (Springer-Verlag, 1994).

[3] Belolipetsky, M: On the number of automorphisms of a nonarithmetic Riemann surface. Siberian Math. J. 38 (1997), 860867.

[4] Belolipetsky, M., Jones, G.A.: A bound for the number of automorphisms of an arithmetic Riemann surface. Math. Proc. Cambridge Philos. Soc. 138 (2005), no. 2, 289–299.

[5] Belolipetsky, M.: Jones, G. A.: Automorphism groups of Riemann surfaces of genus $p+1$, where p is prime. (English summary) Glasg. Math. J. 47 (2005), no. 2, 379–393.

[6] Borel, A.: Commensurability classes and volumes of hyperbolic 3-manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), 133.

[7] Farkas, H.M., Kra, I.: Riemann surfaces, Grad. Texts in Math. 71 Springer-Verlag, 1980.

[8] Hurwitz, A.: Über algebraische Gebilde mit eindeutige Transformationen in sich. Math. Ann. 41 (1893), 403442.

[9] Jones, G.A., Singerman, D.: Complex Functions, An algebraic and geometric viewpoint, Camb. U. Press, Cambridge, 1987, reprinted 1997. ISBN 0-521-31366-X

[10] Katok, S.: Fuchsian Groups, Chicago Lectures in Mathematics, Chicago, 1992. ISBN 0-226-42583-5

[11] Kiley, W.T.: Automorphism groups of compact Riemann surfaces, Trans. Amer. Math. Soc., 150, No. 2 (1970), 557–563.

[12] Maclachlan, C.: A bound for the number of automorphisms of a compact Riemann surface. J. London Math. Soc. 44 (1968), 265272.

[13] Schwarz, H.A., Ueber diejenigen algebraischen Gleichungen zwischen zwei veränderlichen Gr̈ossen, welche eine Schaar rationaler, eindeutig umkehrbarer Transformationen in sich selbst zulassen. J. reine angew. Math. 87, 139145; Ges. Math. Abh. II , pp. 285291 (1890, reprinted Chelsea, 1972).

[14] Takeuchi, K.: Arithmetic triangle groups, J. Math. Soc. Japan 29 (1977), 91–106.

Current address: Department of Mathematics, Texas A&M University, TAMU 3368, TX 77843-3368, USA

E-mail address: `ptretkoff@math.tamu.edu`